Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 982925, 13 pages
doi:10.1155/2012/982925

Research Article

Semilocal Convergence Analysis for Inexact Newton Method under Weak Condition

Xiubin Xu, Yuan Xiao, and Tao Liu

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Correspondence should be addressed to Xiubin Xu, xxu@zjnu.cn

Received 29 May 2012; Accepted 5 August 2012

Academic Editor: Jen-Chih Yao

Under the hypothesis that the first derivative satisfies some kind of weak Lipschitz conditions, a new semilocal convergence theorem for inexact Newton method is presented. Unified convergence criteria ensuring the convergence of inexact Newton method are also established. Applications to some special cases such as the Kantorovich type conditions and γ-conditions are provided and some well-known convergence theorems for Newton's method are obtained as corollaries.

1. Introduction

Let F be a continuously Fréchet differentiable nonlinear operator from a convex subset D of Banach space X to Banach space Y. Finding solutions of a nonlinear operator equation:

$$F(x) = 0 \tag{1.1}$$

in Banach space is a basic and important problem in applied and computational mathematics. A classical method for finding an approximation of a solution of (1.1) is Newton's method which is defined by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad x_0 \in D, \; n = 0, 1, 2, \ldots. \tag{1.2}$$

There is a huge literature on local as well as semilocal convergence for Newton's method under various assumptions (see [1–9]). Besides, there are a lot of works on the weakness of the hypotheses made on the underlying operators, see for example [2, 3, 5–9] and references therein. In particular, Wang in [7, 8] introduced the notions of Lipschitz conditions with L average, under which Kantorovich like convergence criteria and Smale's point estimate theory can be put together to be investigated.

However, Newton's method has two disadvantages. One is to evaluate F' involved, the other is to solve the exact solution of Newton equations:

$$F'(x_n)(x_{n+1} - x_n) = -F(x_n), \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

In many applications, for example, those in Euclidean spaces, computing the exact solutions using a direct method such as Gaussian elimination can be expensive if the number of unknowns is large and may not be justified when x_k is far from the searched solution. While using linear iterative methods to approximate the solutions of (1.3) instead of solving it exactly can reduce some of the costs of Newton's method. One of the methods is inexact Newton method which can be found in [10] and takes the following form:

$$x_{n+1} = x_n + s_n, \qquad F'(x_n)s_n = -F(x_n) + r_n, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where $\{r_n\}$ is a sequence in Y.

As is well known, the convergence behavior of the inexact Newton method depends on the residual controls of $\{r_n\}$ under the hypothesis that F' satisfies different conditions. Some relative results can be found in [10–24], for example.

Under the Lipschitz continuity assumption on F', different residual controls were used. For example, the residual controls $\|r_n\| \le \eta_n \|F(x_n)\|$ were adopted in [10, 12]; in [15] the affine invariant conditions $\|F'(x_0)^{-1} r_n\| \le \eta_n \|F'(x_0)^{-1} F(x_n)\|$ were considered; while in [21] Shen has analyzed the semilocal convergence behavior in some manner such that the relative residuals $\{r_n\}$ satisfy

$$\left\| F'(x_0)^{-1} r_n \right\| \le \eta_n \left\| F'(x_0)^{-1} F(x_n) \right\|^{1+\kappa}, \quad 0 \le \kappa \le 1, \; n = 0, 1, 2, \ldots. \tag{1.5}$$

Assuming that the residuals satisfy $\|P_n r_n\| \le \theta_n \|P_n F(x_n)\|^{1+\kappa}$, where $\{P_n\}$ is a sequence of invertible operators from Y to X, and that $F'(x_0)^{-1}F'$ satisfies the Hölder condition around x_0, Li and Shen established the local and semilocal convergence in [16, 20], respectively. Besides, the γ-condition was also introduced into inexact Newton method in [22] by considering residual controls (1.5) with $\kappa = 1$, that is,

$$\left\| F'(x_0)^{-1} r_n \right\| \le \eta_n \left\| F'(x_0)^{-1} F(x_n) \right\|^{2}, \quad n = 0, 1, 2, \ldots; \tag{1.6}$$

Smale's α-theory for the inexact Newton method was established there.

In the present paper, by considering the residual controls (1.6), we will study the convergence of inexact Newton method under the assumption that F has a continuous derivative in a closed ball $\overline{B(x_0, r)}$, $F'(x_0)^{-1}F'$ exists and $F'(x_0)^{-1}F'$ satisfies the weak Lipschitz condition:

$$\left\| F'(x_0)^{-1} \left(F'(x) - F'(x') \right) \right\| \le \int_{\rho(x)}^{\rho(\overline{xx'})} L(u)\,du, \quad \forall x \in B(x_0, r), \; \forall x' \in \overline{B(x, r - \rho(x))}, \tag{1.7}$$

where r is a positive number, $\rho(x) = \|x - x_0\|$, $\rho(\overline{xx'}) = \rho(x) + \|x' - x\| \le r$, and L is a positive integrable nondecreasing function on $[0, r]$. We also establish the unified convergence criteria, which include Kantorovich type and Smale type convergence criteria as special cases. In particular, in the special case when $\eta_n = 0$ $(n = 0, 1, 2, \ldots)$, (1.4) reduces to Newton's method and our result extends the corresponding one in [7].

The paper is organized as follows. Section 2 gives some lemmas which are used in the proof of our main theorem. In Section 3, the semilocal convergence of inexact Newton method is studied under the weak Lipschitz condition (1.7). Its applications to some special cases are provided in Section 4.

2. Preliminaries

Let X and Y be Banach spaces. Throughout this paper, $R > r$ are two positive numbers, L is a positive integrable nondecreasing function on any involved intervals, and $B(x, R)$ is an open ball in X with center x and radius R. Let $\beta > 0$, $0 \le \lambda < 1$, $\omega \ge 1$, and $\sigma \ge 0$. Define

$$\begin{aligned} \varphi(t) &= \beta - (1-\lambda)t + \sigma t^2 + \omega \int_0^t L(u)(t-u)\,du, \quad 0 \le t \le R, \\ \psi(t) &= \beta - t + \omega \int_0^t L(u)(t-u)\,du, \quad 0 \le t \le R. \end{aligned} \tag{2.1}$$

Obviously,

$$\varphi'(t) = -(1-\lambda) + 2\sigma t + \omega \int_0^t L(u)\,du, \quad 0 \le t \le R, \tag{2.2}$$

$$\psi'(t) = -1 + \omega \int_0^t L(u)\,du, \quad 0 \le t \le R, \tag{2.3}$$

$$\varphi''(t) = 2\sigma + \omega L(t) > 0, \quad 0 \le t \le R. \tag{2.4}$$

Set

$$r_\lambda := \sup\left\{ r \in (0, R) : \omega \int_0^r L(u)\,du + 2\sigma r \le 1 - \lambda \right\}, \tag{2.5}$$

$$b_\lambda := (1-\lambda) r_\lambda - \sigma r_\lambda^2 - \omega \int_0^{r_\lambda} L(u)(r_\lambda - u)\,du. \tag{2.6}$$

Write $\delta = \omega \int_0^R L(u)du + 2\sigma R$. Then

$$r_\lambda = \begin{cases} R, & \text{if } \delta < 1 - \lambda, \\ r'_\lambda, & \text{if } \delta \ge 1 - \lambda, \end{cases} \tag{2.7}$$

where $r'_\lambda \in [0, R]$ is such that $\omega \int_0^{r'_\lambda} L(u)du + 2\sigma r'_\lambda = 1 - \lambda$. Furthermore, it follows that

$$\begin{aligned} b_\lambda &\geq \omega \int_0^{r_\lambda} L(u)u\,du + \sigma r_\lambda^2, \quad \text{if } \delta < 1-\lambda, \\ b_\lambda &= \omega \int_0^{r_\lambda} L(u)u\,du + \sigma r_\lambda^2, \quad \text{if } \delta \geq 1-\lambda. \end{aligned} \tag{2.8}$$

Let

$$t_0 = 0, \qquad t_{n+1} = t_n - \frac{\varphi(t_n)}{\varphi'(t_n)}, \quad n = 0, 1, 2, \ldots. \tag{2.9}$$

The following two lemmas describe some properties about the majorizing function φ and the convergence property of $\{t_n\}$.

Lemma 2.1. *Suppose that $\beta \leq b_\lambda$ and φ is defined by* (2.1)*. Then the function φ is strictly decreasing and has exact one zero t^* on $[0, r_\lambda]$ satisfying $\beta < t^*$.*

Proof. By (2.4) and (2.5), we know φ' is strictly increasing on $[0, r_\lambda]$ and has the values $\varphi'(0) < 0$ and $\varphi'(r_\lambda) \leq 0$. This implies that φ is strictly decreasing on $[0, r_\lambda]$. Note that $\varphi(0) = \beta > 0$ and $\varphi(r_\lambda) \leq 0$ by the definition of b_λ. Thus, $\varphi(t) = 0$ has exact one solution t^* on $[0, r_\lambda]$. Since

$$\varphi(\beta) = \lambda\beta + \sigma\beta^2 + \omega \int_0^\beta L(u)(\beta - u)du > 0, \tag{2.10}$$

we have $\beta < t^*$. The proof is complete. □

Lemma 2.2. *Let t^* be the positive solution of equation $\varphi(t) = 0$ on $[0, r_\lambda]$. Suppose that $\beta \leq b_\lambda$ and the sequence $\{t_n\}$ is defined by* (2.9)*. Then*

$$t_n < t_{n+1} < t^*, \quad n = 0, 1, 2, \ldots. \tag{2.11}$$

Consequently, $\{t_n\}$ is strictly increasing and converges to t^*.

Proof. We prove the lemma by mathematical induction. Note that $0 = t_0 < t_1 = \beta < t^*$. For $n > 1$, assume that

$$t_{n-1} < t_n < t^*. \tag{2.12}$$

Since $\varphi''(t) = \omega L(t) > 0$, $-\varphi'$ is strictly decreasing on $[0, r_\lambda]$. Hence,

$$-\varphi'(t_n) > -\varphi'(t^*) \geq -\varphi'(r_\lambda) = -\varphi'(r_\lambda) + \lambda + 2\sigma r_\lambda \geq 0. \tag{2.13}$$

Moreover, $\varphi(t_n) > 0$ by of Lemma 2.1. It follows that

$$t_{n+1} = t_n - \frac{\varphi(t_n)}{\varphi'(t_n)} > t_n. \tag{2.14}$$

Define a function $N(t)$ on $[0, t^*]$ by

$$N(t) := t - \frac{\varphi(t)}{\varphi'(t)}, \quad t \in [0, t^*]. \tag{2.15}$$

Note that $\varphi'(t) < 0$, $t \in [0, t^*]$, unless $\lambda = 0$, $\sigma = 0$ and $t = t^* = r_\lambda$, for which we adopt the convention that $\lim_{t\to t^*}(\varphi(t)/\varphi'(t)) = 0$ and $N(t^*) = t^* - \lim_{t\to t^*}(\varphi(t)/\varphi'(t)) = t^*$. Hence, the function $N(t)$ is well defined and continuous on $[0, t^*]$.

Moreover, by (2.2) and (2.3), we have

$$N'(t) = 1 - \frac{\varphi'(t)\varphi'(t) - \varphi(t)\varphi''(t)}{\left(\varphi'(t)\right)^2} = \frac{-\varphi'(t)(\lambda + 2\sigma t) + \varphi(t)\varphi''(t)}{\left(\varphi'(t)\right)^2} > 0, \quad t \in [0, t^*). \tag{2.16}$$

Hence, $N(t)$ is monotonically increasing on $[0, t^*)$. This together with (2.9) and (2.14) implies that

$$t_n < t_{n+1} = N(t_n) < N(t^*) = t^*. \tag{2.17}$$

Therefore, by mathematical induction, (2.11) holds. Consequently, $\{t_n\}$ is increasing, bounded, and converges to a point t_λ^*, which satisfies $\varphi(t_\lambda^*) = 0$. Hence, $t^* = t_\lambda^*$. The proof is complete. □

To prove our main result, we need two more lemmas. The first can be found in [23] and the second in [7].

Lemma 2.3. *Suppose that F has a continuous derivative satisfying the weak Lipschitz condition* (1.7). *Let r satisfy $\int_0^r L(u)du \le 1$. Then $F'(x)$ is invertible in the ball $B(x_0, r)$ and*

$$\left\| F'(x)^{-1} F'(x_0) \right\| \le \left(1 - \int_0^{\rho(x)} L(u)du\right)^{-1}. \tag{2.18}$$

Lemma 2.4. *Let $0 \le c < R$ and define*

$$\chi(t) = \frac{1}{t^2}\int_0^t L(c+u)(t-u)du, \quad 0 \le t < R - c. \tag{2.19}$$

Then, χ is increasing on $[0, R-c)$.

3. Semilocal Convergence Analysis

Recall that $F : D \subseteq X \to Y$ is a nonlinear operator with continuous Fréchet derivative. Let $B(x_0, R) \subseteq D$ and $x_0 \in D$ be such that $F'(x_0)^{-1}$ exists. In the present paper, we adopt the residuals $\{r_n\}$ satisfying (1.6) and assume that $\eta = \sup_{n\geq 0}\eta_n < 1$. Thus, if $n \geq 0$ and $\{x_n\}$ is well defined, then

$$\left\| F'(x_0)^{-1} r_n \right\| \leq \eta_n \left\| F'(x_0)^{-1} F(x_n) \right\|^2 \leq \eta \left\| F'(x_0)^{-1} F(x_n) \right\|^2. \tag{3.1}$$

Let

$$\alpha = \left\| F'(x_0)^{-1} F(x_0) \right\|, \qquad \beta = (1+\sqrt{\eta})\alpha. \tag{3.2}$$

Write

$$\omega = 1+\sqrt{\eta}, \qquad \sigma = \frac{\eta(1+\sqrt{\eta})\left(1+\int_0^R L(u)du\right)^2}{(1-\sqrt{\eta})^2}. \tag{3.3}$$

Recall that r_λ is determined by (2.5), $\varphi(t^*) = 0$, and $\{t_n\}$ is generated by (2.9) with ω and σ given in (3.3).

Lemma 3.1. *Let $\{x_n\}$ be a sequence generated by* (1.4)*. Suppose that F satisfies the weak Lipschitz condition* (1.7) *on $B(x_0, t^*) \subseteq B(x_0, R)$ and that $\beta \leq b_\lambda$. For an integer $m \geq 1$, if*

$$\sqrt{\eta}\left\| F'(x_0)^{-1} F(x_{n-1}) \right\| \leq 1, \qquad \|x_n - x_{n-1}\| \leq t_n - t_{n-1} \tag{3.4}$$

hold for each $1 \leq n \leq m$, then the following assertions hold:

$$\begin{gathered}(1+\sqrt{\eta})\left\| F'(x_0)^{-1} F(x_m) \right\| \leq \varphi(t_m); \\ \sqrt{\eta}\left\| F'(x_0)^{-1} F(x_m) \right\| \leq 1.\end{gathered} \tag{3.5}$$

Proof. Assume that (3.4) holds for each $1 \leq n \leq m$. Write $x_{m-1}^{\tau} = x_{m-1} + \tau(x_m - x_{m-1})$, $\tau \in [0,1]$. Applying (1.4), we have

$$\begin{aligned}F(x_m) &= F(x_m) - F(x_{m-1}) - F'(x_{m-1})(x_m - x_{m-1}) + r_{m-1} \\ &= \int_0^1 \left[F'(x_{m-1}^{\tau}) - F'(x_{m-1})\right] d\tau (x_m - x_{m-1}) + r_{m-1}.\end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned}\left\| F'(x_0)^{-1} F(x_m) \right\| &\leq \left\| F'(x_0)^{-1} \int_0^1 \left[F'(x_{m-1}^{\tau}) - F'(x_{m-1})\right] d\tau (x_m - x_{m-1}) \right\| \\ &\quad + \left\| F'(x_0)^{-1} r_{m-1} \right\| \\ &= I_1 + I_2.\end{aligned} \tag{3.7}$$

To estimate I_1, by (3.4), we notice that

$$\begin{aligned} \left\|x_{m-1}^{\tau}-x_0\right\| &= \|x_{m-1}+\tau(x_m-x_{m-1})-x_0\| \\ &\le \sum_{n=1}^{m-1}\|x_n-x_{n-1}\|+\tau\|x_m-x_{m-1}\| \\ &\le t_{m-1}+\tau(t_m-t_{m-1}) \\ &= (1-\tau)t_{m-1}+\tau t_m < t^*. \end{aligned} \tag{3.8}$$

In particular,

$$\|x_{m-1}-x_0\| \le t_{m-1} < t^*, \qquad \|x_m-x_0\| \le t_m < t^*. \tag{3.9}$$

Thus, by the weak Lipschitz condition (1.7), we obtain

$$I_1 \le \int_0^{\|x_m-x_{m-1}\|}(\|x_m-x_{m-1}\|-u)L(\|x_{m-1}-x_0\|+u)du. \tag{3.10}$$

Below we estimate I_2. We firstly notice that (3.1) and (3.4) yield

$$\begin{aligned} \left\|F'(x_0)^{-1}F'(x_{m-1})(x_m-x_{m-1})\right\| &\ge \left\|F'(x_0)^{-1}F(x_{m-1})\right\|-\left\|F'(x_0)^{-1}r_{m-1}\right\| \\ &\ge \left\|F'(x_0)^{-1}F(x_{m-1})\right\|-\eta\left\|F'(x_0)^{-1}F(x_{m-1})\right\|^2 \\ &\ge (1-\sqrt{\eta})\left\|F'(x_0)^{-1}F(x_{m-1})\right\|. \end{aligned} \tag{3.11}$$

Since

$$\begin{aligned} \left\|F'(x_0)^{-1}F'(x_{m-1})\right\| &= \left\|I+F'(x_0)^{-1}\left[F'(x_{m-1})-F'(x_0)\right]\right\| \\ &\le 1+\int_0^{\rho(x_{m-1})}L(u)du \\ &\le 1+\int_0^{R}L(u)du, \end{aligned} \tag{3.12}$$

we have

$$\left\|F'(x_0)^{-1}F(x_{m-1})\right\| \le \frac{\left\|F'(x_0)^{-1}F'(x_{m-1})\right\|\|x_m-x_{m-1}\|}{1-\sqrt{\eta}} \le \frac{1+\int_0^R L(u)du}{1-\sqrt{\eta}}\|x_m-x_{m-1}\|. \tag{3.13}$$

Combining this with (3.1) implies that

$$I_2 \le \eta\left\|F'(x_0)^{-1}F(x_{m-1})\right\|^2 \le \frac{\eta\left(1+\int_0^R L(u)du\right)^2}{(1-\sqrt{\eta})^2}\|x_m-x_{m-1}\|^2. \tag{3.14}$$

Consequently, by (3.7), (3.10), (3.14) and Lemma 2.4, we get

$$\begin{aligned}
(1+\sqrt{\eta})\left\|F'(x_0)^{-1}F(x_m)\right\| &\le (1+\sqrt{\eta})(I_1+I_2)\\
&\le (1+\sqrt{\eta})\int_0^{\|x_m-x_{m-1}\|}(\|x_m-x_{m-1}\|-u)L(\|x_{m-1}-x_0\|+u)du\\
&\quad+\frac{\eta(1+\sqrt{\eta})\left(1+\int_0^R L(u)du\right)^2}{(1-\sqrt{\eta})^2}\|x_m-x_{m-1}\|^2\\
&=\omega\int_0^{\|x_m-x_{m-1}\|}(\|x_m-x_{m-1}\|-u)L(\|x_{m-1}-x_0\|+u)du\\
&\quad+\sigma\|x_m-x_{m-1}\|^2\\
&=\left(\frac{\omega}{\|x_m-x_{m-1}\|^2}\int_0^{\|x_m-x_{m-1}\|}(\|x_m-x_{m-1}\|-u)\right.\\
&\quad\left.\times L(\|x_{m-1}-x_0\|+u)du+\sigma\right)\|x_m-x_{m-1}\|^2\\
&\le\left(\frac{\omega}{(t_m-t_{m-1})^2}\int_0^{t_m-t_{m-1}}(t_m-t_{m-1}-u)L(t_{m-1}+u)du+\sigma\right)\\
&\quad\times(t_m-t_{m-1})^2\\
&=\omega\int_0^{t_m-t_{m-1}}(t_m-t_{m-1}-u)L(t_{m-1}+u)du\\
&\quad+\sigma\left[t_m^2-t_{m-1}^2-2t_{m-1}(t_m-t_{m-1})\right]\\
&=\varphi(t_m)-\varphi(t_{m-1})-\varphi'(t_{m-1})(t_m-t_{m-1}).
\end{aligned}\tag{3.15}$$

Noting that $\varphi'(t)=\psi'(t)+\lambda+2\sigma t$ and $-\varphi(t_{m-1})-\psi'(t_{m-1})(t_m-t_{m-1})=0$, we have

$$\begin{aligned}
(1+\sqrt{\eta})\left\|F'(x_0)^{-1}F(x_m)\right\| &\le \varphi(t_m)-\varphi(t_{m-1})-\varphi'(t_{m-1})(t_m-t_{m-1})\\
&=\varphi(t_m)-(\lambda+2\sigma t_{m-1})(t_m-t_{m-1})\\
&\le\varphi(t_m).
\end{aligned}\tag{3.16}$$

Moreover, since φ is decreasing on $[0,t^*]$, one has

$$(1+\sqrt{\eta})\left\|F'(x_0)^{-1}F(x_m)\right\|\le\varphi(t_m)\le\varphi(t_0)=\beta.\tag{3.17}$$

And therefore

$$\sqrt{\eta}\left\|F'(x_0)^{-1}F(x_m)\right\| \le \frac{\sqrt{\eta}}{1+\sqrt{\eta}}\beta = \sqrt{\eta}\left\|F'(x_0)^{-1}F(x_0)\right\| \le 1. \tag{3.18}$$

That is, (3.5) holds, and the proof is complete. □

We now give the main result.

Theorem 3.2. *Suppose that* $\beta \le \min\{1/\sqrt{\jmath}, \mathrm{b}\breve{}\}$ *and* $\overline{B(x_0, t^*)} \subseteq B(x_0, R)$*, and that* $F'(x_0)^{-1}F'$ *satisfies the weak Lipschitz condition* (1.7) *on* $B(x_0, t^*)$*. Then the sequence* $\{x_n\}$ *generated by the inexact Newton method* (1.4) *converges to a solution* x^* *of* (1.1)*. Moreover,*

$$\|x_n - x^*\| \le t^* - t_n, \quad n = 0, 1, 2, \ldots. \tag{3.19}$$

Proof. We firstly use mathematical induction to prove that (3.4) holds for each $n = 1, 2, \ldots$. For $n = 1$, by the above condition and (3.2), the first inequality in (3.4) holds trivially. While the second one can be proved as follows:

$$\begin{aligned} \|x_1 - x_0\| &\le \left\|F'(x_0)^{-1}F(x_0)\right\| + \left\|F'(x_0)^{-1}r_0\right\| \\ &\le \alpha + \eta\alpha^2 \le \alpha + \sqrt{\eta}\alpha = \left(1+\sqrt{\eta}\right)\alpha = \beta = t_1 - t_0. \end{aligned} \tag{3.20}$$

Assume that (3.4) holds for all $n \le m$. Then, Lemma 3.1 is applicable to concluding that

$$\begin{aligned} \left(1+\sqrt{\eta}\right)\left\|F'(x_0)^{-1}F(x_m)\right\| &\le \varphi(t_m); \\ \sqrt{\eta}\left\|F'(x_0)^{-1}F(x_m)\right\| &\le 1. \end{aligned} \tag{3.21}$$

Hence, by (3.5), together with the weak Lipschitz condition (1.7) and Lemma 2.3, one has

$$\begin{aligned} \|x_{m+1} - x_m\| &\le \left\|F'(x_m)^{-1}F'(x_0)\right\|\left(\left\|F'(x_0)^{-1}F(x_m)\right\| + \left\|F'(x_0)^{-1}r_m\right\|\right) \\ &\le \frac{1}{1-\int_0^{\rho(x_m)} L(u)\,du}\left(\left\|F'(x_0)^{-1}F(x_m)\right\| + \eta\left\|F'(x_m)^{-1}F(x_m)\right\|^2\right) \\ &\le \frac{1+\sqrt{\eta}}{1-\omega\int_0^{\rho(x_m)} L(u)\,du}\left\|F'(x_0)^{-1}F(x_m)\right\| \\ &\le -\frac{\varphi(t_m)}{\varphi'(t_m)} = t_{m+1} - t_m. \end{aligned} \tag{3.22}$$

Therefore, (3.4) holds for $n = m+1$ and so for each $n \ge 1$. Consequently, for $n \ge 0$ and $k \ge 0$,

$$\|x_{k+n} - x_n\| \le \sum_{i=1}^{k}\|x_{i+n} - x_{i+n-1}\| \le \sum_{i=1}^{k}(t_{i+n} - t_{i+n-1}) = t_{k+n} - t_n. \tag{3.23}$$

This together with Lemma 2.2 means that $\{x_n\}$ is a Cauchy sequence and so converges to some x^*. While taking $k \to \infty$ in (3.23), we obtain

$$\|x_n - x^*\| \le t^* - t_n, \quad n = 0, 1, 2, \ldots. \tag{3.24}$$

The proof is complete. □

In the special case when $\eta_n = 0$ $(n = 0, 1, 2, \ldots)$, inexact Newton method (1.4) reduces to Newton's method. Moreover, $\omega = 1$, $\sigma = 0$, $\beta = \|F'(x_0)^{-1}F(x_0)\|$. Thus, Theorem 3.2 reduces to the related theorem of Newton's method.

Corollary 3.3. *Assume that $\beta \le b_\lambda$ and $\overline{B(x_0, t^*)} \subseteq B(x_0, R)$, where $b_\lambda = \int_0^{r_\lambda} L(u)u\,du$ and r_λ satisfying $\int_0^{r_\lambda} L(u)du \le 1 - \lambda$. Suppose that $F'(x_0)^{-1}F'$ satisfies the weak Lipschitz condition* (1.7) *on $B(x_0, t^*)$. Then the sequence $\{x_n\}$ generated by Newton's method* (1.2) *converges to a solution x^* of* (1.1). *Moreover,*

$$\|x_n - x^*\| \le t^* - t_n, \quad n = 0, 1, 2, \ldots, \tag{3.25}$$

where t^* and $\{t_n\}$ are defined in Lemma 2.2 for $\eta = 0$.

In more particular, suppose that $\int_0^R L(u)du > 1$ and $\lambda = 0$. Then Corollary 3.3 reduces to the following result given in (Theorem 3.1, [7]).

Corollary 3.4. *Assume that $\beta \le b_{\lambda_0}$, where $b_{\lambda_0} = \int_0^{r_{\lambda_0}} L(u)u\,du$ and $\int_0^{r_{\lambda_0}} L(u)du = 1$. Suppose that $F'(x_0)^{-1}F'$ satisfies weak Lipschitz condition* (1.7) *on $B(x_0, t^*) \subseteq B(x_0, R)$. Then the sequence $\{x_n\}$ generated by Newton's method* (1.2) *converges to a solution x^* of* (1.1). *Moreover,*

$$\|x_n - x^*\| \le t^* - t_n, \quad n = 0, 1, 2, \ldots, \tag{3.26}$$

where t^* and $\{t_n\}$ are defined in Lemma 2.2 for $\eta = 0$ and $\lambda = 0$.

4. Application

This section is divided into two subsections: we consider the applications of our main results specializing, respectively, in Kantorovich type condition and in γ-condition. In particular, our results reduce some of the corresponding results of Newton's method.

4.1. Kantorovich-Type Condition

Throughout this subsection, let L be a positive constant. By (2.1), we have

$$\begin{gathered} \varphi(t) = \beta - (1-\lambda)t + \left(\sigma + \frac{1}{2}\omega L\right)t^2, \quad t \ge 0, \\ \varphi(t) = \beta - t + \frac{1}{2}\omega L t^2, \quad t \ge 0. \end{gathered} \tag{4.1}$$

By (2.5) and (2.6), we get

$$r_\lambda = \frac{1-\lambda}{\omega L+\sigma}, \qquad b_\lambda = \frac{(1-\lambda)^2\omega L}{2(\omega L+\sigma)}. \tag{4.2}$$

The convergence criterion becomes

$$\left\|F'(x_0)^{-1}F(x_0)\right\| \le \frac{(1-\lambda)^2\omega L}{2(\omega L+\sigma)}. \tag{4.3}$$

Moreover, suppose that $\eta = 0$ and $\lambda = 0$. Then criterion (4.3) reduces to the well-known Kantorovich type criterion $\|F'(x_0)^{-1}F(x_0)\| \le 1/2L$ of Newton's method in [7].

Corollary 4.1. *Let L be a positive constant, $\beta = \|F'(x_0)^{-1}F(x_0)\|$ and $\beta \le b_{\lambda_0}$, where $b_{\lambda_0} = 1/2L$ and $r_{\lambda_0} = 1/L$. Assume that F satisfies the condition:*

$$\left\|F'(x_0)^{-1}\left(F'(x) - F'(x')\right)\right\| \le L\left\|x-x'\right\|, \quad \forall x, x' \in B(x_0, r),\ \|x-x_0\| + \left\|x-x'\right\| \le r, \tag{4.4}$$

where $r = (1-\sqrt{1-2L\beta})/L$. Then the sequence $\{x_n\}$ generated by Newton's method (1.2) *converges to a solution x^* of* (1.1)*, and satisfies*

$$\|x_n - x^*\| \le t^* - t_n, \quad n = 0, 1, 2, \ldots. \tag{4.5}$$

4.2. *γ-Condition*

Throughout this subsection, we assume that $\gamma > 0$ and F has continuous second derivative and satisfies

$$\left\|F'(x_0)^{-1}F''(x)\right\| \le \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3}, \quad \forall x \in B\left(x_0, \frac{1}{\gamma}\right). \tag{4.6}$$

Let

$$L(u) = \frac{2\gamma}{(1-\gamma u)^3}, \quad u \in \left[0, \frac{1}{\gamma}\right). \tag{4.7}$$

Then, by (2.1), we have

$$\begin{aligned}\varphi(t) &= \beta - (1-\lambda)t + \sigma t^2 + \frac{\gamma t^2}{1-\gamma t}, \quad 0 \le t < \frac{1}{\gamma},\\ \psi(t) &= \beta - t + \frac{\gamma t^2}{1-\gamma t}, \quad 0 \le t < \frac{1}{\gamma}.\end{aligned} \tag{4.8}$$

By (2.5) and (2.6), r_λ and b_λ satisfy

$$\omega\left[\frac{1}{(1-\gamma r_\lambda)^2} - 1\right] + \sigma r_\lambda = 1 - \lambda, \qquad b_\lambda = \frac{\gamma r_\lambda^2}{(1-\gamma r_\lambda)^2}. \tag{4.9}$$

The convergence criterion becomes

$$\left\| F'(x_0)^{-1} F(x_0) \right\| \le \frac{\gamma r_\lambda^2}{(1-\gamma r_\lambda)^2}. \tag{4.10}$$

In the more special case, when $\eta = 0$ and $\lambda = 0$, we obtain the criterion $\|F'(x_0)^{-1}F(x_0)\| \le (3-2\sqrt{2})/\gamma$ the same with Newton's method in [7].

Corollary 4.2. *Let γ be a positive constant, $\beta = \|F'(x_0)^{-1}F(x_0)\|$ and $\beta \le b_{\lambda_0}$, where $b_{\lambda_0} = (3-2\sqrt{2})/\gamma$ and $r_{\lambda_0} = (1-(1/\sqrt{2}))(1/\gamma)$. Assume that F satisfies the condition:*

$$\begin{gathered}\left\| F'(x_0)^{-1}\left(F'(x) - F'(x')\right) \right\| \le \frac{1}{(1-\gamma\|x-x_0\| - \gamma\|x'-x_0\|)^2} - \frac{1}{(1-\gamma\|x-x_0\|)^2},\\ \forall x, x' \in B(x_0, r), \quad \|x-x_0\| + \left\|x'-x\right\| \le r,\end{gathered} \tag{4.11}$$

where $r = (1+\beta\gamma - \sqrt{(1+\beta\gamma)^2 - 8\beta\gamma})/4\gamma$. Then the sequence $\{x_n\}$ generated by Newton's method (1.2) *converges to a solution x^* of* (1.1), *and satisfies*

$$\|x_n - x^*\| \le t^* - t_n, \quad n = 0, 1, 2, \ldots. \tag{4.12}$$

Acknowledgment

Supported in part by the National Natural Science Foundation of China (Grants no. 61170109 and no. 10971194) and Zhejiang Innovation Project (Grant no. T200905).

References

[1] I. K. Argyros, "On the Newton-Kantorovich hypothesis for solving equations," *Journal of Computational and Applied Mathematics*, vol. 169, no. 2, pp. 315–332, 2004.

[2] J. A. Ezquerro and M. A. Hernández, "Generalized differentiability conditions for Newton's method," *IMA Journal of Numerical Analysis*, vol. 22, no. 2, pp. 187–205, 2002.

[3] J. A. Ezquerro and M. A. Hernández, “On an application of Newton’s method to nonlinear operators with w-conditioned second derivative,” *BIT. Numerical Mathematics*, vol. 42, no. 3, pp. 519–530, 2002.
[4] J. M. Gutiérrez, “A new semilocal convergence theorem for Newton’s method,” *Journal of Computational and Applied Mathematics*, vol. 79, no. 1, pp. 131–145, 1997.
[5] J. M. Gutiérrez and M. A. Hernández, “Newton’s method under weak Kantorovich conditions,” *IMA Journal of Numerical Analysis*, vol. 20, no. 4, pp. 521–532, 2000.
[6] M. A. Hernández, “The Newton method for operators with Hölder continuous first derivative,” *Journal of Optimization Theory and Applications*, vol. 109, no. 3, pp. 631–648, 2001.
[7] X. H. Wang, “Convergence of Newton’s method and inverse function theorem in Banach space,” *Mathematics of Computation*, vol. 68, no. 225, pp. 169–186, 1999.
[8] X. H. Wang, “Convergence of Newton’s method and uniqueness of the solution of equations in Banach space,” *IMA Journal of Numerical Analysis*, vol. 20, no. 1, pp. 123–134, 2000.
[9] X. H. Wang and C. Li, “Convergence of Newton’s method and uniqueness of the solution of equations in Banach spaces. II,” *Acta Mathematica Sinica, English Series*, vol. 19, no. 2, pp. 405–412, 2003.
[10] R. S. Dembo, S. C. Eisenstat, and T. Steihaug, “Inexact Newton methods,” *SIAM Journal on Numerical Analysis*, vol. 19, no. 2, pp. 400–408, 1982.
[11] I. K. Argyros, “A new convergence theorem for the inexact Newton methods based on assumptions involving the second Fréchet derivative,” *Computers & Mathematics with Applications*, vol. 37, no. 7, pp. 109–115, 1999.
[12] Z. Z. Bai and P. L. Tong, “Affine invariant convergence of the inexact Newton method and Broyden’s method,” *Journal of University of Electronic Science and Technology of China*, vol. 23, no. 5, pp. 535–540, 1994.
[13] J. Chen and W. Li, “Convergence behaviour of inexact Newton methods under weak Lipschitz condition,” *Journal of Computational and Applied Mathematics*, vol. 191, no. 1, pp. 143–164, 2006.
[14] M. G. Gasparo and G. Morini, “Inexact methods: forcing terms and conditioning,” *Journal of Optimization Theory and Applications*, vol. 107, no. 3, pp. 573–589, 2000.
[15] X. P. Guo, “On semilocal convergence of inexact Newton methods,” *Journal of Computational Mathematics*, vol. 25, no. 2, pp. 231–242, 2007.
[16] C. Li and W. P. Shen, “Local convergence of inexact methods under the Hölder condition,” *Journal of Computational and Applied Mathematics*, vol. 222, no. 2, pp. 544–560, 2008.
[17] I. Moret, “A Kantorovich-type theorem for inexact Newton methods,” *Numerical Functional Analysis and Optimization*, vol. 10, no. 3-4, pp. 351–365, 1989.
[18] J. M. Martínez and L. Q. Qi, “Inexact Newton methods for solving nonsmooth equations,” *Journal of Computational and Applied Mathematics*, vol. 60, no. 1-2, pp. 127–145, 1995.
[19] B. Morini, “Convergence behaviour of inexact Newton methods,” *Mathematics of Computation*, vol. 68, no. 228, pp. 1605–1613, 1999.
[20] W. P. Shen and C. Li, “Convergence criterion of inexact methods for operators with Hölder continuous derivatives,” *Taiwanese Journal of Mathematics*, vol. 12, no. 7, pp. 1865–1882, 2008.
[21] W. P. Shen and C. Li, “Kantorovich-type convergence criterion for inexact Newton methods,” *Applied Numerical Mathematics*, vol. 59, no. 7, pp. 1599–1611, 2009.
[22] W. P. Shen and C. Li, “Smale’s α-theory for inexact Newton methods under the γ-condition,” *Journal of Mathematical Analysis and Applications*, vol. 369, no. 1, pp. 29–42, 2010.
[23] M. Wu, “A new semi-local convergence theorem for the inexact Newton methods,” *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 80–86, 2008.
[24] T. J. Ypma, “Local convergence of inexact Newton methods,” *SIAM Journal on Numerical Analysis*, vol. 21, no. 3, pp. 583–590, 1984.