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Review Article

A Survey of Results on the Limit q-Bernstein Operator

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The limit q-Bernstein operator B_q emerges naturally as a modification of the Szász-Mirakyan operator related to the Euler distribution, which is used in the q-boson theory to describe the energy distribution in a q-analogue of the coherent state. At the same time, this operator bears a significant role in the approximation theory as an exemplary model for the study of the convergence of the q-operators. Over the past years, the limit q-Bernstein operator has been studied widely from different perspectives. It has been shown that B_q is a positive shape-preserving linear operator on $C[0,1]$ with $\|B_q\| = 1$. Its approximation properties, probabilistic interpretation, the behavior of iterates, and the impact on the smoothness of a function have already been examined. In this paper, we present a review of the results on the limit q-Bernstein operator related to the approximation theory. A complete bibliography is supplied.

1. Introduction

The limit q-Bernstein operator comes out as an analogue of the Szász-Mirakyan operator related to the Euler probability distribution, also called the q-deformed Poisson distribution (see [1–3]). The latter is used in the q-boson theory, which is a q-deformation of the quantum harmonic oscillator formalism [4]. Namely, the q-deformed Poisson distribution describes the energy distribution in a q-analogue of the coherent state [5]. The q-analogue of the boson operator calculus has proved to be a powerful tool in theoretical physics, providing explicit expressions for the representations of the quantum group $\mathrm{SU}_q(2)$, which itself is by now known to play a profound role in a variety of different problems. Some of these are integrable model field theories, exactly solvable lattice models of statistical mechanics, conformal field theory, and others. Therefore, the properties of the q-deformed Poisson distribution and its related limit q-Bernstein operator have proved to be of paramount value for various applications. What is more, this operator is also decisive for the approximation theory as a model pertinent to the asymptotic behavior for a sequence of the q-operators. Indeed, operators whose nature is similar to that of B_q appear as a limit of a sequence of the various q-operators, see, for example, [6–11]. In this respect, a general approach has been developed by Wang in [12].

To present the subject of this survey, it can serve well to recall some notions related to the q-calculus (cf., e.g., [13]).

Let $q > 0$. For any $k \in \mathbb{Z}_+$, the *q-integer* $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1} \quad (k \in \mathbb{N}), \quad [0]_q := 0, \tag{1}$$

and the *q-factorial* $[k]_q!$ by

$$[k]_q! := [1]_q[2]_q \cdots [k]_q \quad (k = 1, 2, \ldots), \quad [0]_q! := 1. \tag{2}$$

For integers k and n with $0 \le k \le n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{3}$$

In addition, we employ the notation:

$$\begin{gathered} (a-x)_q^n := \prod_{j=0}^{n-1} \left(a - q^j x\right) \left(n \in \mathbb{Z}_+\right), \\ (a-x)_q^{\infty} := \prod_{j=0}^{\infty} \left(a - q^j x\right). \end{gathered} \tag{4}$$

For the sequel, it is also convenient to denote

$$\psi_q(x) = (1-x)_q^\infty. \tag{5}$$

In the case $0 < q < 1$, the function ψ_q is an entire function involved in Euler's identities (see [13, formulae (9.7) and (9.10)]):

$$\begin{aligned} \psi_q(-x) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}x^k}{(1-q)\cdots(1-q^k)}, \\ \frac{1}{\psi_q(x)} &= \sum_{k=0}^{\infty} \frac{x^k}{(1-q)\cdots(1-q^k)} \quad \text{for } |x|<1. \end{aligned} \tag{6}$$

For $0 < q < 1$, q-analogues of the exponential function are given by

$$\begin{aligned} e_q(x) &:= \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \quad |x| < \frac{1}{1-q}, \\ E_q(x) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}x^k}{[k]_q!}. \end{aligned} \tag{7}$$

By the virtue of Euler's identities,

$$\begin{aligned} e_q(x) &= \prod_{j=0}^{\infty}\left(1-(1-q)xq^j\right)^{-1}, \quad |x| < \frac{1}{1-q}, \\ E_q(x) &= \prod_{j=0}^{\infty}\left(1+(1-q)xq^j\right), \end{aligned} \tag{8}$$

whence

$$e_q(x)E_q(-x) = 1. \tag{9}$$

Clearly, for $q = 1$, we have

$$[k]_1 = k, \qquad [k]_1! = k!, \qquad e_1(x) = E_1(x) = e^x. \tag{10}$$

Definition 1. Given $q \in (0,1)$, the limit q-Bernstein operator on $C[0,1]$ is defined by $f \mapsto B_q f$, where

$$\begin{aligned} &\left(B_q f\right)(x) \\ &= B_q(f;x) \\ &:= \begin{cases} E_q\left(-\dfrac{x}{1-q}\right)\cdot \displaystyle\sum_{k=0}^{\infty} \frac{f\left(1-q^k\right)x^k}{(1-q)^k[k]_q!} & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1, \end{cases} \\ &= \begin{cases} (1-x)_q^\infty \cdot \displaystyle\sum_{k=0}^{\infty} \frac{f\left(1-q^k\right)}{(1-q)\cdots(1-q^k)}x^k & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1. \end{cases} \end{aligned} \tag{11}$$

Since

$$(1-x)_q^\infty \sum_{k=0}^{\infty} \frac{x^k}{(1-q)\cdots(1-q^k)} = 1 \quad \text{for } |x|<1, \tag{12}$$

it follows that B_q is a bounded positive linear operator on $C[0,1]$ with $\|B_q\| = 1$. It can be readily seen from the definition that B_q possesses the end-point interpolation property:

$$B_q(f;0) = f(0), \qquad B_q f(1) = f(1). \tag{13}$$

It is commonly known in the field that B_q leaves invariant linear functions and maps a polynomial of degree m to a polynomial of degree m (see also Theorem 26). Additional properties of this operator will be considered in the present paper. Prior to presenting the results on B_q, it is worth discussing the origin of the operator itself.

2. q-Bernstein Polynomials

This section describes the relation between the limit q-Bernstein operator and the theory of q-Bernstein polynomials. Within the framework of this theory, B_q emerges as a limit for a sequence of the q-Bernstein polynomials. These polynomials were introduced by Phillips in 1997 (cf. [14]) who initiated researches in the area. The summary of the results obtained by Phillips and his collaborators is presented in [15, Ch. 7].

Definition 2 (see [14]). The q-Bernstein polynomial of f is

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)p_{nk}(q;x), \quad n = 1,2,\ldots, \tag{14}$$

where

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-1-k}\left(1-q^j x\right), \quad k = 0,1,\ldots n. \tag{15}$$

Note that $B_{n,1}(f;x)$ are classical Bernstein polynomials.

Some of the properties of the classical Bernstein polynomials are known to have been taken after by the q-Bernstein polynomials (see [15]). For example, the q-Bernstein polynomials possess the end-point interpolation property, leave invariant linear functions, admit representation with the help of q-differences, and are degree-reducing on polynomials. Apart from that, the q-Bernstein basic polynomials (15) admit a probabilistic interpretation via q-binomial distribution (see [1, 16, 17]). A comprehensive review of the results on the q-Bernstein polynomials along with an extensive bibliography and a collection of open problems on the subject have all been provided in [18]. Recently, modifications of the q-Bernstein polynomials related to the q-Stirling numbers, q-integral representations, and the p-adic numbers have been investigated by Kim et al. in [19–22].

However, further investigation of the q-Bernstein polynomials demonstrates that their convergence properties are essentially different from those of the classical ones and that the cases $0 < q < 1$ and $q > 1$ are different from one another—a difference whose origin can be traced back to the fact that while, for $0 < q < 1$, the q-Bernstein polynomials are positive linear operators on $C[0,1]$, this is no longer valid for $q > 1$.

The next theorem shows the limit q-Bernstein operator rising naturally when a sequence of the q-Bernstein polynomials in the case as $0 < q < 1$ is considered.

Theorem 3 (see [23]). *Let $q \in (0,1)$.*

(i) *Then, for any $f \in C[0,1]$,*

$$B_{n,q}(f;x) \longrightarrow B_q(f;x) \quad \text{as } n \longrightarrow \infty, \tag{16}$$

uniformly for $x \in [0,1]$.

(ii) *The equality $B_q(f;x) = f(x)$ for $x \in [0,1]$ holds if and only if f is a linear function.*

Remark 4. Wang observed [24] that if $\{M_{n,q}(f;x)\}$, $q \in (0,1)$ is a sequence of the q-Meier-König and Zeller operator considered by Trif (cf. [25]), then for any $f \in C[0,1]$,

$$M_{n,q}(f;x) \longrightarrow B_q(f;x) \quad \text{as } n \longrightarrow \infty, \tag{17}$$

uniformly for $x \in [0,1]$.

It should be emphasized that various analogues of Theorem 3 have been proved for different classes of q-operators, as, for example, in [6, 7, 9, 10]. On the top of that, this theorem has triggered the start of further research on the Korovkin-type theorems (cf. [12, 26]). As it turns out, while many q-versions of the known operators—in particular, q-Bernstein polynomials—do not satisfy the conditions of the Korovkin theorem, they do satisfy the conditions of Wang's Korovkin-type theorem (Theorem 5), which guarantees their uniform convergence on $[0,1]$ to the limit operator.

Theorem 5 (see [12]). *Let L_n be a sequence of positive linear operators on $C[0,1]$ satisfying the following conditions:*

(a) *the sequence $\{L_n(t^2;x)\}$ converges uniformly on $[0,1]$,*

(b) *the sequence $\{L_n(f;x)\}$ is nondecreasing in n for any convex function f and any $x \in [0,1]$.*

Then, there exists an operator L on $C[0,1]$ such that

$$L_n(f;x) \longrightarrow L(f;x) \quad \text{on } [0,1] \text{ as } n \longrightarrow \infty, \tag{18}$$

uniformly on $[0,1]$.

Remark 6. In general, condition (b) cannot be left out completely. The corresponding example is provided in [12, Theorem 1].

Meanwhile, statement (ii) of Theorem 3 is a general property of positive linear operator as stated by the next theorem.

Theorem 7 (see [10]). *Let L be a positive linear operator on $C[0,1]$ which reproduces linear functions. If $L(t^2;x) > x^2$ for $x \in (0,1)$, then $Lf = f$ if and only if f is linear.*

3. Probabilistic Approach

Another approach to B_q is given in terms of probability theory.

Consider a function $\varphi(x)$ with the positive Taylor coefficients analytic in the disc $\{x : |x| < r\}$, $0 < r \le \infty$,

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_0 = 1,\ a_k > 0, \tag{19}$$

and consider a random variable ξ_x $(0 \le x \le r)$, whose values do not depend on x and are taken with the following probabilities:

$$\mathbf{P}\{\xi_x = \alpha_k\} = \frac{a_k x^k}{\varphi(x)} =: p_k(x), \quad k = 0,1,\dots. \tag{20}$$

Let X be the linear space of functions defined on $\{\alpha_k\}$ so that for $f \in X$, $x \in [0,r)$, the mathematical expectation $\mathbf{E}[f(\xi_x)]$ exists. We define a linear operator A_φ on X as follows:

$$(A_\varphi f)(x) := \mathbf{E}[f(\xi_x)] = \sum_{k=0}^{\infty} f(\alpha_k) p_k(x). \tag{21}$$

Suppose that the probability distribution of ξ_x satisfies the following conditions:

(i) $\mathbf{E}[\xi_x] = x$, that is, A_φ leaves invariant linear functions,

(ii) $\mathbf{E}[\xi_x^2] = qx^2 + bx + c$, that is, A_φ takes a square polynomial to a square polynomial.

Example 8. The Poisson distribution with parameter x.

Theorem 9 (see [2]). *Let ξ_x be a random variable whose distribution*

$$\mathbf{P}\{\xi_x = \alpha_k\} = \frac{a_k x^k}{\varphi(x)}, \quad k = 0,1,2,\dots \tag{22}$$

satisfies the conditions above. Then,

$$\begin{gathered} c = 0,\ q > -1, \quad \alpha_k = b\frac{1-q^k}{1-q}, \\ a_k = \frac{(1-q)^k}{b^k(1-q)\cdots(1-q^k)}, \end{gathered} \tag{23}$$

and the function φ has the form:

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{(1-q)^k x^k}{b^k(1-q)\cdots(1-q^k)}. \tag{24}$$

The Theorem means that conditions (i) and (ii) imply a rather specific form of probability distribution.

Consider the following particular cases.

(1) Let $q = b = 1$. Then,

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x, \quad \alpha_k = k, \quad \mathbf{P}\{\xi_x = k\} = \frac{x^k}{k!}e^{-x}, \tag{25}$$

therefore, ξ_x has the Poisson distribution with parameter x. Correspondingly,

$$(A_\varphi f)(x) = \sum_{k=0}^{\infty} f(k)\frac{x^k}{k!}e^{-x}. \tag{26}$$

(2) For $q = 1,\ b = 1/n$, we obtain

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} = e^{nx}, \qquad \alpha_k = \frac{k}{n},$$
$$\mathbf{P}\left\{\xi_x = \frac{k}{n}\right\} = \frac{(nx)^k}{k!} e^{-nx}. \tag{27}$$

In this case,

$$\left(A_\varphi f\right)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} e^{-nx} = S_n(f; x), \tag{28}$$

that is, A_φ coincides with the Szász-Mirakyan operator. By Feller's Lemma [27, v. II, Ch. VII, Section 1, Lemma 1], if $f \in C[0, \infty)$ is bounded, then $S_n(f; x) \to f(x)$ as $n \to \infty$, uniformly on any compact subset of $[0, \infty)$.

(3) Let $0 < q < 1,\ b = 1 - q$. Then,

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{x^k}{(1-q)\cdots(1-q^k)} = \frac{1}{\psi_q(x)}, \quad |x| < 1. \tag{29}$$

Besides, $\alpha_k = 1 - q^k$ and

$$\mathbf{P}\left\{\xi_x = 1 - q^k\right\} = \psi_q(x) \frac{x^k}{(1-q)\cdots(1-q^k)}. \tag{30}$$

Therefore,

$$\left(A_\varphi f\right)(x) = \psi_q(x) \sum_{k=0}^{\infty} \frac{f\left(1-q^k\right)}{(1-q)\cdots(1-q^k)} x^k = B_q(f; x). \tag{31}$$

As we can see, in this way, B_q occurs as an analogue of the Szász-Mirakyan operator.

4. Approximation Properties of B_q

The approximation by operator B_q was first studied by Videnskii in [28]. Let us recollect that the *modulus of continuity* of a function f on $[0, 1]$ is defined by

$$\omega(f; t) := \sup\left\{\left|f(x) - f(y)\right| : |x - y| \le t,\ x, y \in [0, 1]\right\}. \tag{32}$$

The following estimates are valid.

Theorem 10 (see [28]). (i) *If $f \in C[0, 1]$, then*

$$\left|B_q(f; x) - f(x)\right| \le 2\omega\left(f; \frac{1}{2}\sqrt{1-q}\right). \tag{33}$$

Consequently,

$$B_q(f; x) \longrightarrow f(x) \quad \text{as } q \longrightarrow 1^-, \text{ uniformly for } x \in [0, 1]. \tag{34}$$

(ii) *If $f \in C^{(2)}[0, 1]$, then*

$$\left|B_q(f; x) - f(x) - \frac{1-q}{2} f''(x)\, x(1-x)\right| \le K(1-q)\, x(1-x)\, \omega\left(f''; \sqrt{1-q}\right), \tag{35}$$

where K is a positive constant.

Consequently, for $f \in C^{(2)}[0, 1]$,

$$\lim_{q \to 1^-} \frac{B_q(f; x) - f(x)}{1-q} = \frac{f''(x)}{2} x(1-x), \tag{36}$$

uniformly on $[0, 1]$.

The elaboration of these results has been carried out in [29]. Videnskii [28] has also considered the modification of the limit q-Bernstein operator defined for $f \in C^{(2)}[0, 1]$ by

$$\widetilde{B}_q(f; x) := B_q(f; x) - \frac{1-q}{2} x(1-x) B_q\left(f''; x\right) \tag{37}$$

and proved that

$$\left|\widetilde{B}_q(f; x) - f(x)\right| \le K(1-q)\, x(1-x)\, \omega\left(f''; \sqrt{1-q}\right), \quad K > 0. \tag{38}$$

In [30], Mahmudov has introduced a generalization of the limit q-Bernstein operator defined on the space $C^r[0, 1]$ of the r times continuously differentiable functions and proved that, for $r \ge 1$, these operators provide a better degree of the approximation than operators B_q, corresponding to $r = 0$.

The approximation of the analytic functions in complex domains by the limit q-Bernstein operator has been investigated in [31], where the following results have been established.

Theorem 11. *Let $f \in C[0, 1]$ admit an analytic continuation from $[0, 1]$ into $\{z : |z - 1| < 1 + \varepsilon\}$. Then, for any compact set $K \subset D(\varepsilon)$,*

$$B_q(f; z) \longrightarrow f(z), \quad q \longrightarrow 1^-, \text{ uniformly on } K. \tag{39}$$

Corollary 12. *If f is an entire function, then, for any compact set $K \subset \mathbb{C}$,*

$$B_q(f; z) \longrightarrow f(z), \quad q \longrightarrow 1^-, \text{ uniformly on } K. \tag{40}$$

Finally, we provide an estimate for the rate of approximation for functions analytic in $D(r),\ r > 1$.

Theorem 13. *Let $f(z)$ be analytic in a closed disk $\overline{D(r)}$ with $r > 1$. Then, for $z \in \overline{D(r)}$, we have*

$$\left|B_q(f; z) - f(z)\right| \le C_{f,r}(1-q). \tag{41}$$

Remark 14. Clearly, Corollary 12 can also be derived from Theorem 13. Moreover, we obtain that the order of approximation for analytic functions equals $(1-q)$. Using the growth estimates for f, we can estimate $C_{f,r}$ for $r > 1$.

5. Functional-Analytic Properties of the Limit q-Bernstein Operator

To begin with, let us identify the kernel and the image of the limit q-Bernstein operator. The relevant results have been supplied in [23, 32].

Theorem 15. (i) *ker* $B_q = \{f \in C[0,1] : f(1-q^k) = 0$ *for all* $k \in \mathbb{Z}_+\}$ *and* (ii) *im* $B_q = \{f \in C[0,1] : f(x) = \sum_{k=0}^{\infty} a_k x^k$, *where* $\sum_{k=0}^{\infty} a_k$ *converges.*$\}$

Corollary 16. *The image of the limit q-Bernstein operator* $B_q : C[0,1] \to C[0,1]$ *is nonclosed.*

We say that an operator $T : X \to Y$ is bounded below on a subspace $L \subset X$ if there exists a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for each $x \in L$. An easy consequence of Theorem 15 is that B_q is not bounded below on any subspace which does not contain isomorphic copies of c_0.

However, for subspaces containing subspaces isomorphic to c_0, the situation can be different. To be specific, the following result holds.

Theorem 17 (see [33]). *There exists a subspace of* $C[0,1]$ *isomorphic to* c_0 *such that the restriction of* B_q *to this subspace is an isomorphic embedding.*

Further properties of the image of the limit q-Bernstein operator are expressed by the uniqueness theorems below.

In general, for a function $f \in C[0,1]$, its image under B_q depends on q. Plain calculations show that

$$B_q\left(t^2; x\right) = x^2 + (1-q)\, x\,(1-x), \quad (42)$$

which implies that $B_{q_1}(t^2;x) \not\equiv B_{q_2}(t^2;x)$ for distinct q_1 and q_2. However, if f is a linear function, then $B_q(f;x) = f(x)$ regardless of q. It is not difficult to see that the converse statement is also true.

Theorem 18 (see [32]). *If, for any* $q_1, q_2 \in (0,1)$, *we have*

$$B_{q_1}(f;x) \equiv B_{q_2}(f;x), \quad x \in [0,1], \quad (43)$$

then f *is a linear function.*

A stronger assertion may be proved for the images of analytic functions.

Theorem 19. *Let* f *be analytic on* $[0,1]$. *If, for* $q_1 \neq q_2$,

$$B_{q_1}(f;x) \equiv B_{q_2}(f;x), \quad x \in [0,1], \quad (44)$$

then f *is a linear function.*

A closer look can show that this result appears to be sharp and that the statement ceases to be true for infinitely differentiable functions.

Now, let us draw attention to the behavior of the iterates of the limit q-Bernstein operator, which have been studied in [34]. By L, we denote the operator of linear interpolation at 0 and 1, that is,

$$L(f;x) := (1-x)\, f(0) + x f(1). \quad (45)$$

Theorem 20 (see [34]). *If* $\{j_n\}$ *is a sequence of positive integers such that* $j_n \to \infty$, *then, for any* $f \in C[0,1]$,

$$B_q^{j_n}(f;x) \longrightarrow L(f;x) \quad \text{for } x \in [0,1] \text{ as } n \longrightarrow \infty, \quad (46)$$

uniformly on $[0,1]$.

As an immediate consequence of this theorem, we obtain the following statement mentioned in Section 2.

Corollary 21. *Let* $q \in (0,1)$. *Then,* $B_q(f) = f$ *if and only if* $f = L(f)$, *that is,* f *is a linear function.*

6. The Improvement of Analytic Properties under the Limit q-Bernstein Operator

Generally speaking, it can be stated that B_q improves the analytic properties of functions. The first result in this direction is the following:

Theorem 22 (see [23, 35]). (i) *For any* $f \in C[0,1]$, *the function* $B_q(f;x)$ *is continuous on* $[0,1]$ *and admits an analytic continuation into the open unit disc* $\{z : |z| < 1\}$.

(ii) *If* f *is* m $(m \geq 0)$ *times differentiable from the left at 1 and* $f^{(m)}$ *satisfies the Hölder condition at 1, that is,*

$$\left|f^{(m)}(x) - f^{(m)}(1)\right| \leq M|x-1|^{\alpha}, \quad M > 0,\ \alpha \in (0,1], \quad (47)$$

then $B_q(f;x)$ *admits an analytic continuation into the disc* $\{z : |z| < q^{-(m+\alpha)}\}$.

In particular, if f *is infinitely differentiable from the left at 1, then* $B_q(f;z)$ *is an entire function.*

Remark 23. In general, an analytic continuation of $B_q(f;x)$ may not be continuous in the closed unit disc.

For a function F, analytic in a disc $\{z : |z| \leq r\}$, we denote

$$M(r;F) := \max_{|z|\leq r} |F(z)|. \quad (48)$$

Theorem 24 (see [36]). (i) *If* f *is analytic at 1, then* $B_q(f;z)$ *is an entire function and*

$$M\left(r; B_q f\right) \leq C r^m \psi_q(-r), \quad \text{for } C, m > 0,\ r \geq 1. \quad (49)$$

(ii) *If* f *is analytic in* $\{z : |z-1| < 2+\varepsilon\}$, *then*

$$M\left(r; B_q f\right) \leq C \psi_q(-r), \quad \text{for some } C > 0. \quad (50)$$

Note that

$$C_1 \exp\left\{\frac{\ln^2(r/\sqrt{q})}{2\ln(1/q)}\right\} \leq \psi_q(-r) \leq C_2 \exp\left\{\frac{\ln^2(r/\sqrt{q})}{2\ln(1/q)}\right\}. \quad (51)$$

Therefore, for any entire function f, the growth of $B_q(f;z)$ does not exceed the growth of $\psi_q(z)$, showing that for an entire function, whose growth is faster than that of $\psi_q(z)$, the growth of $B_q f$ is slower than that of f. In other terms, the application of B_q to entire functions slows down a rather speedy growth. It turns out that the same phenomenon occurs for all transcendental entire functions regardless of their growth.

Theorem 25 (see [36]). *If f is a transcendental entire function, then*

$$M\left(r; B_q f\right) = o\left(M\left(r; f\right)\right) \quad \text{as } r \longrightarrow \infty. \tag{52}$$

Finally, we state the following noteworthy property of the q-Bernstein operator: it maps binomial $(1-x)^m$ to the corresponding q-binomial $(x;q)_m$.

Theorem 26 (see [36]). *If f is a polynomial of degree m, then $B_q(f;x)$ is also a polynomial of degree m. In addition, the following identity holds*

$$\begin{aligned}&\left(B_q\right)\left((1-x)^m\right)\\ &\quad = (1-x)\left(1-qx\right)\cdots\left(1-q^{m-1}x\right), \quad m = 0,1,2,\ldots.\end{aligned} \tag{53}$$

The results above indicate how the analytic properties of f are transformed under B_q. If f at least satisfies the Hölder condition at 1, then, on the whole, it gets "better", unless f is a polynomial, that is, "too good" to be improved.

The results above can be concluded in the form of a table as follows:

$f^{(m)} \in \operatorname{Lip} \alpha$ at 1 $\Rightarrow$ $B_q f$ admits an analytic continuation into $\{z : |z| < q^{-(m+\alpha)}\}$,

f infinitely differentiable at 1 $\Rightarrow$ $B_q f$ is entire,

f analytic at 1 $\Rightarrow$ $B_q f$ is entire with $M(r;B_q f) \le Cr^a \exp(C\ln^2 r)$,

f transcendental entire $\Rightarrow$ $B_q f$ is transcendental entire with $M(r;B_q f) \le Cr^{-u(r)}\exp(C\ln^2 r)$, $u(r) \to +\infty$ as $r \to \infty$ and $M(r;B_q f) = o(M(r;f))$, $r \to \infty$,

f polynomial, $\deg f = m$ $\Rightarrow$ $B_q f$ polynomial, $\deg B_q f = m$.

One can establish that, to a certain extent, the analytic properties of f may be retrieved from those of $B_q f$. For details, see [37]. Put differently, all "$\Rightarrow$" can be replaced with "$\Leftrightarrow$" provided that we consider the following equivalence relation on $C[0,1]$:

$$f \sim g \Longleftrightarrow f\left(1-q^k\right) = g\left(1-q^k\right), \quad k \in \mathbf{Z}_+. \tag{54}$$

Obviously,

$$f \sim g \Longleftrightarrow B_q f = B_q g. \tag{55}$$

Then, what happens under the application of B_q to continuous functions—those which do not satisfy the Hölder condition on $[0,1]$? In this case, $B_q f$ is a function in $C[0,1]$ which possesses an analytic continuation into the open unit disc, and, as a result, the possible lack of smoothness on $[0,1)$ will be corrected by B_q. One can also inquire about the smoothness at 1. In response to this query, it has been shown that, under some minor restrictions, the operator B_q speeds up the convergence of $f(x)$ to $f(1)$ as $x \to 1^-$. The rate of $f(x)$ approaching $f(1)$ is measured by the local *modulus of continuity* at 1:

$$\Omega\left(f;\delta\right) := \max_{1-\delta\le x\le 1}\left|f\left(x\right) - f\left(1\right)\right|. \tag{56}$$

Theorem 27 (see [36]). *If $f \in C[0,1]$ and $\Omega(f;\delta)$ satisfies the following regularity condition:*

$$\exists b \in (0,1), \quad \lim_{\delta\to 0^+} \frac{\delta\int_{b^{1/\delta}}^{1}\left(\Omega\left(f;t\right)/t\right)dt}{\Omega\left(f;\delta\right)} = 0, \tag{57}$$

then $\Omega(B_q f;\delta) = o(\Omega(f;\delta))$ as $\delta \to 0^+$.

Corollary 28. *If $C_1\delta^\beta \le \Omega(f;\delta) \le C_2(\ln(1/\delta))^{-\alpha}$, $0<\beta<\alpha<1$, then $\Omega(B_q f;\delta) = o(\Omega(f;\delta))$ as $\delta \to 0^+$.*

Remark 29. The condition (57) is rather general. For example, it holds for the functions:

$$\begin{aligned}\Omega\left(\delta\right) &= \delta^\alpha\left(\ln\frac{1}{\delta}\right)^{\beta_1}\left(\ln_2\frac{1}{\delta}\right)^{\beta_2}\cdots\left(\ln_n\frac{1}{\delta}\right)^{\beta_n},\\ &\qquad 0<\alpha<1, \quad \beta_1,\ldots,\beta_n \in \mathbb{R}, \quad n \in \mathbb{N},\\ \Omega\left(\delta\right) &= \left(\ln_k\frac{1}{\delta}\right)^{-\alpha}\left(\ln_{k+1}\frac{1}{\delta}\right)^{\beta_1}\cdots\left(\ln_{k+j}\frac{1}{\delta}\right)^{\beta_j},\\ &\qquad \alpha>0, \quad \beta_1,\ldots,\beta_j \in \mathbb{R}, \quad k,j \in \mathbb{N}.\end{aligned} \tag{58}$$

However, as it is shown in [38], there exist functions without the Hölder conditions at 1 which do not satisfy (57) such that for some $c>0$,

$$\Omega\left(B_q f;\delta\right) \ge c\Omega\left(f;\delta\right), \quad \delta \in [0,1]. \tag{59}$$

7. Concluding Remarks

The limit q-Bernstein operator has remained under scrutiny, and new researches on the subject appear on a regular basis. The aim of the present survey has been not only to exhibit the results related to this operator but also to primarily demonstrate the interrelations of the operator with a variety of mathematical disciplines.

Finally, it is beneficial to formulate an open problem for future investigation.

Problem. (Eigenvalues and eigenfunctions of the limit q-Bernstein operator). Find all $f \in C[0,1]$ so that

$$B_q f = \lambda f, \quad \lambda \in \mathbb{C}\setminus\{0\}. \tag{60}$$

Conjecture. If $B_q f = \lambda f$, $\lambda \ne 0$, then f is a polynomial and $\lambda \in \{q^{m(m-1)/2}\}_{m=0}^{\infty}$.

Comment. The conjecture has been proved under some additional conditions on the smoothness of f at 1 (e.g., for f satisfying the Hölder condition of order α) in [36, Corollary 5.6].

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