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Research Article

Positive Solutions for Boundary Value Problems of Singular Fractional Differential Equations

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In this paper, by using a fixed point theorem, we investigate the existence of a positive solution to the singular fractional boundary value problem ${}^CD_{0+}^{\alpha}u + f(t,u,{}^CD_{0+}^{\nu}u,{}^CD_{0+}^{\mu}u) + g(t,u,{}^CD_{0+}^{\nu}u,{}^CD_{0+}^{\mu}u) = 0$, $u(0) = u'(0) = u''(0) = u'''(0) = 0$, where $3 < \alpha < 4$, $0 < \nu < 1$, $1 < \mu < 2$, ${}^CD_{0+}^{\alpha}$ is Caputo fractional derivative, $f(t,x,y,z)$ is singular at the value 0 of its arguments x, y, z, and $g(t,x,y,z)$ satisfies the Lipschitz condition.

1. Introduction

In recent years, as an extended concept of integral differential equations, fractional differential equations are widely concerned in various fields of science. For examples, see [1–14]. Many results, such as [1, 2, 6, 15, 16], discuss singular fractional boundary value problems.

In [1], the authors discuss positive solutions to the singular Dirichlet problem

$$D_{0+}^{\alpha}u(t) + f(t,u(t),D^{\mu}u(t)) = 0,\quad u(0) = u(1) = 0, \tag{1}$$

where $1 < \alpha < 2$, $0 < \mu \le \alpha - 1$ and f is a Carathéodory function on $[0,1]\times(0,\infty)\times\mathbb{R}$. Here, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. The existence of positive solutions is obtained by the combination of regularization and sequential techniques with the Guo-Krasnosel'skii fixed point theorem on cone.

The singular problem

$$D_{0+}^{\alpha}u(t) + q(t)f\left(u(t),u'(t),\ldots,u^{(n-2)}(t)\right) = 0,\quad n-1 < \alpha \le n,\quad n \ge 2, \tag{2}$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\quad u^{(n-2)}(1) = 0,$$

was discussed in [16], where $f \in C((0,\infty)^{n-1})$ and $q \in L^r[0,1]$ $(r > 0)$ are positive. The existence results of positive solutions are acquired by the use of regularization and sequential techniques with a fixed point theorem for mixed monotone operators on normal cones.

Paper [6] investigates positive solutions of singular fractional boundary value problem

$$D_{0+}^{\alpha}u(t) + f\left(t,u(t),D_{0+}^{\nu}u(t),D_{0+}^{\mu}u(t)\right) = 0,\quad u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{3}$$

where $3 < \alpha \le 4$, $0 < \nu \le 1$, $1 < \mu \le 2$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, and f is a Carathédory function. The existence and multiplicity of positive solutions are obtained by means of Guo-Krasnosel'skii fixed point theorem on cones.

In this paper, we are concerned with the following singular fractional boundary value problem:

$${}^CD_{0+}^{\alpha}u + f\left(t,u,{}^CD_{0+}^{\nu}u,{}^CD_{0+}^{\mu}u\right) + g\left(t,u,{}^CD_{0+}^{\nu}u,{}^CD_{0+}^{\mu}u\right) = 0, \tag{4}$$

$$u(0) = u'(0) = u''(1) = u'''(0) = 0, \tag{5}$$

where $3 < \alpha < 4$, $0 < \nu < 1$, and $1 < \mu < 2$ are real numbers. ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α. f satisfies the Carathéodory condition on $[0,1] \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^3$, $f(t,x,y,z)$ may be singular at the value 0 of all its space variables x, y, z, and $g(t,x,y,z)$ satisfies the Lipschitz condition.

A function $u \in C^2[0,1]$ is called a positive solution of problems (4), (5) if $u > 0$ on $(0,1]$, ${}^{C}D_{0+}^{\alpha} u \in L[0,1]$, and u satisfies boundary condition (5) and equality (4) for a.e. $t \in [0,1]$.

Throughout the paper, denote $\|x\|_1 = \int_0^1 |x(t)|dt$ which is the norm of $L[0,1]$, and $\|x\| = \max\{|x(t)| : t \in [0,1]\}$ is the norm of space $C[0,1]$, while $\|x\|_* = \max\{\|x\|, \|x'\|, \|x''\|\}$ is the norm of $C^2[0,1]$. $AC[0,1]$ and $AC^k[0,1]$ are sets of absolutely continuous functions and functions having absolutely continuous kth derivatives on $[0,1]$, respectively.

The following conditions on f and g in (4) will be used.

(H_1) f is a Carathéodory function on $[0,1]\times\mathcal{D}$, where $\mathcal{D} = (0,\infty)^3$, and there exists a positive constant m such that, for a.e. $t \in [0,1]$ and all $(x,y,z) \in \mathcal{D}$,

$$f(t,x,y,z) \geq m. \tag{6}$$

(H_2) $g \geq 0$ satisfies the following inequality, for a.e. $t \in [0,1]$ and all $(x_1,y_1,z_1), (x_2,y_2,z_2) \in \mathcal{D}$:

$$\begin{aligned} &|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)| \\ &\quad\leq L_1|x_1 - x_2| + L_2|y_1 - y_2| + L_3|z_1 - z_2|, \end{aligned} \tag{7}$$

with

$$\frac{1}{\Gamma(\alpha-2)}\left[L_1 + \frac{L_2}{\Gamma(2-\nu)} + \frac{L_3}{\Gamma(3-\mu)}\right] < 1. \tag{8}$$

(H_3) For a.e. $t \in [0,1]$ and all $(x,y,z) \in \mathcal{D}$,

$$\begin{aligned} &f(t,x,y,z) + g(t,x,y,z) \\ &\quad\leq p(x,y,z) + \gamma(t)h(x,y,z), \end{aligned} \tag{9}$$

where $\gamma \in L[0,1]$, $p \in C(\mathcal{D})$, and $h \in C([0,\infty)^3)$ are positive, p and h are nonincreasing and nondecreasing in all their arguments, respectively,

$$\begin{gathered} \int_0^1 p\left(2Mt^{\alpha}, \frac{M}{12}t^{4-\nu}, \frac{(2-\mu)M}{6}t^{3-\mu}\right)dt < \infty, \\ M = \frac{m}{\Gamma(\alpha+1)}, \qquad \lim_{x\to\infty}\frac{h(x,x,x)}{x} = 0. \end{gathered} \tag{10}$$

We will use regularization and sequential techniques to prove the existence of a positive solution of problems (4), (5). Define χ_n and f_n ($n \in \mathbb{N}$) by the following formulas:

$$\chi_n(t) = \begin{cases} t, & \text{if } t \geq \dfrac{1}{n}; \\ \dfrac{1}{n}, & \text{if } t < \dfrac{1}{n}, \end{cases} \tag{11}$$

for a.e. $t \in [0,1]$ and all $(x,y,z) \in \mathbb{R}^3$,

$$f_n(t,x,y,z) = f(t,\chi_n(x),\chi_n(y),\chi_n(z)). \tag{12}$$

Then, condition (H_1) gives that f_n is a Carathéodory function on $[0,1]\times\mathbb{R}^3$,

$$f_n(t,x,y,z) \geq m, \quad \text{for a.e. } t \in [0,1] \text{ and all } (x,y,z) \in \mathbb{R}^3. \tag{13}$$

Condition (H_3) gives

$$\begin{aligned} f_n(t,x,y,z) &\leq p\left(\frac{1}{n},\frac{1}{n},\frac{1}{n}\right) + \gamma(t)h\left(x+\frac{1}{n}, y+\frac{1}{n}, z+\frac{1}{n}\right), \\ &\qquad \text{for a.e. } t \in [0,1] \text{ and all } (x,y,z) \in [0,\infty)^3, \end{aligned} \tag{14}$$

$$\begin{aligned} f_n(t,x,y,z) &\leq p(x,y,z) + \gamma(t)h\left(x+\frac{1}{n}, y+\frac{1}{n}, z+\frac{1}{n}\right), \\ &\qquad \text{for a.e. } t \in [0,1] \text{ and all } (x,y,z) \in \mathcal{D}. \end{aligned} \tag{15}$$

In Section 3, We will firstly investigate the regular fractional differential equation

$$\begin{aligned} &{}^{C}D_{0+}^{\alpha}u + f_n\left(t,u,{}^{C}D_{0+}^{\nu}u,{}^{C}D_{0+}^{\mu}u\right) \\ &\quad + g\left(t,u,{}^{C}D_{0+}^{\nu}u,{}^{C}D_{0+}^{\mu}u\right) = 0. \end{aligned} \tag{16}$$

2. Preliminaries

Definition 1. The Caputo fractional derivative of order $\beta > 0$ of a function $v \in C[0,1]$ is defined by

$${}^{C}D_{0+}^{\beta}v(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1}v^{(n)}(s)\,ds, \tag{17}$$

provided that the right-hand side is pointwise defined on $[0,1]$, where $n = [\beta]+1$ and $[\beta]$ means the integer part of the number β. Γ is the Euler function.

Definition 2. The fractional integral of order $\alpha > 0$ of a function $y: [0,1] \to \mathbb{R}$ is defined by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds, \tag{18}$$

provided the right-hand side is pointwise defined on $[0,1]$.

Lemma 3 (see [10]). *One has*

$$I_{0+}^{\alpha}: L^1[0,1] \longrightarrow \begin{cases} L^1[0,1], & \textit{if } \alpha \in (0,1), \\ AC^{[\alpha]-1}[0,1], & \textit{if } \alpha \geq 1, \end{cases} \tag{19}$$

where $[\alpha]$ means the integral part of α and $AC^0[0,1] = AC[0,1]$.

Lemma 4 (see [10]). *Suppose that $\alpha > 0$, $\alpha \notin \mathbb{N}$. If $x \in C(0,1]$ and ${}^{C}D_{0+}^{\alpha}x \in L^1[0,1]$, then*

$$x(t) = I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha}x(t) + \sum_{k=0}^{n-1} c_k t^k, \quad \text{for } t \in (0,1], \tag{20}$$

where $n = [\alpha] + 1$ and $c_k \in \mathbb{R}$, $k = 0, 1, \ldots, n-1$.

Lemma 5. *Given $\rho \in L[0,1]$, then for $t \in [0,1]$,*

$$u(t) = \int_0^1 G(t,s)\rho(s)\,ds \tag{21}$$

is the unique solution in $C^2[0,1]$ of the equation

$$D_{0+}^{\alpha}u(t) + \rho(t) = 0, \tag{22}$$

satisfying the boundary condition (5), where $\alpha \in (3,4)$ and

$$G(t,s) = \begin{cases} \dfrac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \le s \le t \le 1, \\ \dfrac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}, & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{23}$$

Proof. By Lemma 4,

$$u(t) = -I_{0+}^{\alpha}\rho(t) + c_0 + c_1 t^1 + c_2 t^2 + c_3 t^3, \quad \text{for } 3 < \alpha < 4, \tag{24}$$

are all solutions of (22) in $C[0,1]$, where $c_j \in \mathbb{R}$. Lemma 3 guarantees that $I_{0+}^{\alpha}\rho \in AC^2[0,1]$, for $3 < \alpha < 4$; therefore,

$$u(t) = -I_{0+}^{\alpha}\rho(t) + c_2 t^2, \quad \text{for } 3 < \alpha < 4, \tag{25}$$

are all solutions of (22) in $C^2[0,1]$, where $c_2, c_4 \in \mathbb{R}$. Considering that the solutions should satisfy $u(0) = u'(0) = u''(1) = u'''(0) = 0$, we get that $c_2 = (1/2\Gamma(\alpha-2))\int_0^1 (1-s)^{\alpha-3}\rho(s)ds$. Consequently,

$$\begin{aligned} u(t) &= \frac{t^2\int_0^1 (1-s)^{\alpha-3}\rho(s)\,ds}{2\Gamma(\alpha-2)} - \frac{\int_0^t (t-s)^{\alpha-1}\rho(s)\,ds}{\Gamma(\alpha)} \\ &= \int_0^1 G(t,s)\rho(s)\,ds \end{aligned} \tag{26}$$

is the unique solution of problems (22), (5). □

Lemma 6. *Let G be as defined in (2.3). Then,*

(1) $G(t,s) \in C([0,1]\times[0,1])$ *and* $G(t,s) > 0$ *on* $(0,1)\times(0,1)$,

(2) $G(t,s) \le 1/\Gamma(\alpha-1)$ *for* $(t,s) \in [0,1]\times[0,1]$,

(3) $\int_0^1 G(t,s)ds \ge (\alpha^2-\alpha-2)t^{\alpha}/2\Gamma(\alpha+1)$ *for* $t \in [0,1]$,

(4) $(\partial/\partial t)G(t,s) \in C([0,1]\times[0,1])$ *and* $(\partial/\partial t)G(t,s) > 0$ *on* $(0,1)\times(0,1)$,

(5) $(\partial/\partial t)G(t,s) \le 1/\Gamma(\alpha-2)$ *for* $(t,s) \in [0,1]\times[0,1]$,

(6) $\int_0^1 (\partial/\partial t)G(t,s)ds \ge (\alpha-2)t^{\alpha-1}/\Gamma(\alpha)$ *for* $t \in [0,1]$,

(7) $(\partial^2/\partial t^2)G(t,s) \in C([0,1]\times[0,1])$ *and* $(\partial^2/\partial t^2)G(t,s) > 0$ *on* $(0,1)\times(0,1)$,

(8) $(\partial^2/\partial t^2)G(t,s) \le 1/\Gamma(\alpha-2)$ *for* $(t,s) \in [0,1]\times[0,1]$,

(9) $\int_0^1 (\partial^2/\partial t^2)G(t,s)ds \ge t(1-t^{\alpha-2})/\Gamma(\alpha-1)$ *for* $t \in [0,1]$.

Proof. (1), (4), and (7) are as follows

$$\begin{gathered} G(t,s) = \begin{cases} \dfrac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \le s \le t \le 1, \\ \dfrac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}, & \text{if } 0 \le t \le s \le 1, \end{cases} \\ \frac{\partial}{\partial t}G(t,s) = \begin{cases} \dfrac{t(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} - \dfrac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & \text{if } 0 \le s \le t \le 1, \\ \dfrac{t(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}, & \text{if } 0 \le t \le s \le 1, \end{cases} \\ \frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} - \dfrac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}, & \text{if } 0 \le s \le t \le 1, \\ \dfrac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}, & \text{if } 0 \le t \le s \le 1. \end{cases} \end{gathered} \tag{27}$$

Because $(\partial^2/\partial t^2)G(t,s) \ge 0$, therefore $(\partial/\partial t)G(t,s) \ge (\partial/\partial t)G(0,s) \ge 0$ and $G(t,s) \ge G(0,s) \ge 0$.

It is obvious that (2), (3), (5), (6), (8), and (9) hold. □

3. Auxiliary Regular Problems (16), (5)

Let $X = C^2[0,1]$, and let

$$P = \left\{x \in X : x(t) \ge 0,\ x'(t) \ge 0,\ x''(t) \ge 0,\ \text{for } t \in [0,1]\right\}. \tag{28}$$

For $x \in P$, we can obtain that

$$\begin{gathered} {}^{C}D_{0+}^{\nu}x \in C[0,1], \qquad {}^{C}D_{0+}^{\mu}x \in C[0,1], \\ {}^{C}D_{0+}^{\nu}x(t) \ge 0, \qquad {}^{C}D_{0+}^{\mu}x(t) \ge 0, \\ \text{for } x \in P, \quad t \in [0,1]. \end{gathered} \tag{29}$$

We define the operators Φ_n and Ψ on P as

$$\begin{aligned} (\Phi_n x)(t) &= \int_0^1 G(t,s) f_n\left(s, x(s), {}^{C}D_{0+}^{\nu}x(s), {}^{C}D_{0+}^{\mu}x(s)\right) ds, \\ &\qquad n = 1, 2, \ldots, \\ (\Psi x)(t) &= \int_0^1 G(t,s)\, g\left(s, x(s), {}^{C}D_{0+}^{\nu}x(s), {}^{C}D_{0+}^{\mu}x(s)\right) ds. \end{aligned} \tag{30}$$

Lemma 7. *$\Phi_n : P \to P$ is a completely continuous operator.*

The proof is similar to Lemma 3.1 of [6], so we omit it.

Lemma 8 (see [17]). *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ wherever $x, y \in M$, (ii) A is compact and continuous, and (iii) B is a contraction mapping. Then, there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 9. *Let (H_1) and (H_2) hold. Then, problems* (16), (5) *have a solution $u_n \in P$ such that*

$$u_n(t) \ge \frac{mt^{\alpha}(\alpha^2 - \alpha - 2)}{2\Gamma(\alpha+1)}, \quad \text{for } t \in [0,1]. \tag{31}$$

Proof. By Lemma 7, $\Phi_n : P \to P$ is a completely continuous operator. Now, for $x, y \in X$, we obtain that

$$\begin{aligned}
&\|(\Psi x) - (\Psi y)\| \\
&= \max_{0\le t\le 1}\left|\int_0^1 G(t,s) \times\left[g\left(s, x(s), {}^{C}D_{0+}^{\nu}x(s), {}^{C}D_{0+}^{\mu}x(s)\right) - g\left(s, y(s), {}^{C}D_{0+}^{\nu}y(s), {}^{C}D_{0+}^{\mu}y(s)\right)\right] ds\right| \\
&\le \left|\max_{0\le t\le 1}\int_0^1 G(t,s)\,ds\right| \times\left(L_1\|x-y\| + L_2\left\|{}^{C}D_{0+}^{\nu}x - {}^{C}D_{0+}^{\nu}y\right\| + L_3\left\|{}^{C}D_{0+}^{\mu}x - {}^{C}D_{0+}^{\mu}y\right\|\right) \\
&\le \left|\max_{0\le t\le 1}\int_0^1 G(t,s)\,ds\right| \times\left(L_1\|x-y\|_* + \frac{L_2}{\Gamma(2-\nu)}\|x-y\|_* + \frac{L_3}{\Gamma(3-\mu)}\|x-y\|_*\right) \\
&\le \frac{\|x-y\|_*}{\Gamma(\alpha-2)}\left(L_1 + \frac{L_2}{\Gamma(2-\nu)} + \frac{L_3}{\Gamma(3-\mu)}\right).
\end{aligned} \tag{32}$$

Therefore, Ψ is a contraction mapping, and it is obvious that $\Phi_n x + \Psi y \in X$, for $x, y \in X$. Thus, all the assumptions of Lemma 8 are satisfied, and the conclusion of Lemma 8 implies that the boundary value problems (16), (5) have at least one solution. □

Lemma 10. *Suppose that (H_1), (H_2), and (H_3) hold and u_n be a solution of problems* (16), (5). *Then, the sequence $\{u_n\}$ is relatively compact in X.*

Proof. Note that, for $t \in [0,1]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
u_n(t) &= \int_0^1 G(t,s) f_n\left(s, x_n(s), {}^{C}D_{0+}^{\nu}x_n(s), {}^{C}D_{0+}^{\mu}x_n(s)\right) ds \\
&\quad + \int_0^1 G(t,s)\, g\left(s, x_n(s), {}^{C}D_{0+}^{\nu}x_n(s), {}^{C}D_{0+}^{\mu}x_n(s)\right) ds.
\end{aligned} \tag{33}$$

And u_n fulfills (31).

Lemma 6 and (13) imply that

$$\begin{aligned}
u_n'(t) &\ge m\int_0^1 \frac{\partial}{\partial t}G(t,s)\,ds \ge \frac{m(\alpha-2)t^{\alpha-1}}{\Gamma(\alpha)}, \\
&\qquad \text{for } t \in [0,1], \quad n \in \mathbb{N}, \\
u_n''(t) &\ge m\int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)\,ds \ge \frac{m\left(1 - t^{\alpha-2}\right)t}{\Gamma(\alpha-1)}, \\
&\qquad \text{for } t \in [0,1], \quad n \in \mathbb{N}.
\end{aligned} \tag{34}$$

So,

$$\begin{aligned}
{}^{C}D_{0+}^{\mu}u_n(t) &= \frac{1}{\Gamma(2-\mu)}\int_0^t (t-s)^{1-\mu}u_n''(s)\,ds \\
&\ge \frac{m}{\Gamma(2-\mu)\Gamma(\alpha-1)}\int_0^t (t-s)^{1-\mu}s\left(1 - s^{\alpha-2}\right)ds, \\
{}^{C}D_{0+}^{\nu}u_n(t) &= \frac{1}{\Gamma(1-\nu)}\int_0^t (t-s)^{-\nu}u_n'(s)\,ds \\
&\ge \frac{m(\alpha-1)}{\Gamma(1-\nu)\Gamma(\alpha)}\int_0^t (t-s)^{-\nu}s^{\alpha-1}\,ds.
\end{aligned} \tag{35}$$

Since

$$\begin{aligned}
&\int_0^t (t-s)^{1-\mu}s\left(1 - s^{\alpha-2}\right)ds \\
&\quad > \int_0^t (t-s)^{1-\mu}s(1-s)\,ds \\
&\quad = \frac{1}{2-\mu}\int_0^t (t-s)^{2-\mu}(1-2s)\,ds \\
&\quad = \frac{1}{2-\mu}\left(\frac{t^{3-\mu}}{3-\mu} - \frac{2t^{4-\mu}}{(3-\mu)(4-\mu)}\right) \\
&\quad = \frac{t^{3-\mu}}{2-\mu}\left(\frac{4-\mu-2t}{(3-\mu)(4-\mu)}\right) \\
&\quad \ge \frac{t^{3-\mu}}{2-\mu}\left(\frac{2-\mu}{(3-\mu)(4-\mu)}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{t^{3-\mu}}{(3-\mu)(4-\mu)}, \\
&\int_0^t (t-s)^{-\nu} s^{\alpha-1} ds \\
&\geq \int_0^t (t-s)^{-\nu} s^3\, ds \\
&= \int_0^t \frac{(t-s)^{1-\nu} \times 3s^2}{(1-\nu)} ds \\
&= \int_0^t \frac{(t-s)^{2-\nu} \times 6s}{(1-\nu)(2-\nu)} ds \\
&= \int_0^t \frac{(t-s)^{3-\nu} \times 6}{(1-\nu)(2-\nu)(3-\nu)} ds \\
&= \frac{6t^{4-\nu}}{(1-\nu)(2-\nu)(3-\nu)(4-\nu)} \\
&\geq \frac{t^{4-\nu}}{(1-\nu)(2-\nu)(3-\nu)(4-\nu)},
\end{aligned} \tag{36}$$

then,

$$\begin{aligned}
{}^{C}D_{0+}^{\mu} u_n(t) &\geq \frac{m(2-\mu)}{\Gamma(5-\mu)\Gamma(\alpha-1)} t^{3-\mu}, \quad \text{for } t \in [0,1],\ n \in \mathbb{N}, \\
{}^{C}D_{0+}^{\nu} u_n(t) &\geq \frac{m(\alpha-1)}{\Gamma(5-\nu)\Gamma(\alpha)} t^{4-\nu}, \quad \text{for } t \in [0,1],\ n \in \mathbb{N}.
\end{aligned} \tag{37}$$

Let

$$m \cdot \min\left\{\frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(\alpha)}, \frac{1}{(\alpha-1)}\right\} := M. \tag{38}$$

It follows from (31) and (37) that, for $t \in [0,1]$, $n \in \mathbb{N}$,

$$\begin{aligned}
u_n(t) \geq 2Mt^{\alpha}, \qquad {}^{C}D_{0+}^{\nu} u_n(t) \geq \frac{M}{12} t^{4-\nu}, \\
{}^{C}D_{0+}^{\mu} u_n(t) \geq \frac{(2-\mu)M}{6} t^{3-\mu}.
\end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned}
&p\left(u_n(t), {}^{C}D_{0+}^{\nu} u_n(t), {}^{C}D_{0+}^{\mu} u_n(t)\right) \\
&\quad \leq p\left(2Mt^{\alpha}, \frac{M}{12} t^{4-\nu}, \frac{(2-\mu)M}{6} t^{3-\mu}\right).
\end{aligned} \tag{40}$$

By Lemma 6, (15), and (39), there hold that

$$\begin{aligned}
0 &\leq u_n''(t) \\
&= \int_0^1 \frac{\partial^2}{\partial t^2} G(t,s) f_n\left(s, u_n(s), {}^{C}D_{0+}^{\nu} u_n(s), {}^{C}D_{0+}^{\mu} u_n(s)\right) ds \\
&\leq \frac{1}{\Gamma(\alpha-2)} \int_0^1 p\left(2Ms^{\alpha}, \frac{M}{12} s^{4-\nu}, \frac{(2-\mu)M}{6} s^{3-\mu}\right) ds \\
&\quad + \frac{1}{\Gamma(\alpha-2)} h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n}\right) \\
&\quad \times \int_0^1 \gamma(s)\, ds \\
&\leq \frac{1}{\Gamma(\alpha-2)} \\
&\quad \times \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n}\right) \|\gamma\|_q\right), \\
0 &\leq u_n'(t) = \int_0^t u_n''(s)\, ds \\
&\leq \frac{1}{\Gamma(\alpha-2)} \\
&\quad \times \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n}\right) \|\gamma\|_q\right), \\
0 &\leq u_n(t) = \int_0^t u_n'(s)\, ds \\
&\leq \frac{1}{\Gamma(\alpha-2)} \\
&\quad \times \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n}\right) \|\gamma\|_q\right),
\end{aligned} \tag{41}$$

where $t \in [0,1]$, $n \in \mathbb{N}$, and $\Lambda = \int_0^1 p(2Ms^{\alpha}, (M/12)s^{4-\nu}, ((2-\mu)M/6)s^{3-\mu})\, ds$.

It follows from (H_2) and the assumption that $\Lambda < \infty$. Hence,

$$\begin{aligned}
\|u_n\|_* \leq{}& \frac{1}{\Gamma(\alpha-2)} \\
&\times \left(\Lambda, + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n},\right.\right. \\
&\left.\left.\frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n}\right) \|\gamma\|_q\right),
\end{aligned} \tag{42}$$

where $n \in \mathbb{N}$. Since $\lim_{x\to\infty} h(x,x,x)/x = 0$, there exists $L > 0$ such that, for $v \geq L$,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha-2)} \\
&\quad \times \left(\Lambda + h\left(v + \frac{1}{n}, \frac{v}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{v}{\Gamma(3-\mu)} + \frac{1}{n}\right) \|\gamma\|_q\right) \\
&\quad < v.
\end{aligned} \tag{43}$$

Consequently, $\|u_n\|_* < L$ for $n \in \mathbb{N}$, so that $\{u_n\}$ is bounded in X. We are now in a position to prove that $\{u_n''\}$ is equicontinuous on $[0,1]$. Let

$$V_1 = h\left(L + \frac{1}{n}, \frac{L}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{L}{\Gamma(3-\mu)} + \frac{1}{n}\right), \quad (44)$$

$$\Theta(t) = p\left(2Mt^{\alpha}, \frac{M}{12}t^{4-\nu}, \frac{(2-\mu)M}{6}t^{3-\mu}\right), \quad \text{for } t \in (0,1]. \quad (45)$$

Then, $\Lambda = \int_0^1 \Theta(t)dt$ and, for a.e. $t \in [0,1]$, all $n \in \mathbb{N}$,

$$\begin{aligned}\Theta(t) + V_1\gamma(t) &\geq f_n\left(t, u_n(t), {}^{C}D_{0+}^{\nu}u_n(t), {}^{C}D_{0+}^{\mu}u_n(t)\right) \\ &\quad + g\left(t, u_n(t), {}^{C}D_{0+}^{\nu}u_n(t), {}^{C}D_{0+}^{\mu}u_n(t)\right)\end{aligned} \quad (46)$$

holds. Suppose that $0 \leq t_1 < t_2 \leq 1$, then

$$\begin{aligned}&\left|u_n''(t_2) - u_n''(t_1)\right| \\ &= \left|\int_0^1 \left(\frac{\partial^2}{\partial t^2}G(t_2,s) - \frac{\partial^2}{\partial t^2}G(t_1,s)\right)\right. \\ &\quad \times \left(f_n\left(s, u_n(s), {}^{C}D_{0+}^{\nu}u_n(s), {}^{C}D_{0+}^{\mu}u_n(s)\right)\right. \\ &\quad \left.\left. + g\left(s, u_n(s), {}^{C}D_{0+}^{\nu}u_n(s), {}^{C}D_{0+}^{\mu}u_n(s)\right)\right) ds\right| \\ &\leq \frac{1}{\Gamma(\alpha-2)}\left[\int_{t_1}^{t_2} (t_2-s)^{\alpha-3}\left(\Theta(s) + V_1\gamma(s)\right) ds\right. \\ &\quad + \int_0^{t_1}\left((t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}\right) \\ &\quad \left.\times \left(\Theta(s) + V_1\gamma(s)\right) ds\right] \\ &\leq \frac{1}{\Gamma(\alpha-2)}\left[(t_2-t_1)^{\alpha-3}\left(\Lambda + V_1\|\gamma\|_q\right)\right. \\ &\quad + \int_0^{t_1}\left((t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}\right) \\ &\quad \left.\times \left(\Theta(s) + V_1\gamma(s)\right) ds\right].\end{aligned} \quad (47)$$

The proof is similar to that of Lemma 7. We choose $\varepsilon > 0$. Then, there exists $\delta_0 > 0$ such that $(t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3} < \varepsilon$, for any $0 \leq t_1 < t_2 \leq 1$, $t_2 - t_1 < \delta_0$, and $0 \leq s \leq t_1$. Suppose that $0 < \delta < \min\{\delta_0, \sqrt[\alpha-3]{\varepsilon}\}$. Then, for $t_1, t_2 \in [0,1]$, $0 < t_2 - t_1 < \delta$, $n \in \mathbb{N}$, we have

$$\left|u_n''(t_2) - u_n''(t_1)\right| \leq \frac{2\varepsilon}{\Gamma(\alpha-2)}\left(\Lambda + V_1\|\gamma\|_q\right). \quad (48)$$

Thus, $\{u_n''\}$ is equicontinuous on $[0,1]$. □

4. Main Result

Theorem 11. *Suppose that (H_1), (H_2), and (H_3) hold. Then, problems (4), (5) has a positive solution u and, for $t \in [0,1]$,*

$$\begin{gathered}u(t) \geq 2Mt^{\alpha}, \qquad {}^{C}D_{0+}^{\nu}u(t) \geq \frac{M}{24}t^{4-\nu}, \\ {}^{C}D_{0+}^{\mu}u(t) \geq \frac{(2-\mu)M}{6}t^{3-\mu}.\end{gathered} \quad (49)$$

Proof. Theorem 9 shows that problems (16), (5) have a solution $u_n \in P$. In addition, Lemma 10 gives that $\{u_n\}$ is relatively compact in X and satisfies inequality (39) for $t \in [0,1]$, $n \in \mathbb{N}$. Assume that $\{u_n\}$ itself is convergent in X and $\lim_{n\to\infty} u_n = u$. Then, $u \in P$ satisfies the boundary condition (5), and $\lim_{n\to\infty} {}^{C}D_{0+}^{\mu}u_n = {}^{C}D_{0+}^{\mu}u$ and $\lim_{n\to\infty} {}^{C}D_{0+}^{\nu}u_n = {}^{C}D_{0+}^{\nu}u$ in $C[0,1]$. Consequently, u satisfies (49). Furthermore,

$$\begin{aligned}&\lim_{n\to\infty} f_n\left(t, u_n(t), {}^{C}D_{0+}^{\nu}u_n(t), {}^{C}D_{0+}^{\mu}u_n(t)\right) \\ &\quad = f\left(t, u(t), {}^{C}D_{0+}^{\nu}u(t), {}^{C}D_{0+}^{\mu}u(t)\right).\end{aligned} \quad (50)$$

Let $K = \sup\{\|u_n\|_* : n \in \mathbb{N}\}$. Then, it follows from $0 < \nu < 1$ and $1 < \mu < 2$

$$\left\|{}^{C}D_{0+}^{\mu}u_n\right\| \leq \frac{K}{\Gamma(3-\mu)}, \qquad \left\|{}^{C}D_{0+}^{\nu}u_n\right\| \leq \frac{K}{\Gamma(2-\nu)} \qquad \text{for } n \in \mathbb{N}. \quad (51)$$

Hence, for a.e. $(t,s) \in [0,1] \times [0,1]$ and all $u_n \in \mathbb{N}$, we have

$$\begin{aligned}0 &\leq G(t,s)\left(f_n\left(s, u_n(s), {}^{C}D_{0+}^{\nu}u_n(s), {}^{C}D_{0+}^{\mu}u_n(s)\right)\right. \\ &\quad \left. + g\left(s, u_n(s), {}^{C}D_{0+}^{\nu}u_n(s), {}^{C}D_{0+}^{\mu}u_n(s)\right)\right) \\ &\leq \frac{1}{\Gamma(\alpha-1)} \\ &\quad \times \left(\Theta(s) + h\left(K + \frac{1}{n}, \frac{K}{\Gamma(2-\nu)} + \frac{1}{n}, \right.\right. \\ &\quad \left.\left. \frac{K}{\Gamma(3-\mu)} + \frac{1}{n}\right)\gamma(s)\right),\end{aligned} \quad (52)$$

where Θ is defined by (45). Putting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have, for $t \in [0,1]$,

$$\begin{aligned}u(t) &= \int_0^1 G(t,s) \\ &\quad \times \left(f\left(s, u(s), {}^{C}D_{0+}^{\nu}u(s), {}^{C}D_{0+}^{\mu}u(s)\right)\right. \\ &\quad \left. + g\left(s, u(s), {}^{C}D_{0+}^{\nu}u(s), {}^{C}D_{0+}^{\mu}u(s)\right)\right) ds.\end{aligned} \quad (53)$$

Consequently, u is a positive solution of problems (4), (5) and satisfies inequality (49). The proof is complete. □

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