Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 165420, 11 pages
http://dx.doi.org/10.1155/2013/165420

Research Article

Some Almost Generalized (ψ, ϕ)-Contractions in G-Metric Spaces

Hassen Aydi,[1] Sana Hadj Amor,[2] and Erdal Karapinar[3]

[1] *Department of Mathematics, Jubail College of Education, Dammam University, P.O. Box 12020, Industrial Jubail 31961, Saudi Arabia*
[2] *Laboratoire Physique Mathématique, Fonctions Spéciales et Applications (MAPSFA) LR11ES35, Ecole Supérieure des Sciences et de la Technologie de Hammam Sousse, Université de Sousse, Rue Lamine el Abbassi, 4011 Hammam Sousse, Tunisia*
[3] *Department of Mathematics, Atilim University, İncek, 06836 Ankara, Turkey*

Correspondence should be addressed to Hassen Aydi; hassen.aydi@isima.rnu.tn

Received 20 July 2013; Accepted 27 August 2013

Academic Editor: Chi-Ming Chen

In this paper, we introduce some almost generalized (ψ, ϕ)-contractions in the setting of G-metric spaces. We prove some fixed points results for such contractions. The presented theorems improve and extend some known results in the literature. An example is also presented.

1. Introduction and Preliminaries

In 2006, a new structure of a generalized metric space was introduced by Mustafa and Sims [1] as an appropriate notion of a generalized metric space called a G-metric space. Fixed point theory in this space was initiated in [2]. Particularly, Banach contraction mapping principle was established in this work. Since then the fixed point theory in G-metric spaces has been studied and developed by many authors, see [1–29].

The following definitions and results will be needed in the sequel.

Definition 1 (see [1]). Let X be a nonempty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

($G1$) $G(x, y, z) = 0$ if $x = y = z$,

($G2$) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

($G3$) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

($G5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \quad (1)$$

Example 2. Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \quad (2)$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x), \quad (3)$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 3 (see [1]). Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X; therefore, we say that x_n is G-convergent to $x \in X$ if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$; that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n \to +\infty} x_n = x$.

Proposition 4 (see [1]). *Let (X, G) be a G-metric space. The following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x,*

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$,

(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$.

Definition 5 (see [1]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$; that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 6 (see [1]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *the sequence $\{x_n\}$ is G-Cauchy,*

(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.*

Definition 7 (see [1]). A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 8. Let (X, G) be a G-metric space. A mapping $F : X \to X$ is said to be G-continuous if for any G-convergent sequence $\{x_n\}$ to x, then $\{F(x_n)\}$ is G-convergent to $F(x)$.

Now, let $\mathcal{F}$ denote the set of functions $f : [0, +\infty) \to [0, +\infty)$ such that $f(t) = 0$ if and only if $t = 0$. We denote by Ψ and Φ the subsets of $\mathcal{F}$ such that

$$\Psi = \{\psi \in \mathcal{F} : \psi \text{ is continuous and nondecreasing}\},$$
$$\Phi = \{\phi \in \mathcal{F} : \phi \text{ is lower semicontinuous}\}. \tag{4}$$

There are a lot of fixed point theorems for different type contractions in the literature. In particular, Berinde [30–32] introduced the concept of an almost contraction in metric spaces and studied many interesting fixed point theorems for a Ćirić strong almost contraction. For other fixed point results on generalized almost contractions, see [33–37]. In this paper, we introduce some almost generalized (ψ, ϕ)-contractions in the setting of G-metric spaces, and we establish some fixed points results for such contractions.

2. Main Results

Let (X, G) be a G-metric space. First, we consider the following expressions:

$$\begin{aligned} M(x, y, z) &= \max\{G(x, Tx, y), G(y, T^2x, Ty), G(Tx, T^2x, Ty), \\ &\quad G(x, Tx, z), G(z, T^2x, Tz), \\ &\quad G(Tx, T^2x, Tz), G(x, y, z)\}, \\ N(x, y, z) &= \min\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ &\quad G(z, Tx, Tx), G(y, Tz, Tz)\}, \end{aligned} \tag{5}$$

for all $x, y, z \in X$.

Our first result for almost generalized (ψ, ϕ)-contractions is the following.

Theorem 9. *Let (X, G) be a complete G-metric space. Let $T : X \to X$ be a self-mapping. Suppose there exist $\psi \in \Psi$, $\phi \in \Phi$, and $L \geq 0$ such that for all $x, y, z \in X$,*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(M(x, y, z)) - \phi(M(x, y, z)) + LN(x, y, z). \tag{6}$$

Then T has a unique fixed point; say $u \in X$.

Proof. Let $x_0 \in X$, and define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \quad \text{for any } n \in \mathbb{N}. \tag{7}$$

If for some $n \in \mathbb{N}$, $x_{n+1} = x_n$, then $x_n = Tx_n$, and the proof is completed. Thus, we may assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By (6) we have

$$\begin{aligned} \psi(G(x_n, x_{n+1}, x_{n+1})) &= \psi(G(Tx_{n-1}, Tx_n, Tx_n)) \\ &\leq \psi(M(x_{n-1}, x_n, x_n)) \\ &\quad - \phi(M(x_{n-1}, x_n, x_n)) + LN(x_{n-1}, x_n, x_n), \end{aligned} \tag{8}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\{G(x_{n-1}, Tx_{n-1}, x_n), G(x_n, T^2x_{n-1}, Tx_n), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, Tx_n), G(x_{n-1}, Tx_{n-1}, x_n), \\ &\quad G(x_n, T^2x_{n-1}, Tx_n), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, Tx_n), G(x_{n-1}, x_n, x_n)\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_{n-1}, x_n, x_n)\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}, \\ N(x_{n-1}, x_n, x_n) &= \min\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\ &\quad G(x_n, Tx_n, Tx_n), G(x_n, Tx_{n-1}, Tx_{n-1}), \\ &\quad G(x_n, Tx_n, Tx_n)\} \end{aligned}$$

$$
\begin{aligned}
&= \min \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\
&\qquad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), \\
&\qquad G(x_n, x_{n+1}, x_{n+1})\} \\
&= \min \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), 0\} = 0.
\end{aligned} \tag{9}
$$

From (8) and (9), we get

$$
\begin{aligned}
&\psi(G(x_n, x_{n+1}, x_{n+1})) \\
&\quad \le \psi(\max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}) \\
&\qquad - \phi(\max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}).
\end{aligned} \tag{10}
$$

If

$$
\begin{aligned}
&\max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\
&\quad = G(x_n, x_{n+1}, x_{n+1}),
\end{aligned} \tag{11}
$$

then by (10) we have

$$
\begin{aligned}
&\psi(G(x_n, x_{n+1}, x_{n+1})) \\
&\quad \le \psi(G(x_n, x_{n+1}, x_{n+1})) - \phi(G(x_n, x_{n+1}, x_{n+1})).
\end{aligned} \tag{12}
$$

Thus $\phi(G(x_n, x_{n+1}, x_{n+1})) = 0$, and hence $G(x_n, x_{n+1}, x_{n+1}) = 0$. Therefore, $x_n = x_{n+1}$ which is a contradiction. So,

$$
\max \{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n). \tag{13}
$$

Therefore, we get

$$
G(x_n, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) \quad \forall n \in \mathbb{N}, \tag{14}
$$

and (10) becomes

$$
\begin{aligned}
&\psi(G(x_n, x_{n+1}, x_{n+1})) \\
&\quad \le \psi(G(x_{n-1}, x_n, x_n)) - \phi(G(x_{n-1}, x_n, x_n)) \\
&\qquad \forall n \in \mathbb{N}.
\end{aligned} \tag{15}
$$

Thus, by (14), the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is monotone nonincreasing. It follows that $G(x_n, x_{n+1}, x_{n+1}) \to \alpha$ as $n \to +\infty$ for some $\alpha \ge 0$. Next we claim that $\alpha = 0$. On taking limit as $n \to +\infty$ in (15), we obtain

$$
\psi(\alpha) \le \psi(\alpha) - \phi(\alpha). \tag{16}
$$

Hence $\phi(\alpha) = 0$ and we get $\alpha = 0$. Hence

$$
\lim_{n \to +\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{17}
$$

Next, we show that $\{x_n\}$ is a G-Cauchy sequence. On contrary, assume that $\{x_n\}$ is not a G-Cauchy sequence. Then, there is an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) \ge k$ such that

$$
\begin{aligned}
&G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \ge \varepsilon, \\
&G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) < \varepsilon.
\end{aligned} \tag{18}
$$

Using (18) and $(G5)$, we have

$$
\begin{aligned}
\varepsilon &\le G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&\le G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) \\
&< \varepsilon + 2G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}).
\end{aligned} \tag{19}
$$

Taking limit as $k \to +\infty$ and using (17), we have

$$
\lim_{k \to +\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{20}
$$

By $(G5)$, we get

$$
\begin{aligned}
&G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&\quad \le G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
&\qquad + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \\
&\quad \le G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
&\qquad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\quad \le G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) \\
&\qquad + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\qquad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\
&\quad \le 3G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&\qquad + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}).
\end{aligned} \tag{21}
$$

On taking limit as $k \to +\infty$ in the above inequality and using (17) and (20), we obtain

$$
\lim_{k \to +\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{22}
$$

Again, by $(G5)$, we have

$$
\begin{aligned}
&G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}) \\
&\quad \le G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+2}) \\
&\quad \le G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{m(k)+2}, x_{m(k)+1}, x_{m(k)+1}) \\
&\qquad + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\
&\quad \le G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\qquad + G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\
&\qquad + G(x_{m(k)+2}, x_{n(k)+1}, x_{n(k)+1})
\end{aligned}
$$

$$
\begin{aligned}
&\leq G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right) + 2G\left(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}\right) \\
&\quad + G\left(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}\right) + G\left(x_{n(k)+1}, x_{n(k)}, x_{n(k)}\right) \\
&\quad + G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}\right) \\
&\leq G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right) + 2G\left(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}\right) \\
&\quad + G\left(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}\right) \\
&\quad + 2G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right) + G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}\right).
\end{aligned}
\tag{23}
$$

On taking limit as $k \to +\infty$ in the above inequality and using (17) and (22), we obtain

$$
\lim_{k \to +\infty} G\left(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}\right) = \varepsilon. \tag{24}
$$

We have also, by $(G5)$,

$$
\begin{aligned}
&G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right) \\
&\quad \leq G\left(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}\right) + G\left(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}\right), \\
&G\left(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}\right) \\
&\quad \leq G\left(x_{m(k)+1}, x_{m(k)}, x_{m(k)}\right) + G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right).
\end{aligned}
\tag{25}
$$

On taking limit as $k \to +\infty$ in the above inequality and using (17) and $(G3)$, we obtain

$$
\begin{aligned}
&\lim_{k \to +\infty} G\left(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}\right) \\
&\quad = \lim_{k \to +\infty} G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right).
\end{aligned}
\tag{26}
$$

We have also, by $(G3)$ and $(G5)$,

$$
\begin{aligned}
&G\left(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}\right) \\
&\quad \leq G\left(x_{n(k)}, x_{m(k)+1}, x_{n(k)+1}\right) \\
&\quad \leq G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right) + G\left(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}\right), \\
&\qquad G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) \leq G\left(x_{m(k)}, x_{n(k)}, x_{m(k)+1}\right).
\end{aligned}
\tag{27}
$$

Now, on taking limit as $k \to +\infty$ in (27), and using (17), (20), and (26), we obtain

$$
\begin{aligned}
&\lim_{k \to +\infty} G\left(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}\right) \\
&\quad = \lim_{k \to +\infty} G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right) = \varepsilon.
\end{aligned}
\tag{28}
$$

Furthermore, by $(G3)$ and $(G5)$, we get

$$
\begin{aligned}
G\left(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}\right) &\leq G\left(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}\right) \\
&\leq G\left(x_{m(k)+2}, x_{m(k)+1}, x_{m(k)+1}\right) \\
&\quad + G\left(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}\right) \\
&\leq 2G\left(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}\right) \\
&\quad + G\left(x_{n(k)+1}, x_{n(k)}, x_{n(k)}\right) \\
&\quad + G\left(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}\right) \\
&\leq 2G\left(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}\right) \\
&\quad + 2G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right) \\
&\quad + G\left(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}\right).
\end{aligned}
\tag{29}
$$

On taking limit as $k \to +\infty$ in the above inequality, and using (17), (20), (22), and (28), we obtain

$$
\lim_{k \to +\infty} G\left(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}\right) = \varepsilon. \tag{30}
$$

Now, we have

$$
\begin{aligned}
&\psi\left(G\left(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}\right)\right) \\
&\quad = \psi\left(G\left(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right)\right) \\
&\quad \leq \psi\left(M\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right)\right) \\
&\qquad - \phi\left(M\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right)\right) \\
&\qquad + LN\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) \\
&\quad \leq \psi\left(\max\left\{G\left(x_{m(k)}, Tx_{m(k)}, x_{n(k)}\right),\right.\right. \\
&\qquad\qquad G\left(x_{n(k)}, T^2x_{m(k)}, Tx_{n(k)}\right), \\
&\qquad\qquad G\left(Tx_{m(k)}, T^2x_{m(k)}, Tx_{n(k)}\right), \\
&\qquad\qquad G\left(x_{m(k)}, Tx_{m(k)}, x_{n(k)}\right), \\
&\qquad\qquad G\left(x_{n(k)}, T^2x_{m(k)}, Tx_{n(k)}\right), \\
&\qquad\qquad G\left(Tx_{m(k)}, T^2x_{m(k)}, Tx_{n(k)}\right), \\
&\qquad\qquad \left.\left. G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right)\right\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\phi(\max\{G(x_{m(k)},Tx_{m(k)},x_{n(k)}),\\
&\qquad G(x_{n(k)},T^2x_{m(k)},Tx_{n(k)}),\\
&\qquad G(Tx_{m(k)},T^2x_{m(k)},Tx_{n(k)}),\\
&\qquad G(x_{m(k)},Tx_{m(k)},x_{n(k)}),\\
&\qquad G(x_{n(k)},T^2x_{m(k)},Tx_{n(k)}),\\
&\qquad G(Tx_{m(k)},T^2x_{m(k)},Tx_{n(k)}),\\
&\qquad G(x_{m(k)},x_{n(k)},x_{n(k)})\})\\
&+L(\min\{G(x_{m(k)},Tx_{m(k)},Tx_{m(k)}),\\
&\qquad G(x_{n(k)},Tx_{n(k)},Tx_{n(k)}),\\
&\qquad G(x_{n(k)},Tx_{n(k)},Tx_{n(k)}),\\
&\qquad G(x_{n(k)},Tx_{m(k)},Tx_{m(k)}),\\
&\qquad G(x_{n(k)},Tx_{n(k)},Tx_{n(k)})\})\\
&\le\psi(\max\{G(x_{m(k)},x_{m(k)+1},x_{n(k)}),\\
&\qquad G(x_{n(k)},x_{m(k)+2},x_{n(k)+1}),\\
&\qquad G(x_{m(k)+1},x_{m(k)+2},x_{n(k)+1}),\\
&\qquad G(x_{m(k)},x_{n(k)},x_{n(k)})\})\\
&-\phi(\max\{G(x_{m(k)},x_{m(k)+1},x_{n(k)}),\\
&\qquad G(x_{n(k)},x_{m(k)+2},x_{n(k)+1}),\\
&\qquad G(x_{m(k)+1},x_{m(k)+2},x_{n(k)+1}),\\
&\qquad G(x_{m(k)},x_{n(k)},x_{n(k)})\})\\
&+L(\min\{G(x_{m(k)},x_{m(k)+1},x_{m(k)+1}),\\
&\qquad G(x_{n(k)},x_{n(k)+1},x_{n(k)+1}),\\
&\qquad G(x_{n(k)},x_{m(k)+1},x_{m(k)+1})\}).
\end{aligned}
\tag{31}
$$

Letting $k\to+\infty$, and using (17), (20), (22), (24), (28), and (30), and the properties of ϕ and ψ, we have

$$\psi(\varepsilon)\le\psi(\varepsilon)-\phi(\varepsilon). \tag{32}$$

Thus $\phi(\varepsilon)=0$ and hence $\varepsilon=0$, a contradiction. Thus $\{x_n\}$ is a G-Cauchy sequence in X.

Now, since (X,G) is G-complete, there are $x\in X$ such that $\{x_n\}$ is G-convergent to x; that is

$$\lim_{n\to+\infty}G(x_n,x_n,x)=\lim_{n\to+\infty}G(x_n,x,x)=0. \tag{33}$$

By (6), we get

$$
\begin{aligned}
&\psi(G(x_{n+1},x_{n+1},Tx))\\
&\quad=\psi(G(Tx_n,Tx_n,Tx))\\
&\quad\le\psi(M(x_n,x_n,x))-\phi(M(x_n,x_n,x))+LN(x_n,x_n,x),
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
M(x_n,x_n,x)&=\max\{G(x_n,Tx_n,x_n),G(x_n,T^2x_n,Tx_n),\\
&\qquad G(Tx_n,T^2x_n,Tx_n),G(x_n,Tx_n,x),\\
&\qquad G(x,T^2x_n,Tx),\\
&\qquad G(Tx_n,T^2x_n,Tx),G(x_n,x_n,x)\}\\
&=\max\{G(x_n,x_{n+1},x_n),G(x_n,x_{n+2},x_{n+1}),\\
&\qquad G(x_{n+1},x_{n+2},x_{n+1}),G(x_n,x_{n+1},x),\\
&\qquad G(x,x_{n+2},Tx),G(x_{n+1},x_{n+2},Tx),\\
&\qquad G(x_n,x_n,x)\},\\
N(x_n,x_n,x)&=\min\{G(x_n,Tx_n,Tx_n),G(x_n,Tx_n,Tx_n),\\
&\qquad G(x,Tx,Tx),G(x,Tx_n,Tx_n),\\
&\qquad G(x_n,Tx,Tx)\}\\
&=\min\{G(x_n,x_{n+1},x_{n+1}),G(x,Tx,Tx),\\
&\qquad G(x,x_{n+1},x_{n+1}),G(x_n,Tx,Tx)\}.
\end{aligned}
\tag{35}
$$

Letting $n\to+\infty$ in (35), we get

$$
\begin{aligned}
&\lim_{n\to+\infty}M(x_n,x_n,x)=G(x,x,Tx),\\
&\lim_{n\to+\infty}N(x_n,x_n,x)=0.
\end{aligned}
\tag{36}
$$

On letting $n\to+\infty$ in (34), and using the properties of ψ and ϕ and (36), we obtain

$$\psi(G(x,x,Tx))\le\psi(G(x,x,Tx))-\phi(G(x,x,Tx)). \tag{37}$$

Therefore $G(x,x,Tx)=0$ and hence $x=Tx$. Thus x is a fixed point of T.

Now our purpose is to check that such point is unique. Suppose that there are two fixed points of T; say $x,y\in X$ such that $x\neq y$. By (6), we have

$$
\begin{aligned}
&\psi(G(x,x,y))\\
&\quad=\psi(G(Tx,Tx,Ty))\\
&\quad\le\psi(M(x,x,y))-\phi(M(x,x,y))+LN(x,x,y),
\end{aligned}
\tag{38}
$$

where

$$\begin{aligned} M(x,x,y) &= \max\{G(x,Tx,x), G(x,T^2x,Tx), \\ &\quad G(Tx,T^2x,Tx), G(x,Tx,y), \\ &\quad G(y,T^2x,Ty), \\ &\quad G(Tx,T^2x,Ty), G(x,x,y)\} \\ &= \max\{G(x,x,y), G(y,x,y)\}, \\ N(x,x,y) &= \min\{G(x,Tx,Tx), G(x,Tx,Tx), \\ &\quad G(y,Ty,Ty), G(y,Tx,Tx), \\ &\quad G(x,Ty,Ty)\} = 0. \end{aligned} \quad (39)$$

Similarly, we can prove that

$$\begin{aligned} &\psi(G(x,y,y)) \\ &\quad = \psi(G(Tx,Ty,Ty)) \\ &\quad \le \psi(M(x,y,y)) - \phi(M(x,y,y)) + LN(x,y,y), \end{aligned} \quad (40)$$

where

$$\begin{aligned} M(x,y,y) &= \max\{G(x,x,y), G(y,x,y)\}, \\ N(x,y,y) &= 0. \end{aligned} \quad (41)$$

If

$$\max\{G(x,x,y), G(y,x,y)\} = G(x,x,y). \quad (42)$$

By (38) and (39), we get

$$\psi(G(x,x,y)) \le \psi(G(x,x,y)) - \phi(G(x,x,y)), \quad (43)$$

a contradiction. Then

$$\max\{G(x,x,y), G(y,x,y)\} = G(x,y,y). \quad (44)$$

By (40) and (41), we get

$$\psi(G(x,y,y)) \le \psi(G(x,y,y)) - \phi(G(x,y,y)), \quad (45)$$

a contradiction. Thus, $x = y$, and hence the fixed point of T is unique. □

As consequence of Theorem 9, we present the following corollaries.

Corollary 10. *Let (X,G) be a complete G-metric space. Let $T : X \to X$ be a self mapping. Suppose there exist $k \in [0,1)$ and $L \ge 0$ such that for all $x, y, z \in X$*

$$G(Tx,Ty,Tz) \le kM(x,y,z) + LN(x,y,z). \quad (46)$$

Then T has a unique fixed point; say $u \in X$.

Proof. It suffices to take $\psi(t) = t$ and $\phi(t) = (1-k)(t)$ in Theorem 9. □

Corollary 11. *Let (X,G) be a complete G-metric space. Let $T : X \to X$ be a self mapping. Suppose there exist $k \in [0,1)$ and $L \ge 0$ such that for all $x, y, z \in X$*

$$G(Tx,Ty,Tz) \le kG(x,y,z) + LN(x,y,z). \quad (47)$$

Then T has a unique fixed point; say $u \in X$.

Proof. Note that $G(x,y,z) \le M(x,y,z)$ in Corollary 10. □

Remark 12. Corollary 11 is a generalization of Mustafa's result [2].

Our second main result is given as follows.

Theorem 13. *Let (X,G) be a complete G-metric space. Let $T : X \to X$ be a self mapping. Suppose there exist $\psi \in \Psi$, $\phi \in \Phi$ and $L \ge 0$ such that for all $x, y \in X$:*

$$\begin{aligned} &\psi(G(Tx,Ty,T^2x)) \\ &\quad \le \psi(M^*(x,y,x)) - \phi(M^*(x,y,x)) + LN^*(x,y,x), \end{aligned} \quad (48)$$

where

$$\begin{aligned} &M^*(x,y,x) \\ &\quad = \max\{G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty), \\ &\qquad G(x,Tx,Tx), G(Tx,T^2x,T^2x), G(x,y,Tx)\}, \\ &N^*(x,y,x) \\ &\quad = \min\{G(x,Tx,Tx), G(y,Ty,Ty), G(y,Tx,Tx)\} \end{aligned} \quad (49)$$

for all $x, y \in X$. Then T has a unique fixed point; say $u \in X$.

Proof. Let x_0 be an arbitrary point in X and define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \quad \text{for any } n \in \mathbb{N}. \quad (50)$$

If for some $n \in \mathbb{N}$, $x_{n+1} = x_n$, then $x_n = Tx_n$ and the proof is completed. Thus, we may assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. By (48), we have

$$\begin{aligned} &\psi(G(x_n,x_{n+1},x_{n+1})) \\ &\quad = \psi(G(Tx_{n-1},Tx_n,T^2x_{n-1})) \\ &\quad \le \psi(M^*(x_{n-1},x_n,x_{n-1})) - \phi(M^*(x_{n-1},x_n,x_{n-1})) \\ &\qquad + LN^*(x_{n-1},x_n,x_{n-1}), \end{aligned} \quad (51)$$

where

$$\begin{aligned} &M^*(x_{n-1}, x_n, x_{n-1}) \\ &= \max\{G(x_{n-1}, Tx_{n-1}, x_n), G(x_n, T^2x_{n-1}, Tx_n), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, Tx_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ &\quad G(Tx_{n-1}, T^2x_{n-1}, T^2x_{n-1}), G(x_{n-1}, x_n, Tx_{n-1})\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n)\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}, \\ &N^*(x_{n-1}, x_n, x_{n-1}) \\ &\quad = \min\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\ &\quad\quad G(x_n, Tx_{n-1}, Tx_{n-1})\} \\ &\quad = \min\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\} \\ &\quad = 0. \end{aligned} \tag{52}$$

From (51) and (52), we get

$$\begin{aligned} &\psi(G(x_n, x_{n+1}, x_{n+1})) \\ &\quad \le \psi(\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}) \\ &\quad\quad - \phi(\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}). \end{aligned} \tag{53}$$

If

$$\begin{aligned} &\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\ &\quad = G(x_n, x_{n+1}, x_{n+1}), \end{aligned} \tag{54}$$

then by (53) we have

$$\begin{aligned} &\psi(G(x_n, x_{n+1}, x_{n+1})) \\ &\quad \le \psi(G(x_n, x_{n+1}, x_{n+1})) - \phi(G(x_n, x_{n+1}, x_{n+1})). \end{aligned} \tag{55}$$

Thus $\psi(G(x_n, x_{n+1}, x_{n+1})) = 0$ and hence $G(x_n, x_{n+1}, x_{n+1}) = 0$. Therefore, $x_n = x_{n+1}$, which is a contradiction. So,

$$\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n). \tag{56}$$

Therefore, we get

$$G(x_n, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) \quad \forall n \in \mathbb{N}, \tag{57}$$

and (53) becomes

$$\begin{aligned} \psi(G(x_n, x_{n+1}, x_{n+1})) &\le \psi(G(x_{n-1}, x_n, x_n)) \\ &\quad - \phi(G(x_{n-1}, x_n, x_n)) \\ &\quad \forall n \in \mathbb{N}. \end{aligned} \tag{58}$$

Thus, by (57), the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is monotone nonincreasing. It follows that $G(x_n, x_{n+1}, x_{n+1}) \to g$ as $n \to +\infty$ for some $g \ge 0$. Next we claim that $g = 0$. On taking limit as $n \to +\infty$ in (58), we obtain

$$\psi(g) \le \psi(g) - \phi(g). \tag{59}$$

Hence $\phi(g) = 0$ and we get $g = 0$. Then

$$\lim_{n\to+\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{60}$$

Next, we show that $\{x_n\}$ is a G-Cauchy sequence. On contrary, assume that $\{x_n\}$ is not a G-Cauchy sequence. Then, there is an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) \ge k$ such that

$$\begin{aligned} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) &\ge \varepsilon, \\ G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) &< \varepsilon. \end{aligned} \tag{61}$$

Using (61) and ($G5$), we have

$$\begin{aligned} \varepsilon &\le G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\le G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \varepsilon + 2G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}). \end{aligned} \tag{62}$$

Taking limit as $k \to +\infty$ and using (60), we have

$$\lim_{k\to+\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{63}$$

By ($G5$), and using (60) and (63), we obtain, as stated in the proof of Theorem 9,

$$\lim_{k\to+\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{64}$$

Again, by ($G5$), and using (60) and (64), similarly as in the proof of Theorem 9, we obtain

$$\lim_{k\to+\infty} G(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}) = \varepsilon. \tag{65}$$

Furthermore, by ($G3$) and ($G5$), we get

$$\begin{aligned} &G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \\ &\quad \le G(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}) \\ &\quad \le G(x_{m(k)+2}, x_{m(k)+1}, x_{m(k)+1}) \\ &\quad\quad + G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) \\ &\quad \le 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\ &\quad\quad + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}) \\ &\quad \le 2G(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}) \\ &\quad\quad + 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) \\ &\quad\quad + G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}). \end{aligned} \tag{66}$$

By the proof of Theorem 9, we have stated that

$$\lim_{k\to+\infty} G\left(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)}\right) = \varepsilon. \quad (67)$$

So, on taking limit as $k \to +\infty$ in (66), and using (60), (64), and (67), we obtain

$$\lim_{k\to+\infty} G\left(x_{m(k)+1}, x_{m(k)+2}, x_{n(k)+1}\right) = \varepsilon. \quad (68)$$

Furthermore, by $(G3)$ and $(G5)$, we get

$$\begin{aligned}
&G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}\right) \\
&\quad \leq G\left(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\right) + G\left(x_{n(k)}, x_{n(k)}, x_{m(k)+1}\right) \\
&\quad \leq G\left(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\right) + G\left(x_{m(k)+1}, x_{m(k)}, x_{m(k)}\right) \\
&\qquad + G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) \\
&\quad \leq G\left(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\right) + 2G\left(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}\right) \\
&\qquad + G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right), \\
&G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) \\
&\quad \leq G\left(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}\right) + G\left(x_{m(k)+1}, x_{n(k)}, x_{n(k)}\right) \\
&\quad \leq G\left(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}\right) + G\left(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}\right) \\
&\qquad + G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}\right).
\end{aligned} \quad (69)$$

On taking limit as $k \to +\infty$ in above inequalities, and using (60) and (63), we have

$$\lim_{k\to+\infty} G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}\right) = \varepsilon. \quad (70)$$

Also, by $(G3)$ and $(G5)$, we get

$$\begin{aligned}
&G\left(x_{m(k)+1}, x_{n(k)}, x_{n(k)}\right) \\
&\quad \leq G\left(x_{m(k)+1}, x_{m(k)}, x_{m(k)}\right) + G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) \\
&\quad \leq 2G\left(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}\right) + G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right), \\
&G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) \\
&\quad \leq G\left(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}\right) + G\left(x_{m(k)+1}, x_{n(k)}, x_{n(k)}\right).
\end{aligned} \quad (71)$$

On taking limit as $k \to +\infty$ in above inequalities, and using (60), (63), and (70), we obtain

$$\lim_{k\to+\infty} G\left(x_{m(k)+1}, x_{n(k)}, x_{n(k)}\right) = \varepsilon. \quad (72)$$

Now, we have

$$\begin{aligned}
&\psi\left(G\left(x_{n(k)}, x_{m(k)+2}, x_{n(k)+1}\right)\right) \\
&\quad = \psi\left(G\left(Tx_{n(k)-1}, Tx_{m(k)+1}, T^2x_{n(k)-1}\right)\right) \\
&\quad \leq \psi\left(M^*\left(x_{n(k)-1}, x_{m(k)+1}, x_{n(k)-1}\right)\right) \\
&\qquad - \phi\left(M^*\left(x_{n(k)-1}, x_{m(k)+1}, x_{n(k)-1}\right)\right) \\
&\qquad + LN^*\left(x_{n(k)-1}, x_{m(k)+1}, x_{n(k)-1}\right) \\
&\quad \leq \psi\left(\max\left\{G\left(x_{n(k)-1}, Tx_{n(k)-1}, x_{m(k)+1}\right),\right.\right. \\
&\qquad\qquad G\left(x_{m(k)+1}, T^2x_{n(k)-1}, Tx_{m(k)+1}\right), \\
&\qquad\qquad G\left(Tx_{n(k)-1}, T^2x_{n(k)-1}, Tx_{m(k)+1}\right), \\
&\qquad\qquad G\left(x_{n(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}\right), \\
&\qquad\qquad G\left(Tx_{n(k)-1}, T^2x_{n(k)-1}, T^2x_{n(k)-1}\right), \\
&\qquad\qquad \left.\left.G\left(x_{n(k)-1}, x_{m(k)+1}, Tx_{n(k)-1}\right)\right\}\right) \\
&\qquad - \phi\left(\max\left\{G\left(x_{n(k)-1}, Tx_{n(k)-1}, x_{m(k)+1}\right),\right.\right. \\
&\qquad\qquad G\left(x_{m(k)+1}, T^2x_{n(k)-1}, Tx_{m(k)+1}\right), \\
&\qquad\qquad G\left(Tx_{n(k)-1}, T^2x_{n(k)-1}, Tx_{m(k)+1}\right), \\
&\qquad\qquad G\left(x_{n(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}\right), \\
&\qquad\qquad G\left(Tx_{n(k)-1}, T^2x_{n(k)-1}, T^2x_{n(k)-1}\right), \\
&\qquad\qquad \left.\left.G\left(x_{n(k)-1}, x_{m(k)+1}, Tx_{n(k)-1}\right)\right\}\right) \\
&\qquad + L\min\left\{G\left(x_{n(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}\right),\right. \\
&\qquad\qquad G\left(x_{m(k)+1}, Tx_{m(k)+1}, Tx_{m(k)+1}\right), \\
&\qquad\qquad \left.G\left(x_{m(k)+1}, Tx_{n(k)-1}, Tx_{n(k)-1}\right)\right\} \\
&\quad \leq \psi\left(\max\left\{G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}\right),\right.\right. \\
&\qquad\qquad G\left(x_{m(k)+1}, x_{n(k)+1}, x_{m(k)+2}\right), \\
&\qquad\qquad G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}\right), \\
&\qquad\qquad G\left(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\right), \\
&\qquad\qquad \left.\left.G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right)\right\}\right) \\
&\qquad - \phi\left(\max\left\{G\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)+1}\right),\right.\right. \\
&\qquad\qquad G\left(x_{m(k)+1}, x_{n(k)+1}, x_{m(k)+2}\right), \\
&\qquad\qquad G\left(x_{n(k)}, x_{n(k)+1}, x_{m(k)+2}\right), \\
&\qquad\qquad G\left(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\right), \\
&\qquad\qquad \left.\left.G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right)\right\}\right) \\
&\qquad + L\min\left\{G\left(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\right),\right. \\
&\qquad\qquad G\left(x_{m(k)+1}, x_{m(k)+2}, x_{m(k)+2}\right), \\
&\qquad\qquad \left.G\left(x_{m(k)+1}, x_{n(k)}, x_{n(k)}\right)\right\}.
\end{aligned} \quad (73)$$

Letting $k \to +\infty$, and using (60), (65), (68), (70), and (72), and the properties of ϕ and ψ, we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon). \quad (74)$$

Thus $\phi(\varepsilon) = 0$ and hence $\varepsilon = 0$, a contradiction. Thus $\{x_n\}$ is a G-Cauchy sequence in X.

Now, since (X, G) is G-complete, there is $x \in X$ such that $\{x_n\}$ is G-convergent to x; that is

$$\lim_{n\to+\infty} G(x_n, x_n, x) = \lim_{n\to+\infty} G(x_n, x, x) = 0. \quad (75)$$

By (48), we get

$$\begin{aligned}\psi(G(x_n, Tx, x_{n+1})) &= \psi(G(Tx_{n-1}, Tx, T^2x_{n-1})) \\ &\leq \psi(M^*(x_{n-1}, x, x_{n-1})) \\ &\quad - \phi(M^*(x_{n-1}, x, x_{n-1})) \\ &\quad + LN^*(x_{n-1}, x, x_{n-1}),\end{aligned} \quad (76)$$

where

$$\begin{aligned}&M^*(x_{n-1}, x, x_{n-1}) \\ &\quad = \max\{G(x_{n-1}, Tx_{n-1}, x), G(x, T^2x_{n-1}, Tx), \\ &\qquad G(Tx_{n-1}, T^2x_{n-1}, Tx), \\ &\qquad G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ &\qquad G(Tx_{n-1}, T^2x_{n-1}, T^2x_{n-1}), \\ &\qquad G(x_{n-1}, x, Tx_{n-1})\} \\ &\quad = \max\{G(x_{n-1}, x_n, x), G(x, x_{n+1}, Tx), \\ &\qquad G(x_n, x_{n+1}, Tx), G(x_{n-1}, x_n, x_n), \\ &\qquad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x, x_n)\}, \\ &N^*(x_{n-1}, x, x_{n-1}) \\ &\quad = \min\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ &\qquad G(x, Tx, Tx), G(x, Tx_{n-1}, Tx_{n-1})\} \\ &\quad = \min\{G(x_{n-1}, x_n, x_n), \\ &\qquad G(x, Tx, Tx), G(x, x_n, x_n)\}.\end{aligned} \quad (77)$$

Letting $n \to +\infty$ in (77), we get

$$\begin{aligned}\lim_{n\to+\infty} M^*(x_{n-1}, x, x_{n-1}) &= G(x, x, Tx), \\ \lim_{n\to+\infty} N^*(x_{n-1}, x, x_{n-1}) &= 0.\end{aligned} \quad (78)$$

On letting $n \to +\infty$ in (76), and using the properties of ψ and ϕ and (78), we obtain

$$\psi(G(x, x, Tx)) \leq \psi(G(x, x, Tx)) - \phi(G(x, x, Tx)). \quad (79)$$

Therefore $G(x, x, Tx) = 0$ and hence $x = Tx$. Thus x is a fixed point of T.

Now our purpose is to check that such point is unique. Suppose that there are two fixed points of T; say $x, y \in X$ such that $x \neq y$. By (48), we have

$$\begin{aligned}\psi(G(x, y, x)) &= \psi(G(Tx, Ty, T^2x)) \\ &\leq \psi(M^*(x, y, x)) - \phi(M^*(x, y, x)) \\ &\quad + LN^*(x, y, x),\end{aligned} \quad (80)$$

where

$$\begin{aligned}&M^*(x, y, x) \\ &\quad = \max\{G(x, Tx, y), G(y, T^2x, Ty), G(Tx, T^2x, Ty), \\ &\qquad G(x, Tx, Tx), G(Tx, T^2x, T^2x), G(x, y, Tx)\} \\ &\quad = \max\{G(x, x, y), G(y, x, y)\}, \\ &N^*(x, y, x) \\ &\quad = \min\{G(x, Tx, Tx), G(y, Ty, Ty), G(y, Tx, Tx)\} = 0.\end{aligned} \quad (81)$$

Similarly, we can prove that

$$\begin{aligned}&\psi(G(y, x, y)) \\ &\quad = \psi(G(Ty, Tx, T^2y)) \\ &\quad \leq \psi(M^*(y, x, y)) - \phi(M^*(y, x, y)) + LN^*(y, x, y),\end{aligned} \quad (82)$$

where

$$\begin{aligned}M^*(y, x, y) &= \max\{G(x, x, y), G(y, x, y)\}, \\ N^*(y, x, y) &= 0.\end{aligned} \quad (83)$$

If

$$\max\{G(x, x, y), G(y, x, y)\} = G(x, x, y). \quad (84)$$

By (80) and (81), we get

$$\psi(G(x, x, y)) \leq \psi(G(x, x, y)) - \phi(G(x, x, y)), \quad (85)$$

a contradiction. Then

$$\max\{G(x, x, y), G(y, x, y)\} = G(x, y, y). \quad (86)$$

By (82) and (83), we get

$$\psi(G(x, y, y)) \leq \psi(G(x, y, y)) - \phi(G(x, y, y)), \quad (87)$$

a contradiction. Thus, $x = y$, and hence the fixed point of T is unique. □

Finally, we provide the following example.

Example 14. Let $X = [0,1]$ and $G : X \times X \times X \to \mathbb{R}$ be defined by

$$G(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x,y,z\} & \text{otherwise.} \end{cases} \quad (88)$$

Then (X, G) is a G-complete G-metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{4}x & \text{if } 0 \le x < \frac{1}{3}, \\ \frac{1}{8}x^4 & \text{if } \frac{1}{3} \le x \le 1. \end{cases} \quad (89)$$

Take $\psi(t) = (1/2)t$ and $\phi(t) = (1/4)t$ for all $t \in [0, +\infty[$. We examine the following cases.

(i) Let $0 \le x, y < 1/3$. Then

$$\begin{aligned} &G(Tx, T^2x, Ty) \\ &\quad = \max\left\{\frac{1}{4}x, \frac{1}{16}x, \frac{1}{4}y\right\} = \frac{1}{4}\max\left\{x, \frac{1}{4}x, y\right\} \\ &\quad \le \frac{1}{4}\max\{x,y\} = \psi(M^*(x,y,x)) - \phi(M^*(x,y,x)), \end{aligned} \quad (90)$$

where $M^*(x,y,x) = \max\{x,y\}$.

(ii) Let $1/3 \le x, y < 1$. Then

$$\begin{aligned} &G(Tx, T^2x, Ty) \\ &\quad = \max\left\{\frac{1}{8}x^4, \frac{1}{64}x^{16}, \frac{1}{8}y^4\right\} \le \frac{1}{4}\max\left\{x, \frac{1}{8}x^4, y\right\} \\ &\quad \le \frac{1}{4}\max\{x,y\} = \psi(M^*(x,y,x)) - \phi(M^*(x,y,x)), \end{aligned} \quad (91)$$

where $M^*(x,y,z) = \max\{x,y\}$.

(iii) Let $0 \le x < 1/3 \le y < 1$. Then

$$\begin{aligned} &G(Tx, T^2x, Ty) \\ &\quad = \max\left\{\frac{1}{4}x, \frac{1}{16}x, \frac{1}{8}y^4\right\} \le \frac{1}{4}\max\left\{x, \frac{1}{4}x, y\right\} \\ &\quad \le \frac{1}{4}y = \psi(M^*(x,y,x)) - \phi(M^*(x,y,x)), \end{aligned} \quad (92)$$

where $M^*(x,y,x) = y$.

(iv) Let $0 \le y < 1/3 \le x < 1$. Then

$$\begin{aligned} &G(Tx, T^2x, Ty) \\ &\quad = \max\left\{\frac{1}{8}x^4, \frac{1}{64}x^{16}, \frac{1}{4}y\right\} \le \frac{1}{4}\max\left\{x, \frac{1}{8}x^4, y\right\} \\ &\quad \le \frac{1}{4}x = \psi(M^*(x,y,x)) - \phi(M^*(x,y,x)), \end{aligned} \quad (93)$$

where $M^*(x,y,x) = x$. Then

$$\begin{aligned} G(Tx, T^2x, Ty) &\le \psi(M^*(x,y,x)) - \phi(M^*(x,y,x)) \\ &\quad + LN^*(x,y,x) \end{aligned} \quad (94)$$

for all $L \ge 0$. Then the conditions of Theorem 13, hold and T has a unique fixed point. Notice that $u = 0$ is the desired fixed point of T.

References

[1] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[2] Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory [Ph.D. thesis]*, The University of Newcastle, Callaghan, Australia, 2005.

[3] M. Abbas, W. Sintunavarat, and P. Kumam, "Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 31, 14 pages, 2012.

[4] M. Abbas, T. Nazir, and P. Vetro, "Common fixed point results for three maps in G-metric spaces," *Filomat*, vol. 25, no. 4, pp. 1–17, 2011.

[5] R. P. Agarwal and E. Karapınar, "Remarks on some coupled fixed point theorems in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, article 2, 2013.

[6] H. Aydi, W. Shatanawi, and C. Vetro, "On generalized weak G-contraction mapping in G-metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 11, pp. 4223–4229, 2011.

[7] H. Aydi, E. Karapınar, and W. Shatanawi, "Tripled fixed point results in generalized metric spaces," *Journal of Applied Mathematics*, vol. 2012, Article ID 314279, 10 pages, 2012.

[8] Z. Mustafa, H. Aydi, and E. Karapınar, "On common fixed points in G-metric spaces using (E.A) property," *Computers & Mathematics with Applications. An International Journal*, vol. 64, no. 6, pp. 1944–1956, 2012.

[9] H. Aydi, M. Postolache, and W. Shatanawi, "Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces," *Computers & Mathematics with Applications*, vol. 63, no. 1, pp. 298–309, 2012.

[10] H. Aydi, B. Damjanović, B. Samet, and W. Shatanawi, "Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2443–2450, 2011.

[11] Y. J. Cho, B. E. Rhoades, R. Saadati, B. Samet, and W. Shatanawi, "Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type," *Fixed Point Theory and Applications*, vol. 2012, article 8, 14 pages, 2012.

[12] B. S. Choudhury and A. Kundu, "A coupled coincidence point result in partially ordered metric spaces for compatible mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 8, pp. 2524–2531, 2010.

[13] B. S. Choudhury and P. Maity, "Coupled fixed point results in generalized metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 73–79, 2011.

[14] H. S. Ding and E. Karapınar, "A note on some coupled fixed point theorems on G-metric space," *Journal of Inequalities and Applications*, vol. 2012, article 170, 2012.

[15] Z. Mustafa, H. Aydi, and E. Karapınar, "On common fixed points in G-metric spaces using (E.A) property," *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1944–1956, 2012.

[16] Z. Mustafa, H. Obiedat, and F. Awawdeh, "Some fixed point theorem for mapping on complete G-metric spaces," *Fixed Point*

Theory and Applications, vol. 2008, Article ID 189870, 12 pages, 2008.

[17] Z. Mustafa, M. Khandagji, and W. Shatanawi, "Fixed point results on complete G-metric spaces," *Studia Scientiarum Mathematicarum Hungarica*, vol. 48, no. 3, pp. 304–319, 2011.

[18] Z. Mustafa and B. Sims, "Fixed point theorems for contractive mappings in complete G-metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 917175, 10 pages, 2009.

[19] Z. Mustafa, W. Shatanawi, and M. Bataineh, "Existence of fixed point results in G-metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 283028, 10 pages, 2009.

[20] Z. Mustafa and H. Obiedat, "A fixed point theorem of Reich in G-metric spaces," *Cubo*, vol. 12, no. 1, pp. 83–93, 2010.

[21] H. K. Nashine, "Coupled common fixed point results in ordered G-metric spaces," *Journal of Nonlinear Science and its Applications*, vol. 5, no. 1, pp. 1–13, 2012.

[22] W. Shatanawi, "Coupled fixed point theorems in generalized metric spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 3, pp. 441–447, 2011.

[23] W. Shatanawi, "Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 181650, 9 pages, 2010.

[24] W. Shatanawi, "Some fixed point theorems in ordered G-metric spaces and applications," *Abstract and Applied Analysis*, vol. 2011, Article ID 126205, 11 pages, 2011.

[25] W. Shatanawi, "Coupled fixed point theorems in generalized metric spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 3, pp. 441–447, 2011.

[26] W. Shatanawi, M. Abbas, and T. Nazir, "Common coupled coincidence and coupled fixed point results in two generalized metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 80, p. 13, 2011.

[27] N. Tahat, H. Aydi, E. Karapınar, and W. Shatanawi, "Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 48, 9 pages, 2012.

[28] B. Samet, C. Vetro, and F. Vetro, "Remarks on G-metric spaces," *International Journal of Analysis*, vol. 2013, Article ID 917158, 6 pages, 2013.

[29] M. Jleli and B. Samet, "Remarks on G-metric spaces and fixed point theorems," *Fixed Point Theory and Applications*, vol. 2012, article 210, 7 pages, 2012.

[30] V. Berinde, "Approximating fixed points of weak contractions using the Picard iteration," *Nonlinear Analysis Forum*, vol. 9, no. 1, pp. 43–53, 2004.

[31] V. Berinde, "General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces," *Carpathian Journal of Mathematics*, vol. 24, no. 2, pp. 10–19, 2008.

[32] V. Berinde, "Some remarks on a fixed point theorem for Ćirić-type almost contractions," *Carpathian Journal of Mathematics*, vol. 25, no. 2, pp. 157–162, 2009.

[33] A. Aghajani, S. Radenović, and J. R. Roshan, "Common fixed point results for four mappings satisfying almost generalized (S, T)-contractive condition in partially ordered metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5665–5670, 2012.

[34] G. V. R. Babu, M. L. Sandhya, and M. V. R. Kameswari, "A note on a fixed point theorem of Berinde on weak contractions," *Carpathian Journal of Mathematics*, vol. 24, no. 1, pp. 8–12, 2008.

[35] L. Ćirić, M. Abbas, R. Saadati, and N. Hussain, "Common fixed points of almost generalized contractive mappings in ordered metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5784–5789, 2011.

[36] M. Păcurar, "Sequences of almost contractions and fixed points in b-metric spaces," *Analele Universitatii de Vest, Timisoara Seria Matematica Informatica*, vol. 48, no. 3, pp. 125–137, 2010.

[37] W. Shatanawi and A. Al-Rawashdeh, "Common fixed points of almost generalized (ψ, ϕ)-contractive mappings in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 80, 14 pages, 2012.