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Research Article

Generalized Yosida Approximations Based on Relatively A-Maximal m-Relaxed Monotonicity Frameworks

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We introduce and study a new notion of relatively A-maximal m-relaxed monotonicity framework and discuss some properties of a new class of generalized relatively resolvent operator associated with the relatively A-maximal m-relaxed monotone operator and the new generalized Yosida approximations based on relatively A-maximal m-relaxed monotonicity framework. Furthermore, we give some remarks to show that the theory of the new generalized relatively resolvent operator and Yosida approximations associated with relatively A-maximal m-relaxed monotone operators generalizes most of the existing notions on (relatively) maximal monotone mappings in Hilbert as well as Banach space and can be applied to study variational inclusion problems and first-order evolution equations as well as evolution inclusions.

1. Introduction

In order to generalize other existing results on linear convergence, including Rockafellar's theorem (1976) on linear convergence using the proximal point algorithm in a real Hilbert space setting, Verma [1] introduced a new application-oriented notion of relatively A-maximal monotonicity (so-called H-monotonicity in [2] when the relative monotone operator is an identity operator) framework, and then it was applied to the approximation solvability of the following general class of inclusion problems:

$$0 \in M(x), \tag{1}$$

where $M : \mathscr{H} \to 2^{\mathscr{H}}$ is a multivalued operator on a real Hilbert space $\mathscr{H}$. Furthermore, the author pointed out that "More significantly our approach based on the relatively maximal monotonicity works more smoothly even to the maximal monotone mapping M and corresponding classical resolvent of M than that of the results readily available in literature, and the general linear convergence results on the generalized proximal point algorithm based on the relatively maximal monotonicity can further be applied to theory of the Douglas-Rachford splitting methods as well as to first-order evolution equations based of Yosida approximations."

It seems that the obtained results can be applied to even more relaxed proximal point algorithm, where the Yosida approximation does have a more broader role to the Douglas-Rachford splitting methods [3] and further to first-order evolution equations based on the relatively maximal monotonicity [4]. Let us begin with the result of Eckstein and Bertsekas [3] on the Douglas-Rachford splitting method and the relaxed proximal point algorithm for maximal monotone mappings. For more details, we recommend [5, 6] and the references therein.

The notion of monotone operators was introduced independently by Zarantonello [7] and Minty [8]. Interest in such mappings stems mainly from their firm connection with the following first-order evolution equation:

$$\frac{dx}{dt} = -M(x), \qquad x(0) = x_0, \tag{2}$$

which is the model in terms of many physical problems. What is most interesting and important of the accretive mapping was mainly from the fact that problem (1) is solvable if M is an accretive and locally Lipschitz single-valued operator in an appropriate Banach space. Further, if $\mathscr{H}$ is a real Hilbert space and $M : \mathrm{dom}(M) \subset \mathscr{H} \to \mathscr{H}$ is an operator such that

M is monotone and $R(I+M)=\mathscr{H}$, then, based on the Yosida approximation

$$M_\rho = \frac{1}{\rho}\left(I-(I+\rho M)^{-1}\right), \tag{3}$$

for each given $x_0 \in \operatorname{dom}(M)$, there exists exactly one continuous function $x:[0,1)\to\mathscr{H}$ such that the evolution equation (1) holds for all $t\in(0,\infty)$ (see [9]), where the derivative $dx/dt = x'(t)$ exists in the sense of weak convergence, that is,

$$\frac{x(t+h)-x(t)}{h} \rightharpoonup x'(t) \quad \text{as } h \longrightarrow 0. \tag{4}$$

Recently, several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function, can be put in operator form as

$$x + KF(x) = 0, \tag{5}$$

where K and F are monotone operators (see [10–14] and the references therein for more information).

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $2^{\mathscr{H}}$ denote the family of all the nonempty subsets of $\mathscr{H}$. A multifunction $G:\mathscr{H}\to 2^{\mathscr{H}}$ is said to be a monotone operator if

$$\begin{gathered}\langle x-y, u-v\rangle \geq 0\\ \text{whenever } x,y\in\mathscr{H},\ u\in G(x),\ v\in G(y).\end{gathered} \tag{6}$$

It is said to be maximal monotone if, in addition, the graph

$$\operatorname{Graph}(G) = \{(x,u)\in\mathscr{H}\times\mathscr{H} \mid u\in G(x)\} \tag{7}$$

is not properly contained in the graph of any other monotone operators.

Such operators have been studied extensively because of their role in convex analysis, certain partial differential equations, and differential inclusions. A fundamental problem is that of determining an element x such that $0\in G(x)$, which includes minimization or maximization of functions, variational inequality problems, quasivariational inequality problems, minimax problems, and decision and management sciences. Furthermore, general maximal monotonicity has played a crucial role by providing a powerful framework to develop and use suitable proximal point algorithms in studying convex programming and variational inequalities in the literature. See, for example, [1–16] and the references therein.

Inspired and motivated by the research works going on this field, the purpose of this paper is to introduce and study a new notion of relatively A-maximal m-relaxed monotonicity framework and discuss some properties of a new class of generalized relatively resolvent operator associated with the relatively A-maximal m-relaxed monotone operator and the new generalized Yosida approximations based on relatively A-maximal m-relaxed monotonicity framework. Furthermore, some remarks will be given to show that the theory of the new generalized relatively resolvent operator and Yosida approximations associated with relatively A-maximal m-relaxed monotone operators generalizes most of the existing notions on (relatively) maximal monotone mappings in Hilbert as well as Banach space and can be applied to study variational inclusion problems and first-order evolution equations (inclusions).

2. Relatively A-Maximal m-Relaxed Monotonicity

Let $\mathscr{H}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively, and let $2^{\mathscr{H}}$ denote the family of all the nonempty subsets of $\mathscr{H}$.

In the sequel, let us recall some concepts and lemmas.

Definition 1. Let $B:\mathscr{H}\to\mathscr{H}$ be a single-valued nonlinear operator. Then operator $f:\mathscr{H}\to\mathscr{H}$ is said to be

(i) ξ-strongly monotone if there exists a constant $\xi>0$ such that

$$\langle f(x)-f(y), x-y\rangle \geq \xi\|x-y\|^2, \quad \forall x,y\in\mathscr{H}, \tag{8}$$

(ii) α-strongly monotone with respect to B if there exists a constant $\alpha>0$ such that

$$\langle f(x)-f(y), B(x)-B(y)\rangle \geq \alpha\|x-y\|^2, \quad \forall x,y\in\mathscr{H}, \tag{9}$$

(iii) cocoercive with respect to B if

$$\begin{gathered}\langle f(x)-f(y), B(x)-B(y)\rangle \geq \|f(x)-f(y)\|^2,\\ \forall x,y\in\mathscr{H},\end{gathered} \tag{10}$$

(iv) l-cocoercive with respect to B if there exists a constant $l>0$ such that

$$\begin{gathered}\langle f(x)-f(y), B(x)-B(y)\rangle \geq l\|f(x)-f(y)\|^2,\\ \forall x,y\in\mathscr{H},\end{gathered} \tag{11}$$

(v) μ-Lipschitz continuous if there exists a constant $\mu>0$ such that

$$\|f(x)-f(y)\| \leq \mu\|x-y\|, \quad \forall x,y\in\mathscr{H}; \tag{12}$$

particularly, f is called nonexpansive when $\mu=1$.

Example 2 (see [15]). Let $T:\mathscr{H}\to\mathscr{H}$ be a nonexpansive operator. Then $I-T$ is $1/2$-cocoercive with respect to I, where I is the identity.

Definition 3. Let $H,A,B:\mathscr{H}\to\mathscr{H}$ be single-valued operators. Then multivalued operator $M:\mathscr{H}\to 2^{\mathscr{H}}$ is said to be

(i) monotone with respect to B if

$$\langle u-v, B(x)-B(y)\rangle \geq 0, \quad \forall (x,u),(y,v)\in\operatorname{Graph}(M), \tag{13}$$

(ii) strictly monotone with respect to B if M is monotone with respect to B and equality holds only if $x = y$ for all $x, y \in \mathscr{H}$,

(iii) r-strongly monotone with respect to B if there exists a constant $r > 0$ such that

$$\langle u - v, B(x) - B(y) \rangle \geq r\|x - y\|^2, \quad \forall (x,u), (y,v) \in \text{Graph}(M), \tag{14}$$

(iv) m-relaxed monotone with respect to B if there exists a constant $m > 0$ such that

$$\langle u - v, B(x) - B(y) \rangle \geq -m\|x - y\|^2, \quad \forall (x,u), (y,v) \in \text{Graph}(M), \tag{15}$$

(v) c-cocoercive with respect to B if there exists a constant $c > 0$ such that

$$\langle u - v, B(x) - B(y) \rangle \geq c\|u - v\|^2, \quad \forall (x,u), (y,v) \in \text{Graph}(M), \tag{16}$$

(vi) relatively maximal monotone with respect to B if and only if M is monotone with respect to B and $R(I + \rho M) = \mathscr{H}$ for every $\rho > 0$,

(vii) relatively A-maximal monotone with respect to B if M is monotone with respect to B and $(A+\rho M)(\mathscr{H}) = \mathscr{H}$ for all $\rho > 0$.

Example 4 (see [6]). (1) Let $\mathscr{H} = (-\infty, +\infty)$, $M(x) = -x$ and $B(x) = -(1/2)x$ for all $x \in \mathscr{H}$. Then M is (relatively) monotone with respect to B but not monotone.

(2) Let $\mathscr{H}$ a real Hilbert space and $M : \mathscr{H} \to 2^{\mathscr{H}}$ be a maximal monotone operator. Then the Yosida approximation $M_\rho = \rho^{-1}(I - R_\rho^M)$ is relatively monotone with respect to the resolvent operator $R_\rho^M = (I + \rho M)^{-1}$.

Definition 5. Let $A, B : \mathscr{H} \to \mathscr{H}$ be two single-valued operators. Then multivalued operator $M : \mathscr{H} \to 2^{\mathscr{H}}$ is said to be relatively A-maximal m-relaxed monotone with respect to B if

(a) M is m-relaxed monotone with respect to B,

(b) $(A + \rho M)(\mathscr{H}) = \mathscr{H}$ for all $\rho > 0$.

This is equivalent to stating that M is relatively A-maximal m-relaxed monotone with respect to B if M is m-relaxed monotone with respect to B and $(A+\rho M)$ is maximal monotone.

Remark 6. Obviously, if $B = I$, then the relatively A-maximal m-relaxed monotonicity becomes the A-monotonicity (so-called A-maximal m-relaxed monotonicity or A-maximal relaxed monotonicity [16]) introduced and studied in [15]. Further, if $m = 0$, that is, M is relatively 0-relaxed monotone (in fact, monotone with respect to B), then the relatively A-maximal m-relaxed monotonicity reduces to relatively A-maximal monotonicity [1] (also referred to as H-maximal monotonicity relative to H in [5] where $B = A = H$ and M is a single-valued operator). Therefore, the class of relatively A-maximal m-relaxed monotone operators provides unifying frameworks for classes of (relatively) maximal monotone operators and (relatively) H-maximal monotone operators. For details about these operators, we refer the reader to [1–6, 9] and the references therein.

Theorem 7. *Let $\mathscr{H}$ be a real Hilbert space, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone single-valued operator with respect to $B : \mathscr{H} \to \mathscr{H}$, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively A-maximal m-relaxed monotone operator with respect to B with $r > m$. Then the operator $(A + \rho M)^{-1}$ is single-valued for $\rho > 0$.*

Proof. For any element $u \in \mathscr{H}$, let $x, y \in (A+\rho M)^{-1}(u)$. Then we have $u - A(x) \in \rho M(x)$ and $u - A(y) \in \rho M(y)$. Since M is relatively A-maximal m-relaxed monotone with respect to B, we have

$$\langle u - A(x) - (u - A(y)), B(x) - B(y) \rangle \geq -m\|x - y\|^2, \tag{17}$$

that is,

$$r\|x - y\|^2 \leq \langle A(x) - A(y), B(x) - B(y) \rangle \leq m\|x - y\|^2. \tag{18}$$

It follows that $x = y$ for $r > m$. □

Remark 8. If $m = 0$, that is, M is 0-relaxed monotone, then we can obtain the same result that the operator $(A+\rho M)^{-1}$ is single valued for $\rho > 0$. However, the strongly monotonicity of B is not used but is applied in Proposition 2.1 of [1].

Definition 9. Let $\mathscr{H}$ be a real Hilbert space, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone single-valued operator with respect to $B : \mathscr{H} \to \mathscr{H}$, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively A-maximal m-relaxed monotone operator with respect to B with $r > m$. Then the generalized relatively resolvent operator $J_{\rho,M}^{A,B} : \mathscr{H} \to \mathscr{H}$ is defined by

$$J_{\rho,M}^{A,B}(x) = (A + \rho M)^{-1}(x), \quad \forall x \in \mathscr{H}. \tag{19}$$

Theorem 10. *Let $\mathscr{H}$ be a real Hilbert space, nonlinear operator, $B : \mathscr{H} \to \mathscr{H}$ β-Lipschitz continuous, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone single-valued operator with respect to B, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively A-maximal m-relaxed monotone operator with respect to B with $r > m$. Then the generalized relatively resolvent operator associated with M is $\beta/(r - \rho m)$-Lipschitz continuous with positive constant $\rho < r/m$.*

Proof. For any $x, y \in \mathscr{H}$, by the definition of the resolvent operator $J_{\rho,M}^{A,B}$, we now know

$$\begin{aligned} &\rho^{-1}\left(x - A\left(J_{\rho,M}^{A,B}(x)\right)\right) \in M\left(J_{\rho,M}^{A,B}(x)\right), \\ &\rho^{-1}\left(y - A\left(J_{\rho,M}^{A,B}(y)\right)\right) \in M\left(J_{\rho,M}^{A,B}(y)\right). \end{aligned} \tag{20}$$

Since M is m-relaxed monotone with respect to B, we have

$$\begin{aligned}\rho^{-1}&\left\langle x - A\left(J_{\rho,M}^{A,B}(x)\right) - \left(y - A\left(J_{\rho,M}^{A,B}(y)\right)\right),\right.\\ &\left. B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\rangle \\ &\geq -m\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\|^2.\end{aligned}\tag{21}$$

This implies

$$\begin{aligned}\|x-y\|&\cdot\beta\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\| \\ &\geq \|x-y\|\cdot\left\|B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\| \\ &\geq \left\langle x-y, B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\rangle \\ &\geq \left\langle A\left(J_{\rho,M}^{A,B}(x)\right) - A\left(J_{\rho,M}^{A,B}(y)\right),\right.\\ &\qquad\left. B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\rangle \\ &\quad - \rho m\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\|^2 \\ &\geq (r-\rho m)\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\|^2,\end{aligned}\tag{22}$$

and so

$$\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\| \leq \frac{\beta}{r-\rho m}\|x-y\|,\tag{23}$$

where $\rho \in (0, r/m)$ is a constant. This completes the proof. □

Remark 11. (1) If A, M, and $\mathscr{H}$ are the same as in Theorem 10 and $B : \mathscr{H} \to \mathscr{H}$ is a nonexpansive operator, then the generalized relatively resolvent operator associated with M is $1/(r-\rho m)$-Lipschitz continuous with constant $\rho \in (0, r/m)$.

(2) If $m = 0$, A, B, and $\mathscr{H}$ are the same as in Theorem 10 and M is a relatively A-maximal monotone operator with respect to B, then the generalized resolvent operator associated with M and defined by $R_{\rho,B}^{M,A} = (A+\rho M)^{-1}$ is β/r-Lipschitz continuous.

(3) Moreover, Theorem 10 reduces to Proposition 2.11 in [5] when $m = 0$ and M is a single-valued operator and $B = A$ is r-strongly monotone.

Lemma 12. *An α-strongly monotone and μ-Lipschtiz continuous operator $f : \mathscr{H} \to \mathscr{H}$ is α/μ^2-cocoercive.*

Proof. By the monotonicity and Lipschtiz continuity of f, we have

$$\begin{aligned}&\|f(x) - f(y)\| \leq \mu\|x-y\|, \quad \forall x, y \in \mathscr{H},\\ &\langle f(x) - f(y), x-y\rangle \geq \alpha\|x-y\|^2 \\ &\qquad\geq \frac{\alpha}{\mu^2}\|f(x) - f(y)\|^2, \quad \forall x, y \in \mathscr{H}.\end{aligned}\tag{24}$$

□

Theorem 13. *Let $\mathscr{H}$ be a real Hilbert space, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone operator with respect to $B : \mathscr{H} \to \mathscr{H}$ and σ-Lipschitz continuous, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively A-maximal m-relaxed monotone operator with respect to B with $r > m$. Then the generalized relatively resolvent operator associated with M satisfies*

$$\begin{aligned}&\left\langle x-y, B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\rangle \\ &\quad\geq \frac{r}{\sigma^2}\left\|A\left(J_{\rho,M}^{A,B}(x)\right) - A\left(J_{\rho,M}^{A,B}(y)\right)\right\|^2 \\ &\quad - \rho m\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\|^2.\end{aligned}\tag{25}$$

Proof. By the definition of the resolvent operator $J_{\rho,M}^{A,B}$, we now know, for any $x, y \in \mathscr{H}$,

$$\begin{aligned}&\rho^{-1}\left(x - A\left(J_{\rho,M}^{A,B}(x)\right)\right) \in M\left(J_{\rho,M}^{A,B}(x)\right),\\ &\rho^{-1}\left(y - A\left(J_{\rho,M}^{A,B}(y)\right)\right) \in M\left(J_{\rho,M}^{A,B}(y)\right).\end{aligned}\tag{26}$$

Since M is m-relaxed monotone with respect to B, we have

$$\begin{aligned}\rho^{-1}&\left\langle x - A\left(J_{\rho,M}^{A,B}(x)\right) - \left(y - A\left(J_{\rho,M}^{A,B}(y)\right)\right),\right.\\ &\left. B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\rangle \\ &\geq -m\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\|^2.\end{aligned}\tag{27}$$

It follows from Lemma 12 that

$$\begin{aligned}&\left\langle x-y, B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\rangle \\ &\quad\geq \left\langle A\left(J_{\rho,M}^{A,B}(x)\right) - A\left(J_{\rho,M}^{A,B}(y)\right),\right.\\ &\qquad\left. B\left(J_{\rho,M}^{A,B}(x)\right) - B\left(J_{\rho,M}^{A,B}(y)\right)\right\rangle \\ &\quad - \rho m\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\|^2 \\ &\quad\geq \frac{r}{\sigma^2}\left\|A\left(J_{\rho,M}^{A,B}(x)\right) - A\left(J_{\rho,M}^{A,B}(y)\right)\right\|^2 \\ &\quad - \rho m\left\|J_{\rho,M}^{A,B}(x) - J_{\rho,M}^{A,B}(y)\right\|^2.\end{aligned}\tag{28}$$

□

Corollary 14. *Let $\mathscr{H}$ be a real Hilbert space, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone operator with respect to $B : \mathscr{H} \to \mathscr{H}$ and σ-Lipschitz continuous, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively A-maximal monotone operator with respect to B. Then the generalized relatively resolvent operator associated with M and defined by*

$$R_{\rho,B}^{M,A} = (A+\rho M)^{-1}\tag{29}$$

satisfies

$$\begin{aligned}&\left\langle x-y, B\left(R_{\rho,B}^{M,A}(x)\right) - B\left(R_{\rho,B}^{M,A}(y)\right)\right\rangle \\ &\quad\geq \frac{r}{\sigma^2}\left\|A\left(R_{\rho,B}^{M,A}(x)\right) - A\left(R_{\rho,B}^{M,A}(y)\right)\right\|.\end{aligned}\tag{30}$$

Corollary 15. *Let $\mathscr{H}$ be a real Hilbert space, $B : \mathscr{H} \to \mathscr{H}$ t-strongly monotone and β-Lipschitz continuous, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively maximal monotone operator with respect to B. Then the relatively classical resolvent operator associated with M and defined by*

$$R_{\rho,B}^{M} = (I + \rho M)^{-1} \tag{31}$$

satisfies

$$\begin{aligned}&\left\langle x - y, B\left(R_{\rho,B}^{M}(x)\right) - B\left(R_{\rho,B}^{M}(y)\right)\right\rangle \\ &\quad \geq \frac{t}{\beta^2}\left\|B\left(R_{\rho,B}^{M}(x)\right) - B\left(R_{\rho,B}^{M}(y)\right)\right\|.\end{aligned} \tag{32}$$

Proof. By the similar method as in Theorem 7, we can know that the relatively classical resolvent operator $R_{\rho,B}^{M}$ is single valued via the strongly monotonicity of B (see, [1, Proposition 2.3]). It follows from the definition of the resolvent operator $R_{\rho,B}^{M}$ that, for any $x, y \in \mathscr{H}$,

$$\begin{aligned}&\rho^{-1}\left(x - R_{\rho,B}^{M}(x)\right) \in M\left(R_{\rho,B}^{M}(x)\right),\\ &\rho^{-1}\left(y - R_{\rho,B}^{M}(y)\right) \in M\left(R_{\rho,B}^{M}(y)\right).\end{aligned} \tag{33}$$

Since M is monotone with respect to B, we have

$$\begin{aligned}&\rho^{-1}\left\langle x - R_{\rho,B}^{M}(x) - \left(y - R_{\rho,B}^{M}(y)\right),\right.\\ &\qquad\left. B\left(R_{\rho,B}^{M}(x)\right) - B\left(R_{\rho,B}^{M}(y)\right)\right\rangle \geq 0,\end{aligned} \tag{34}$$

that is,

$$\begin{aligned}&\left\langle x - y, B\left(R_{\rho,B}^{M}(x)\right) - B\left(R_{\rho,B}^{M}(y)\right)\right\rangle \\ &\quad \geq \left\langle R_{\rho,B}^{M}(x) - R_{\rho,B}^{M}(y), B\left(R_{\rho,B}^{M}(x)\right) - B\left(R_{\rho,B}^{M}(y)\right)\right\rangle \\ &\quad \geq \frac{t}{\beta^2}\left\|B\left(R_{\rho,B}^{M}(x)\right) - B\left(R_{\rho,B}^{M}(y)\right)\right\|.\end{aligned} \tag{35}$$

□

Remark 16. Corollaries 14 and 15 are improved Propositions 2.2 and 2.4 in [1], respectively. In deed, the strongly monotonicity of B in Corollary 14 is not used and the cocoercivity is simplified in Corollaries 14 and 15.

3. Generalized Yosida Approximations

In this section, based on Theorems 10 and 13, we shall introduce generalized Yosida approximation of relatively A-maximal m-relaxed operator and give some properties of the generalized Yosida approximation.

Definition 17. Let $\mathscr{H}$ be a real Hilbert space, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone operator with respect to $B : \mathscr{H} \to \mathscr{H}$, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively A-maximal m-relaxed monotone operator with respect to B with $r > m$. Then the generalized Yosida approximation $M_{\rho,m}^{A,B}$ of relatively maximal m-relaxed monotone operator M with respect to B is defined by

$$M_{\rho,m}^{A,B} = \rho^{-1}\left(A - B \circ J_{\rho,M}^{A,B} \circ A\right), \tag{36}$$

where $J_{\rho,M}^{A,B} = (A + \rho M)^{-1}$ is the generalized resolvent operator associated with relatively A-maximal m-relaxed monotone operator M with respect to B.

Definition 18. Let $\mathscr{H}$ be a real Hilbert space, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone single-valued operator with respect to $B : \mathscr{H} \to \mathscr{H}$, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively A-maximal monotone operator with respect to B. Then the generalized Yosida approximation $M_{\rho,B}^{A}$ of relatively maximal monotone operator M with respect to B is defined by

$$M_{\rho,B}^{A} = \rho^{-1}\left(A - B \circ R_{\rho,B}^{M,A} \circ A\right), \tag{37}$$

where $R_{\rho,B}^{M,A} = (A + \rho M)^{-1}$ is the generalized resolvent operator associated with relatively A-maximal monotone operator M with respect to B.

Definition 19. Let $\mathscr{H}$ be a real Hilbert space, $B : \mathscr{H} \to \mathscr{H}$ t-strongly monotone, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ a relatively maximal monotone operator with respect to B. Then the Yosida approximation of relatively maximal monotone operator M with respect to B is defined by

$$M_{\rho,B} = \rho^{-1}\left(I - B \circ R_{\rho,B}^{M}\right), \tag{38}$$

where $R_{\rho,B}^{M} = (I + \rho M)^{-1}$ is the relatively classical resolvent operator associated with relatively maximal monotone operator M with respect to B.

Based on definition of generalized Yosida approximation and Theorem 10, now we give some property of the generalized Yosida approximation.

Theorem 20. *Let $\mathscr{H}$ be a real Hilbert space, nonlinear operator $B : \mathscr{H} \to \mathscr{H}$ β-Lipschtiz continuous, $A : \mathscr{H} \to \mathscr{H}$ an r-strongly monotone operator with respect to B and σ-Lipschtiz continuous, and $M : \mathscr{H} \to 2^{\mathscr{H}}$ relatively A-maximal m-relaxed monotone with respect to B with $r > m$. Then the generalized Yosida approximation $M_{\rho,m}^{A,B}$ of M is $(\sigma/\rho)(1 + \beta^2/(r - \rho m))$-Lipschitz continuous, where $\rho < r/m$ is a positive constant.*

Proof. For any $x, y \in \mathscr{H}$, it follows from Theorem 10 that we have

$$\begin{aligned}&\left\|M_{\rho,m}^{A,B}(x) - M_{\rho,m}^{A,B}(y)\right\| \\ &\quad = \rho^{-1}\left\|\left[A(x) - B\left(J_{\rho,M}^{A,B}(A(x))\right)\right]\right.\\ &\qquad\qquad \left. - \left[A(y) - B\left(J_{\rho,M}^{A,B}(A(y))\right)\right]\right\| \\ &\quad \leq \rho^{-1}\left[\left\|A(x) - A(y)\right\|\right.\\ &\qquad \left. + \left\|B\left(J_{\rho,M}^{A,B}(A(x))\right) - B\left(J_{\rho,M}^{A,B}(A(y))\right)\right\|\right] \\ &\quad \leq \rho^{-1}\left[\sigma\left\|x - y\right\| + \beta\left\|J_{\rho,M}^{A,B}(A(x)) - J_{\rho,M}^{A,B}(A(y))\right\|\right]\end{aligned}$$

$$\leq \rho^{-1}\left[\sigma\|x-y\|+\frac{\beta^2}{r-\rho m}\|A(x)-A(y)\|\right]$$
$$\leq \rho^{-1}\sigma\left(1+\frac{\beta^2}{r-\rho m}\right)\|x-y\|. \tag{39}$$

From Theorem 20 and (2) of Remark 11, we have the following results. □

Corollary 21. *Let $\mathscr{H}$ be a real Hilbert space, $B:\mathscr{H}\to\mathscr{H}$ β-Lipschtiz continuous, $A:\mathscr{H}\to\mathscr{H}$ an r-strongly monotone operator with respect to B and σ-Lipschtiz continuous, and $M:\mathscr{H}\to 2^{\mathscr{H}}$ relatively A-maximal monotone with respect to B. Then the generalized Yosida approximation $M^A_{\rho,B}$ of M is $(\sigma/\rho)(1+\beta^2/r)$-Lipschitz continuous.*

Corollary 22. *Let $\mathscr{H}$ be a real Hilbert space, $B:\mathscr{H}\to\mathscr{H}$ t-strongly monotone and β-Lipschtiz continuous, and $M:\mathscr{H}\to 2^{\mathscr{H}}$ be relatively maximal monotone with respect to B. Then the Yosida approximation $M_{\rho,B}$ of M is $\rho^{-1}(1+\beta^2/t)$-Lipschitz continuous.*

Proof. For any $x,y\in\mathscr{H}$, it follows from the definition of the resolvent operator $R^M_{\rho,B}$ that

$$\rho^{-1}\left(x-R^M_{\rho,B}(x)\right)\in M\left(R^M_{\rho,B}(x)\right),$$
$$\rho^{-1}\left(y-R^M_{\rho,B}(y)\right)\in M\left(R^M_{\rho,B}(y)\right). \tag{40}$$

Since M is monotone with respect to B, we have

$$\rho^{-1}\left\langle x-R^M_{\rho,B}(x)-\left(y-R^M_{\rho,B}(y)\right),\right.$$
$$\left.B\left(R^M_{\rho,B}(x)\right)-B\left(R^M_{\rho,B}(y)\right)\right\rangle\geq 0. \tag{41}$$

That is

$$\|x-y\|\cdot\beta\left\|R^M_{\rho,B}(x)-R^M_{\rho,B}(y)\right\|$$
$$\geq\|x-y\|\cdot\left\|B\left(R^M_{\rho,B}(x)\right)-B\left(R^M_{\rho,B}(y)\right)\right\|$$
$$\geq\left\langle x-y,B\left(R^M_{\rho,B}(x)\right)-B\left(R^M_{\rho,B}(y)\right)\right\rangle$$
$$\geq\left\langle R^M_{\rho,B}(x)-R^M_{\rho,B}(y),B\left(R^M_{\rho,B}(x)\right)-B\left(R^M_{\rho,B}(y)\right)\right\rangle$$
$$\geq t\left\|R^M_{\rho,B}(x)-R^M_{\rho,B}(y)\right\|^2. \tag{42}$$

Thus, the rest of proof can be obtained from the proof of Theorem 20 and it is omitted. □

Theorem 23. *Let $\mathscr{H}$ be a real Hilbert space, $A:\mathscr{H}\to\mathscr{H}$ an r-strongly monotone operator with respect to nonlinear operator $B:\mathscr{H}\to\mathscr{H}$ and σ-Lipschtiz continuous, and $M:\mathscr{H}\to 2^{\mathscr{H}}$ relatively A-maximal m-relaxed monotone with respect to B with $r>m$. Then for all $x,y\in\mathscr{H}$, we have*

$$\left\langle M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y),A(x)-A(y)\right\rangle$$
$$\geq\rho\left\|M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y)\right\|^2$$
$$+\frac{r}{\rho\sigma^2}\left\|A\left(J^{A,B}_{\rho,M}(A(x))\right)-A\left(J^{A,B}_{\rho,M}(A(y))\right)\right\|$$
$$-m\left\|J^{A,B}_{\rho,M}(A(x))-J^{A,B}_{\rho,M}(A(y))\right\|^2$$
$$-\frac{1}{\rho}\left\|B\left(J^{A,B}_{\rho,M}(A(x))\right)-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\|^2, \tag{43}$$

where $M^{A,B}_{\rho,m}=\rho^{-1}(A-B\circ J^{A,B}_{\rho,M}\circ A)$ is the generalized Yosida approximation of relatively A-maximal m-relaxed monotone operator M with respect to B for $J^{A,B}_{\rho,M}=(A+\rho M)^{-1}$ and

$$\left\langle M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y),B\left(J^{A,B}_{\rho,M}(A(x))\right)\right.$$
$$\left.-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\rangle$$
$$\geq\frac{r}{\rho\sigma^2}\left\|A\left(J^{A,B}_{\rho,M}(A(x))\right)-A\left(J^{A,B}_{\rho,M}(A(y))\right)\right\|$$
$$-\frac{1}{\rho}\left\|B\left(J^{A,B}_{\rho,M}(A(x))\right)-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\|^2$$
$$-m\left\|J^{A,B}_{\rho,M}(A(x))-J^{A,B}_{\rho,M}(A(y))\right\|^2. \tag{44}$$

Proof. From Lemma 12 and Theorem 13 that, we get

$$\left\langle A(x)-A(y),B\left(J^{A,B}_{\rho,M}(A(x))\right)-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\rangle$$
$$\geq\frac{r}{\sigma^2}\left\|A\left(J^{A,B}_{\rho,M}(A(x))\right)-A\left(J^{A,B}_{\rho,M}(A(y))\right)\right\|$$
$$-\rho m\left\|J^{A,B}_{\rho,M}(A(x))-J^{A,B}_{\rho,M}(A(y))\right\|^2,$$
$$\left\langle M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y),A(x)-A(y)\right\rangle$$
$$=\left\langle M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y),\rho\left[M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y)\right]\right.$$
$$\left.+\left[B\left(J^{A,B}_{\rho,M}(A(x))\right)-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right]\right\rangle$$
$$=\rho\left\|M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y)\right\|^2$$
$$+\left\langle M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y),B\left(J^{A,B}_{\rho,M}(A(x))\right)\right.$$
$$\left.-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\rangle$$
$$=\rho\left\|M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y)\right\|^2$$
$$+\rho^{-1}\left\langle A(x)-A(y),B\left(J^{A,B}_{\rho,M}(A(x))\right)\right.$$
$$\left.-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\rangle$$

$$
\begin{aligned}
&-\rho^{-1}\left\langle B\left(J^{A,B}_{\rho,M}(A(x))\right)-B\left(J^{A,B}_{\rho,M}(A(y))\right),\right.\\
&\qquad\left.B\left(J^{A,B}_{\rho,M}(A(x))\right)-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\rangle\\
&\geq\rho\left\|M^{A,B}_{\rho,m}(x)-M^{A,B}_{\rho,m}(y)\right\|^2\\
&\quad+\frac{r}{\rho\sigma^2}\left\|A\left(J^{A,B}_{\rho,M}(A(x))\right)-A\left(J^{A,B}_{\rho,M}(A(y))\right)\right\|\\
&\quad-m\left\|J^{A,B}_{\rho,M}(A(x))-J^{A,B}_{\rho,M}(A(y))\right\|^2\\
&\quad-\frac{1}{\rho}\left\|B\left(J^{A,B}_{\rho,M}(A(x))\right)-B\left(J^{A,B}_{\rho,M}(A(y))\right)\right\|^2.
\end{aligned}
\tag{45}
$$

□

Remark 24. If B is β-Lipschtiz continuous, A, M, and $\mathscr{H}$ are the same as in Theorem 20, and positive constant $\rho\in(0,(r^2-\sigma^2\beta^2)/m\beta\sigma^2]$, then it is easy to see that $M^{A,B}_{\rho,m}$ is σ/ρ-Lipschtiz continuous, which is more application-enhanced than that of $(\sigma/\rho)(1+\beta^2/(r-\rho m))$ in Theorem 20.

From Theorem 23 and Corollaries 14 and 15, we have the following results.

Corollary 25. *Let $\mathscr{H}$ be a real Hilbert space, $A:\mathscr{H}\to\mathscr{H}$ r-strongly monotone with respect to nonlinear operator $B:\mathscr{H}\to\mathscr{H}$ and σ-Lipschitz continuous, and $M:\mathscr{H}\to 2^{\mathscr{H}}$ relatively A-maximal monotone with respect to B. Then for all $x,y\in\mathscr{H}$, we have*

$$
\begin{aligned}
&\left\langle M^{A}_{\rho,B}(x)-M^{A}_{\rho,B}(y),A(x)-A(y)\right\rangle\\
&\quad\geq\rho\left\|M^{A}_{\rho,B}(x)-M^{A}_{\rho,B}(y)\right\|^2\\
&\qquad+\frac{r}{\rho\sigma^2}\left\|A\left(R^{M,A}_{\rho,B}(A(x))\right)-A\left(R^{M,A}_{\rho,B}(A(y))\right)\right\|\\
&\qquad-\frac{1}{\rho}\left\|B\left(R^{M,A}_{\rho,B}(A(x))\right)-B\left(R^{M,A}_{\rho,B}(A(y))\right)\right\|^2,
\end{aligned}
\tag{46}
$$

where $M^{A}_{\rho,B}=\rho^{-1}(A-B\circ R^{M,A}_{\rho,B}\circ A)$ is the generalized Yosida approximation of relatively A-maximal monotone operator M with respect to B for $R^{M,A}_{\rho,B}=(A+\rho M)^{-1}$ and

$$
\begin{aligned}
&\left\langle M^{A}_{\rho,B}(x)-M^{A}_{\rho,B}(y),B\left(R^{M,A}_{\rho,B}(A(x))\right)-B\left(R^{M,A}_{\rho,B}(A(y))\right)\right\rangle\\
&\quad\geq\frac{r}{\rho\sigma^2}\left\|A\left(R^{M,A}_{\rho,B}(A(x))\right)-A\left(R^{M,A}_{\rho,B}(A(y))\right)\right\|\\
&\qquad-\frac{1}{\rho}\left\|B\left(R^{M,A}_{\rho,B}(A(x))\right)-B\left(R^{M,A}_{\rho,B}(A(y))\right)\right\|^2.
\end{aligned}
\tag{47}
$$

Remark 26. If M is a single-valued operator, $B=A$ is r-strongly monotone and σ-Lipschitz continuous, and $r=\sigma^2$, that is, A is cocoercive, then Corollary 25 is equivalent to Proposition 3.3 in [5] without the condition that "$A\circ R^{M,A}_{\rho,A}\circ A$ is cocoercive with respect to A," where $R^{M,A}_{\rho,A}=(A+\rho M)^{-1}$.

Corollary 27. *Let $\mathscr{H}$ be a real Hilbert space, $B:\mathscr{H}\to\mathscr{H}$ t-strongly monotone and β-Lipschtiz continuous, and $M:\mathscr{H}\to 2^{\mathscr{H}}$ relatively maximal monotone with respect to B. Then for all $x,y\in\mathscr{H}$, we have*

$$
\begin{aligned}
&\left\langle M_{\rho,B}(x)-M_{\rho,B}(y),x-y\right\rangle\\
&\quad\geq\rho\left\|M_{\rho,B}(x)-M_{\rho,B}(y)\right\|^2\\
&\qquad+\frac{t-\beta^2}{\rho\beta^2}\left\|B\left(R^{M}_{\rho,B}(x)\right)-B\left(R^{M}_{\rho,B}(y)\right)\right\|^2,
\end{aligned}
\tag{48}
$$

where $M_{\rho,B}=\rho^{-1}(I-B\circ R^{M}_{\rho,B})$ is the Yosida approximation of relatively maximal monotone operator M with respect to B for $R^{M}_{\rho,B}=(I+\rho M)^{-1}$ and

$$
\begin{aligned}
&\left\langle M_{\rho,B}(x)-M_{\rho,B}(y),B\left(R^{M}_{\rho,B}(x)\right)-B\left(R^{M}_{\rho,B}(y)\right)\right\rangle\\
&\quad\geq\frac{t-\beta^2}{\rho\beta^2}\left\|B\left(R^{M}_{\rho,B}(x)\right)-B\left(R^{M}_{\rho,B}(y)\right)\right\|^2.
\end{aligned}
\tag{49}
$$

Remark 28. (1) If $B=I$ and $t=\beta=1$ in Corollary 27, then we have the classical theory of maximal monotone operators. This is equivalent to stating that Corollary 27 represents a generalization to (2) of Example 4. This would also clarify the notational as well as theoretical differences between the classical resolvent and relatively classical resolvent.

(2) From Corollary 27, we know that if $\beta^2\leq t\leq\beta$, then we have

$$
\left\langle M_{\rho,B}(x)-M_{\rho,B}(y),x-y\right\rangle\geq\rho\left\|M_{\rho,B}(x)-M_{\rho,B}(y)\right\|^2,
\tag{50}
$$

that is,

$$
\left\|M_{\rho,B}(x)-M_{\rho,B}(y)\right\|\leq\frac{1}{\rho}\left\|x-y\right\|,
\tag{51}
$$

and so $M_{\rho,B}$ is $1/\rho$-Lipschtiz continuous. Thus, more value-added application can be gained than that of $\rho^{-1}(1+\beta^2/t)$ in Corollary 22.

4. Concluding Remarks

The purpose of this paper is to introduce and study a new notion of relatively A-maximal m-relaxed monotonicity framework and to discuss some properties of a class of new generalized relatively resolvent operator associated with the relatively A-maximal m-relaxed monotone operator and the new generalized Yosida approximations based on relatively A-maximal m-relaxed monotonicity framework. Because the relatively A-maximal m-relaxed monotonicity includes (relatively) A-maximal monotonicity, (relatively) H-maximal monotonicity, and (relatively) maximal monotonicity as special cases, the theory of the new generalized relatively resolvent operator and Yosida approximations associated with relatively A-maximal m-relaxed monotone operators generalizes most of the existing notions on (relatively) maximal monotone mappings to Hilbert as well as Banach space settings, and its applications range from nonlinear variational inequalities, equilibrium problems, optimization and control theory, management and decision sciences, and mathematical programming to engineering sciences. Therefore, the following two fields' problems are worth studying in further research.

On the one hand, we note that the classical Yosida approximation associated with classical maximal monotonicity played a prominent role during the proof of the result of applying the Douglas-Rachford splitting method for finding a zero of the sum of two monotone mappings. Hence, it follows from Theorem 13 that we can generalize and improve the main linear convergence results (i.e., Theorem 23) to the variation inclusion problem (1.1) in [1, 3, 6] under the framework of relatively A-maximal m-relaxed monotonicity.

On the other hand, Theorems 20 and 23, that is, the generalized Yosida regularization/approximation results, can be applied to the solvability of the first-order differential evolution inclusions of the following form:

$$x'(t) + M(x(t)) \ni 0 \quad \text{for } 0 < t < \infty, \qquad x(0) = x_0, \tag{52}$$

where $M : \mathscr{H} \rightarrow 2^{\mathscr{H}}$ is relatively A-maximal m-relaxed monotone, $x : [0, \infty) \rightarrow \mathscr{H}$ is such that (52) holds, and the derivative $x'(t)$ exists in the sense of the weak convergence. Further, the problem (52) becomes problem (2) when the operator $M : \mathscr{H} \rightarrow \mathscr{H}$ is single valued, and the real problems could arise due to the presence of the relatively relaxed monotonicity achieving the uniqueness of the solution.

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