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Research Article

Solving Fractional-Order Logistic Equation Using a New Iterative Method

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A fractional version of logistic equation is solved using new iterative method proposed by Daftardar-Gejji and Jafari (2006). Convergence of the series solutions obtained is discussed. The solutions obtained are compared with Adomian decomposition method and homotopy perturbation method.

1. Introduction

The following model describing growth of population was first studied by Pierre Verhulst in 1938 [1]

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{1.1}$$

where $N(t)$ is population at time t, and $r > 0$ is Malthusian parameter describing growth rate and K is carrying capacity. Defining $x = N/K$ gives the following differential equation:

$$\frac{dx}{dt} = rx(1 - x) \tag{1.2}$$

which is called as logistic equation.

Logistic equation of fractional order has been discussed in the literature [2, 3]. El-Sayed et al. [2] have investigated the equation $D^\alpha x(t) = rx(t)(1 - x(t))$, where D^α is Caputo fractional derivative of order $0 < \alpha \leq 1$. Momani and Qaralleh [3] have employed Adomian decomposition method (ADM) for solving fractional population growth model in a closed system.

In the present paper we use New Iterative Method (NIM) introduced by Daftardar-Gejji and Jafari [4] to solve fractional version of logistic equation.

NIM is useful for solving a general functional equation of the form

$$u = f + L(u) + N(u), \tag{1.3}$$

where f is a given function, L and N linear and nonlinear operators, respectively. The NIM has fairly simple algorithm and does not require any knowledge of involved concepts such as Adomian polynomials, homotopy, or Lagrange multipliers. Rigorous convergence analysis of NIM has been worked out recently [5]. This method has been applied by present authors successfully for solving partial differential equations [6], evolution equations [7], and fractional diffusion-wave equations [8].

NIM has been further explored by many researchers. Several numerical methods with higher order convergence can be generated using NIM. M. A. Noor and K. I. Noor [9, 10] have developed a three-step predictor-corrector method for solving nonlinear equation $f(x) = 0$. Further, they have shown that this method has fourth-order convergence [11]. Some new methods [12, 13] are proposed by these authors using NIM. Mohyud-Din et al. [14] solved Hirota-Satsuma coupled KdV system using NIM. These authors [15] also have applied NIM in solutions of some fifth order boundary value problems. Noor and Mohyud-Din [16] have used NIM to solve Helmholtz equations. NIM is applied to solve homogeneous and inhomogeneous advection problems [17], diffusion equations [18], Schrödinger equations [19], time fractional partial differential equations [20], and so on. Yaseen and Samraiz [21] proposed modified NIM and used it to solve Klein-Gordon equations. Srivastava and Rai [22] have proposed a new mathematical model for oxygen delivery through a capillary to tissues in terms of multiterm fractional diffusion equation. They have solved the multi-term fractional diffusion equation using NIM and ADM and have shown that the results are in perfect agreement.

Recently Usman et al. [23] have solved a wide range of physical problems using NIM and various other methods and have shown that NIM has better performance as compared to other methods.

In the present paper, we solve fractional-order logistic equation. We compare the results obtained by NIM, ADM [3], and homotopy perturbation method (HPM) with exact solution. Further, we propose sufficient condition for the convergence of NIM solution of fractional order logistic equation.

2. Preliminaries and Notations

In this section, we set up notation and recall some basic definitions from fractional calculus [24].

Definition 2.1. Riemann-Liouville fractional integration of order α is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} f(y) dy, \quad t > 0. \tag{2.1}$$

Definition 2.2. Caputo fractional derivative of order α is defined as

$$D^{\alpha} f(t) = I^{m-\alpha}\left(\frac{d^m f(t)}{dt^m}\right), \quad 0 \leq m-1 < \alpha \leq m. \tag{2.2}$$

Note that for $0 \leq m-1 < \alpha \leq m,\ a \geq 0$ and $\gamma > -1$

$$\begin{gathered} I^{\alpha}(t-b)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-b)^{\gamma+\alpha}, \\ \left(I^{\alpha} D^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\frac{t^k}{k!}. \end{gathered} \tag{2.3}$$

3. Fractional-Order Logistic Equation

Consider

$$D^{\alpha} x(t) = r x(t)(1-x(t)), \quad t > 0,\ r > 0,\ 0 < \alpha \leq 1 \tag{3.1}$$

with initial condition

$$x(0) = x_0. \tag{3.2}$$

Operating I^{α} on both sides of (3.1) and using (3.2), we get

$$x(t) = x_0 + r I^{\alpha}(x(t)) - r I^{\alpha}\left(x^2(t)\right). \tag{3.3}$$

3.1. Adomian Decomposition Method

Adomian decomposition method [25] is one of the most powerful methods used to solve (1.3). In this method, solution u is assumed to be of the form $\sum_{i=0}^{\infty} u_i$, and the nonlinear operator N is represented as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} A_i, \tag{3.4}$$

where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{n} u_k \lambda^k\right)_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{3.5}$$

are the Adomian polynomials. The terms of the series are given as

$$u_{n+1} = L(u_n) + A_n, \quad n = 0, 1, 2, \ldots. \tag{3.6}$$

For convergence of this series, we refer the reader to [26]. Fractional order logistic equation is solved using ADM in [3].

3.2. *New Iterative Method*

Recently Daftardar-Gejji and Jafari [4, 8] have used a different decomposition of the term $N(u)$. They have decomposed $N(u)$ as follows:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = \underbrace{N(u_0)}_{G_0} + \underbrace{N(u_0+u_1) - N(u_0)}_{G_1} + \underbrace{N(u_0+u_1+u_2) - N(u_0+u_1)}_{G_2} + \cdots . \tag{3.7}$$

The terms of the series $u = \sum_{i=0}^{\infty} u_i$ are determined by the following recurrence relation:

$$\begin{aligned} u_0 &= f, \\ u_1 &= L(u_0) + G_0, \\ u_{n+1} &= L(u_n) + G_n, \quad n = 1, 2, 3, \ldots . \end{aligned} \tag{3.8}$$

Discussion regarding convergence of this series and comparison of ADM and NIM can be found in [5].

In view of NIM, the solution of (3.3) is given by $x(t) = \sum_{i=0}^{\infty} x_i$ where

$$\begin{aligned} x_1 &= rI^{\alpha}(x_0) - rI^{\alpha}\left(x_0^2\right), \\ x_{n+1} &= rI^{\alpha}(x_n) - rI^{\alpha}\left(\sum_{i=0}^{n} x_i\right)^2 + rI^{\alpha}\left(\sum_{i=0}^{n-1} x_i\right)^2, \quad n = 1, 2, \ldots . \end{aligned} \tag{3.9}$$

The k-term approximate solution is given by

$$u_k = \sum_{i=0}^{k-1} x_i. \tag{3.10}$$

3.3. *Homotopy Perturbation Method*

Homotopy perturbation method (HPM) proposed by He [27, 28] is a useful technique to solve nonlinear problems. In view of HPM, we construct the following homotopy for solving fractional order logistic equation (3.1):

$$D^{\alpha} x = prx(1 - x), \tag{3.11}$$

where $p \in [0,1]$ is an embedding parameter. For $p = 0$, the homotopy (3.11) gives a linear equation $D^{\alpha}x = 0$ and for $p = 1$, it gives the original equation (3.1). The solution of homotopy equation (3.11) is assumed to be

$$x(t) = x_0 + px_1 + p^2x_2 + p^3x_3 + \cdots . \tag{3.12}$$

The solution of (3.1) is obtained from series (3.12) by substituting $p = 1$. Substituting (3.12) in (3.11) and equating like powers of p, we get

$$\begin{gathered}
D^{\alpha}x_0 = 0, \\
D^{\alpha}x_1 = r\left(x_0 - x_0^2\right), \\
D^{\alpha}x_2 = r(x_1 - 2x_0x_1), \\
D^{\alpha}x_3 = r\left(x_2 - x_1^2 - 2x_0x_2\right),
\end{gathered} \tag{3.13}$$

and so on. The linear equations given in (3.13) give

$$\begin{gathered}
x_1 = r\left(x_0 - x_0^2\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
x_2 = 0, \\
x_3 = -r^3\left(x_0 - x_0^2\right)^2\frac{t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}, \\
x_4 = 0,
\end{gathered} \tag{3.14}$$

and so forth. It can be seen from (3.11) that the HPM solutions are the same as ADM solutions given in [3].

4. Convergence of NIM Solutions

In this section, we provide sufficient condition for the convergence of NIM solution series (3.10).

Let $f(t,x) = rx(1-x)$ be defined on a rectangle

$$R\colon\ |t| \le a,\ |x - x_0| \le b,\ (a > 0,\ b > 0). \tag{4.1}$$

It is clear that $|f|$ is bounded on R. We have $|f(t,x)| \le M$, for some real number M on R.

Lemma 4.1. *For the initial value problem*

$$D^\alpha x(t) = f(t, x(t)), \quad 0 < \alpha \le 1, \\ x(0) = x_0 \tag{4.2}$$

the m-term approximate NIM solution (3.10) *exists on the interval $I = [-\chi, \chi]$, where $\chi = \min\{a, (\Gamma(\alpha+1)b/M)^{1/\alpha}\}$, on rectangle R. Moreover, the points $(t, u_m(t)) \in R$, for all $m = 1, 2, \ldots$ whenever $t \in I$.*

Proof. We prove the lemma by induction on m.

Clearly, the result is true for $m = 1$. Now,

$$u_2(t) = x_0(t) + x_1(t). \tag{4.3}$$

Since the points $(t, x_0(t)) \in R$, for all $t \in I$,

$$\begin{aligned} |u_2(t) - x_0(t)| &\le \left|I^\alpha f(t, x_0(t))\right| \\ &\le M\frac{|t|^\alpha}{\Gamma(\alpha+1)} \le b. \end{aligned} \tag{4.4}$$

Thus, $(t, u_2(t)) \in R$.

We assume that the result is true for positive integer $m-1$ and proves for m. Using NIM solutions, we get

$$\begin{aligned} |u_m(t) - x_0(t)| &\le \left|I^\alpha f\left(t, \sum_{j=0}^{m-2} x_j(t)\right)\right| \\ &= \left|I^\alpha f(t, u_{m-1}(t))\right|. \end{aligned} \tag{4.5}$$

Induction hypothesis implies that the points $(t, u_{m-1}(t)) \in R$, for all $t \in I$. Hence

$$|u_m(t) - x_0(t)| \le M\frac{|t|^\alpha}{\Gamma(\alpha+1)} \le b. \tag{4.6}$$

This implies that the points $(t, u_m(t)) \in R$, for all $m = 1, 2, \ldots$ when $t \in I$. □

Theorem 4.2. *The NIM solution series converges on the interval $I = [-\chi, \chi]$ to a solution of the IVP* (4.2).

Proof. Note that $f(t,x) = rx(1-x)$ is continuous function defined on a rectangle R and $|f| \leq M$ on R. Further

$$\begin{aligned}
\left|f(t,x) - f(t,y)\right| &= |r|\left|1-(x+y)\right|\left|x-y\right| \\
&\leq (|r||1-2(b+x_0)|)\left|x-y\right| \\
&= K\left|x-y\right|.
\end{aligned} \tag{4.7}$$

Thus, f is Lipschitz in the second variable on R with Lipschitz constant $K = |r||1-2(b+x_0)|$. Now

$$\begin{aligned}
|x_1(t)| &\leq M\frac{|t|^{\alpha}}{\Gamma(\alpha+1)}, \\
|x_2(t)| &\leq I^{\alpha}\left|f(t,x_0(t)+x_1(t)) - f(t,x_0(t))\right| \\
&\leq I^{\alpha}K|x_1(t)| \\
&\leq MK\frac{|t|^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned} \tag{4.8}$$

It can be proved by induction that

$$|x_m(t)| \leq \frac{M}{K}\frac{\left(|t|^{\alpha}K\right)^m}{\Gamma(m\alpha+1)}, \tag{4.9}$$

$m = 1, 2, \ldots$.

This shows that the mth term of the NIM solution series is bounded by $((M/K)$times) the mth term of the Mittag-Leffler function

$$E_{\alpha}\left(K|t|^{\alpha}\right). \tag{4.10}$$

Thus, the NIM series is convergent under given conditions. □

5. Illustrative Examples

Mathematica 8 has been used for solving the following problems.

Example 5.1. Consider a fractional-order logistic equation

$$\begin{gathered}
D^{\alpha}x(t) = \frac{1}{2}x(t)(1-x(t)), \quad t > 0,\ 0 < \alpha \leq 1, \\
x(0) = \frac{1}{2}.
\end{gathered} \tag{5.1}$$

Equivalent integral equation of (5.1) is

$$x(t) = \frac{1}{2} + \frac{1}{2} I^{\alpha}(x(t)) - \frac{1}{2} I^{\alpha}\left(x^{2}(t)\right). \tag{5.2}$$

In view of NIM,

$$\begin{aligned}
x_0 &= 0.5, \\
x_1 &= 0.5 I^{\alpha}(x_0) - 0.5 I^{\alpha}\left(x_0^2\right) = \frac{0.125 t^{\alpha}}{\Gamma(\alpha+1)}, \\
x_2 &= 0.5 I^{\alpha}(x_1) - 0.5 I^{\alpha}(x_0 + x_1)^2 + 0.5 I^{\alpha}\left(x_0^2\right) = -\frac{0.0078125 t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}, \\
x_3 &= \frac{0.0009765625 t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&\quad - \frac{0.000030517578125 t^{7\alpha}\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)},
\end{aligned} \tag{5.3}$$

and so on. Few terms of ADM [3] and HPM solution are

$$\begin{gathered}
x_0 = 0.5, \\
x_1 = \frac{0.125 t^{\alpha}}{\Gamma(\alpha+1)}, \\
x_2 = 0, \\
x_3 = -\frac{0.0078125 t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}, \\
x_4 = 0, \\
x_5 = \frac{0.0009765625 t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)}.
\end{gathered} \tag{5.4}$$

Figure 1 shows the solutions of (5.2) for $\alpha = 1$ by NIM (5-term solution), ADM, and HPM (6-term solution) and exact solution $x(t) = \exp((1/2)t)/(1+\exp((1/2)t))$. In Figures 2 and 3 we compare ADM, HPM, and NIM solutions of (5.2) for $\alpha = 0.5$ and $\alpha = 0.7$, respectively. Figure 4 shows NIM solutions for different values of α.

Example 5.2. Consider the following logistic equation of fractional order

$$D^{\alpha}x(t) = \frac{1}{4}x(t)(1 - x(t)), \quad t > 0,\ 0 < \alpha \le 1, \tag{5.5}$$

$$x(0) = \frac{1}{3}. \tag{5.6}$$

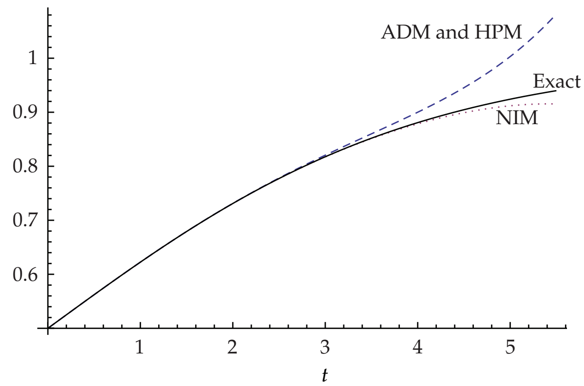

Figure 1: (Example 5.1, $\alpha = 1$).

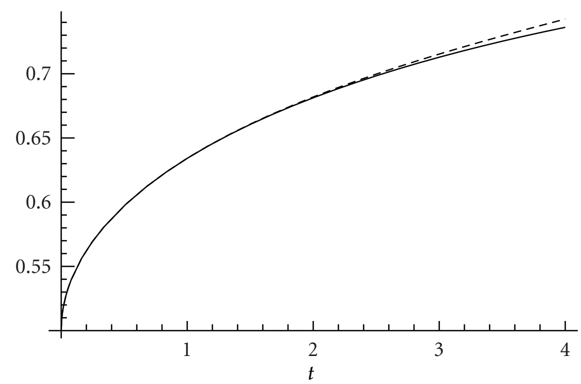

Figure 2: ($\alpha = 0.5$, dashed = ADM & HPM, solid = NIM).

Applying I^{α} on both sides of (5.5) and using (5.6), we get

$$x(t) = \frac{1}{3} + \frac{1}{4} I^{\alpha}(x(t)) - \frac{1}{4} I^{\alpha}\left(x^2(t)\right). \tag{5.7}$$

In Figure 5, we compare the solution of (5.7) for $\alpha = 1$ by NIM (5-term solution), ADM [3], and HPM (6-term solution) and exact solution $\exp((1/4)t)/(2+\exp((1/4)t))$. Figure 6 shows NIM solutions of (5.7) for different values of α.

6. Conclusions

Fractional-order logistic equation is solved using a new iterative method (NIM). From Figures 1–6, it is clear that NIM solutions are more stable than ADM and HPM. Further the condition for the convergence of NIM solution series is also provided. It can be concluded that NIM is a useful technique for solving nonlinear problems.

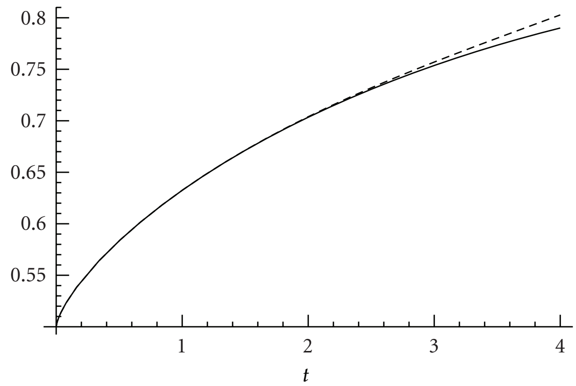

Figure 3: ($\alpha = 0.7$, dashed = ADM & HPM, solid = NIM).

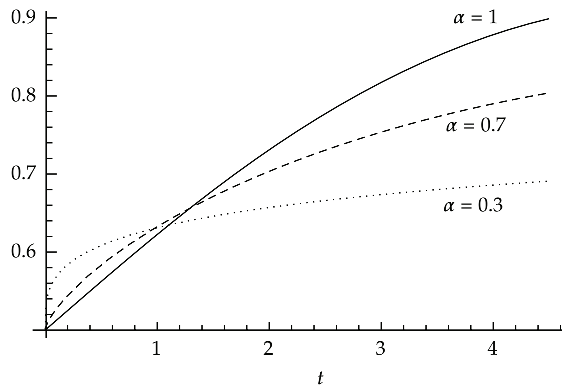

Figure 4: (Example 5.1, NIM solutions).

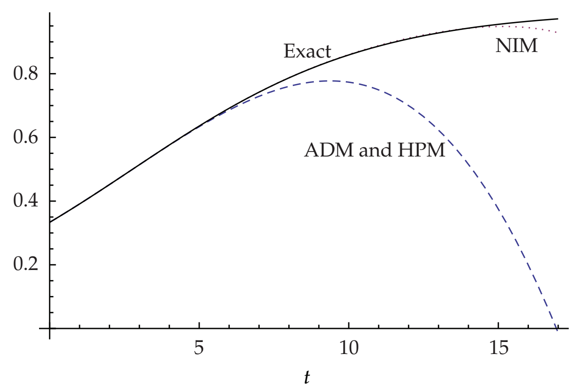

Figure 5: (Example 5.2, $\alpha = 1$).

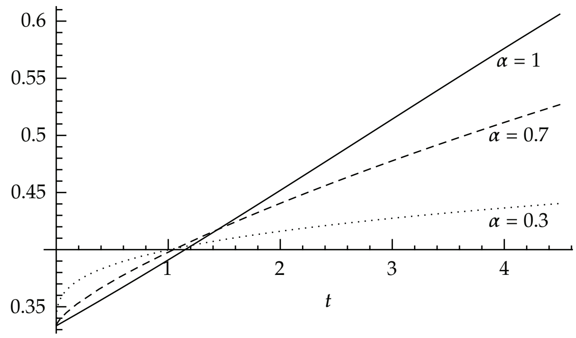

Figure 6: (NIM solutions).

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