

Research Article

Matroidal Structure of Rough Sets from the Viewpoint of Graph Theory

Jianguo Tang,^{1,2} Kun She,¹ and William Zhu³

¹ School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China

² School of Computer Science and Engineering, XinJiang University of Finance and Economics, Urumqi 830012, China

³ Lab of Granular Computing, Zhangzhou Normal University, Zhangzhou 363000, China

Correspondence should be addressed to William Zhu, williamfengzhu@gmail.com

Received 4 February 2012; Revised 30 April 2012; Accepted 18 May 2012

Academic Editor: Mehmet Sezer

Copyright © 2012 Jianguo Tang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Constructing structures with other mathematical theories is an important research field of rough sets. As one mathematical theory on sets, matroids possess a sophisticated structure. This paper builds a bridge between rough sets and matroids and establishes the matroidal structure of rough sets. In order to understand intuitively the relationships between these two theories, we study this problem from the viewpoint of graph theory. Therefore, any partition of the universe can be represented by a family of complete graphs or cycles. Then two different kinds of matroids are constructed and some matroidal characteristics of them are discussed, respectively. The lower and the upper approximations are formulated with these matroidal characteristics. Some new properties, which have not been found in rough sets, are obtained. Furthermore, by defining the concept of lower approximation number, the rank function of some subset of the universe and the approximations of the subset are connected. Finally, the relationships between the two types of matroids are discussed, and the result shows that they are just dual matroids.

1. Introduction

Rough sets provide an important tool to deal with data characterized by uncertainty and vagueness. Since it was proposed by Pawlak [1, 2], rough sets have been generalized from different viewpoints such as the similarity relation [3, 4] or the tolerance relations [5] instead of the equivalence relation, and a covering over the universe instead of a partition [6–10], and the neighborhood instead of the equivalence class [11–14]. Besides, using some other mathematical theories, such as fuzzy sets [15–19], boolean algebra [20–23], topology [24–27], lattice theory [28–30], and modal logic [31], to study rough sets has become another kind

of important generalizations of rough sets. Specially, matroids also have been used to study rough sets recently [32, 33].

Matroids, as a simultaneous generalization of graph theory and linear algebra, was proposed by Whitney in [34]. The original purpose of this theory is to formalize the similarities between the ideas of independence and rank in graph theory and those of linear independence and dimension in the study of vector spaces [35]. It has been found that matroids are effective to simplify various ideas in graph theory and are useful in combinatorial optimization problems.

In the existing works on the combination of rough sets and matroids, Zhu and Wang [32] constructed a matroid by defining the concepts of upper approximation number in rough sets. Then they studied the generalized rough sets with matroidal approaches. As a result, some unique properties are obtained in this way. Wang et al. [33] studied the covering-based rough sets with matroids. Two matroidal structures of covering-based rough sets are established.

In this paper, we attempt to make a further contribution to studying rough sets with matroids. As we see in Section 2.3, it is somewhat hard to understand matroids. And this will also arise in the combination of matroids and rough sets. So, in order to give an intuitive interpreting to the combination, we will study it from the viewpoint of graph theory. There are at least two kinds of graphic ways, which can be used to build relationships between matroids and rough sets. The complete graph and the cycle. More specifically, for a partition over the universe, any equivalence class of the partition can be regarded as a complete graph or a cycle. Thus a partition is transformed to a graph composed of these complete graphs or circles induced by the equivalence classes of the partition. And we can establish a matroid in terms of the graph. Afterwards, some characteristics of the matroid are formulated and some new properties, which are hard to be found via the rough sets way, are obtained. With these characteristics and properties, a matroidal structure of rough sets is constructed. Finally, the relationships between the two kinds of matroids established from the viewpoints of complete graph and cycle are discussed.

The rest of this paper is organized as follows. In Section 2, we review some basic knowledge about rough sets, matroids, and graph theory. In Section 3, we analyze the relationships between rough set theory and graph theory from the viewpoints of complete graph and cycle, respectively. In Sections 4 and 5, two kinds of matroids are established in terms of the analytical results of Section 3. And two kinds of the matroidal structures of rough sets are constructed. In Section 6, the relationships between the two kinds of matroids are discussed.

2. Preliminary

For a better understanding to this paper, in this section, some basic knowledge of rough sets, graph theory, and matroids are introduced.

2.1. Rough Sets

Let U be a nonempty and finite set called universe, \mathbf{R} a family of equivalence relations over U , then the relational system $K = (U, \mathbf{R})$ is called a *knowledge base* [1]. If $\emptyset \neq \mathbf{Q} \subseteq \mathbf{R}$, then $\cap \mathbf{Q}$ is also an equivalence relation [1]. And $\cap \mathbf{Q}$ is called an *indiscernibility relation* and denoted by $\text{IND}(\mathbf{Q})$ [1]. If $R \in \mathbf{R}$, then U/R represents the *partition* of U induced by R . That is in

the partition $U/R = \{T_1, T_2, \dots, T_n\}$, for all $T_i \in U/R$, $T_i \subseteq U$ and $T_i \neq \emptyset$, $T_i \cap T_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \dots, n$, and $\cup T_i = U$. Each T_i in U/R is an equivalence class, and it can also be denoted by $[x]_R$ if $x \in T_i$.

For any subset $X \subseteq U$, the lower and the upper approximations of X with respect to R are defined as follows [1]:

$$\begin{aligned}\underline{R}(X) &= \cup \{T \in U/R : T \subseteq X\}, \\ \overline{R}(X) &= \cup \{T \in U/R : T \cap X \neq \emptyset\}.\end{aligned}\tag{2.1}$$

Set $BN_R(X) = \overline{R}(X) - \underline{R}(X)$ is called the R -boundary of X or the boundary region of X with respect to R [1]. If $\underline{R}(X) = \overline{R}(X)$, that is, $BN_R(X) = \emptyset$, then X is R -definable, or X is called a *definable set* with respect to R ; else, if $\underline{R}(X) \neq \overline{R}(X)$, that is, $BN_R(X) \neq \emptyset$, then X is rough with respect to R , or X is called a *rough set* with respect to R [1]. The lower and the upper approximations satisfy duality, that is [1],

$$(P1) \text{ for all } X \subseteq U, \underline{R}(X) = \sim \overline{R}(\sim X),$$

$$(P2) \text{ for all } X \subseteq U, \overline{R}(X) = \sim \underline{R}(\sim X),$$

where $\sim X$ represents the set $U - X$.

Neighborhood and upper approximation number are another two important concepts, which will be used in this paper. They are defined as follows.

Definition 2.1 (Neighborhood [36]). Let R be a relation on U . For all $x \in U$, $RN_R(x) = \{y \in U : xRy\}$ is called the successor neighborhood of x in R . When there is no confusion, we omit the subscript R .

Definition 2.2 (Upper approximation number [32]). Let R be a relation on U . For all $X \subseteq U$, $f_R^*(X) = |\{RN_R(x) : x \in U \wedge RN_R(x) \cap X \neq \emptyset\}|$ is called the upper approximation number of X with respect to R .

2.2. Graph Theory

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set $V^{(2)}$ of unordered pairs of V [37]. The set V is the set of *vertices* and E is the set of *edges*. If G is a graph, then $V = V(G)$ is the vertex set of G , and $E = E(G)$ is the edge set. An *empty graph* is a graph whose edge set is empty. An edge $\{u, v\}$ is said to join the vertices u and v and is denoted by uv . Thus uv and vu mean exactly the same edge; the vertices u and v are the *endpoints* of this edge. If $uv \in E(G)$, then u and v are *adjacent* and are *neighbors*. A *loop* [38] is an edge whose endpoints are equal. *Parallel edges* are edges having the same pair of endpoints. The degree of vertex v in a graph G , denoted by $d_G(v)$ or $d(v)$, is the number of edges incident to v , except that each loop at v counts twice.

A *simple graph* is a graph having no loops or parallel edges [38]. An *isomorphism* [38] from a simple graph G to a simple graph H is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. That is to say " G is *isomorphic* to H ," denoted by $G \cong H$ if there is an isomorphism from G to H .

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ [37]. In this case, we write $G' \subseteq G$. If G' contains all edges of G that join two vertices in V' then G' is said

to be the subgraph *induced by* V' and is denoted by $G[V']$. Thus, a subgraph G' of G is an induced subgraph if $G' = G[V(G')]$.

2.3. Matroids

Definition 2.3 (Matroid [39]). A matroid M is a pair (E, \mathcal{D}) , where E (called the ground set) is a finite set and \mathcal{D} (called the independent sets) is a family of subsets of E satisfying the following axioms:

$$(I1) \emptyset \in \mathcal{D};$$

$$(I2) \text{ if } I \in \mathcal{D} \text{ and } I' \subseteq I, \text{ then } I' \in \mathcal{D};$$

$$(I3) \text{ if } I_1, I_2 \in \mathcal{D} \text{ and } |I_1| < |I_2|, \text{ then } \exists e \in I_2 - I_1 \text{ such that } I_1 \cup \{e\} \in \mathcal{D},$$

where $|\cdot|$ represents the cardinality of “.”.

The matroid M is generally denoted by $M = M(E, \mathcal{D})$. $E(M)$ represents the ground set of M and $\mathcal{D}(M)$ the independent sets of M . Each element of $\mathcal{D}(M)$ is called an *independent set* of M . If a subset X of E is not an independent set, then it is called a dependent set. The family of all dependent sets of M is denoted by $\mathfrak{D}(M)$, that is,

$$\mathfrak{D}(M) = 2^E - \mathcal{D}(M). \quad (2.2)$$

Example 2.4. Let $E = \{a, b, c\}$, $\mathcal{D} = \{\{a, b\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$. Then (E, \mathcal{D}) is a matroid, which satisfies the axioms (I1)~(I3). And each element of \mathcal{D} is an independent set. $\{a, b, c\}$ and $\{a, c\}$ are only two dependent sets of (E, \mathcal{D}) .

Next, we will introduce some characteristics of a matroid. For a better understanding to them, some operations will be firstly introduced as follows.

Let E be a set and $\mathcal{A} \subseteq 2^E$. Then [39]:

$$\text{Max}(\mathcal{A}) = \{X \in \mathcal{A} : \forall Y \in \mathcal{A}, \text{ if } X \subseteq Y \text{ then } X = Y\},$$

$$\text{Min}(\mathcal{A}) = \{X \in \mathcal{A} : \forall Y \in \mathcal{A}, \text{ if } Y \subseteq X \text{ then } X = Y\},$$

$$\text{Opp}(\mathcal{A}) = \{X \subseteq E : X \notin \mathcal{A}\},$$

$$\text{Com}(\mathcal{A}) = \{X \subseteq E : E - X \in \mathcal{A}\}.$$

(2.3)

Definition 2.5 (Circuit [39]). Let M be a matroid. A minimal dependent set is called a circuit of M , and the set of all circuits of M is denoted by $\mathcal{C}(M)$, that is, $\mathcal{C}(M) = \text{Min}(\text{Opp}(\mathcal{D}))$.

A circuit in a matroid $M(E, \mathcal{D})$ is a set which is not independent but has the property that every proper subset of it is independent. In Example 2.4, $\mathcal{C}(M) = \{\{a, c\}\}$.

Theorem 2.6 (Circuit axioms [39]). *Let \mathcal{C} be a family of subsets of E . Then there exists $M(E, \mathcal{D})$ such that $\mathcal{C} = \mathcal{C}(M)$ if and only if \mathcal{C} satisfies the following properties:*

$$(C1) \emptyset \notin \mathcal{C};$$

$$(C2) \text{ if } C_1, C_2 \in \mathcal{C} \text{ and } C_1 \subseteq C_2, \text{ then } C_1 = C_2;$$

$$(C3) \text{ if } C_1, C_2 \in \mathcal{C}, C_1 \neq C_2, \text{ and } \exists e \in C_1 \cap C_2, \text{ then } \exists C_3 \in \mathcal{C} \text{ such that } C_3 \subseteq (C_1 \cup C_2) - \{e\}.$$

Definition 2.7 (Base [39]). Let M be a matroid. A maximal independent set of M is called a base of M ; the set of all bases of M is denoted by $\mathcal{B}(M)$, that is, $\mathcal{B}(M) = \text{Max}(\mathcal{I})$.

In Example 2.4, according to Definition 2.7, we can get that $\mathcal{B}(M) = \{\{a, b\}, \{b, c\}\}$.

It is obvious that all bases of a matroid have the same cardinality, which is called the rank of the matroid.

Definition 2.8 (Rank function [39]). Let $M = M(E, \mathcal{I})$ be a matroid. Then the rank function r_M of M is defined as: for all $X \subseteq E$,

$$r_M(X) = \max\{|I| : I \in \mathcal{I} \wedge I \subseteq X\}. \quad (2.4)$$

A matroid can be determined by its base, its rank function, or its circuit. For a set, $I \subseteq E$ is independent if and only if it is contained in some base, if and only if it satisfies $r_M(I) = |I|$, or if and only if it contains no circuit. It is possible to axiomatize matroids in terms of their sets of bases, their rank functions, or their sets of circuits [40].

Definition 2.9 (Closure [39]). Let $M = M(E, \mathcal{I})$ be a matroid. For all $X \subseteq E$, the closure operator cl_M of M is defined as follows:

$$\text{cl}_M(X) = \{e \in E : r_M(X) = r_M(X \cup \{e\})\}. \quad (2.5)$$

If $e \in \text{cl}_M(X)$, we say that e depends on X . The closure of X is composed of these elements of E that depend on X . If $\text{cl}_M(X) = X$, then X is called a closed set of M .

Definition 2.10 (Hyperplane [39]). Let $M = (E, \mathcal{I})$ be a matroid. $H \subseteq E$ is called a hyperplane of M if H is a closed set of M and $r_M(H) = r_M(E) - 1$. And $\mathcal{H}(M)$ represents the family of all hyperplanes of M .

3. The Viewpoint of Graph Theory in Rough Sets

Graph theory provides an intuitive way to interpret and comprehend a number of practical and theoretical problems. Here, we will make use of it to interpret rough sets. There are at least two different ways to understand rough sets from the viewpoint of graph theory: the complete graph and the cycle. This will be analyzed in detail in the following subsections.

3.1. The Complete Graph

Definition 3.1 (Complete graph [38]). A complete graph is a simple undirected graph whose vertices are pairwise adjacent. A complete graph whose cardinality of vertex set is equal to n is denoted by K_n .

In rough sets, an equivalence relation can generally be regarded as an indiscernibility relation. That means any two different elements in the same equivalence class are indiscernible. In order to interpret this phenomenon from the viewpoint of graph theory, we can consider the two elements as two vertices and the indiscernibility relation between them as an edge connecting the two vertices. Then an equivalence class is represented by a complete graph. For a better understanding of it, an example is served as follows.

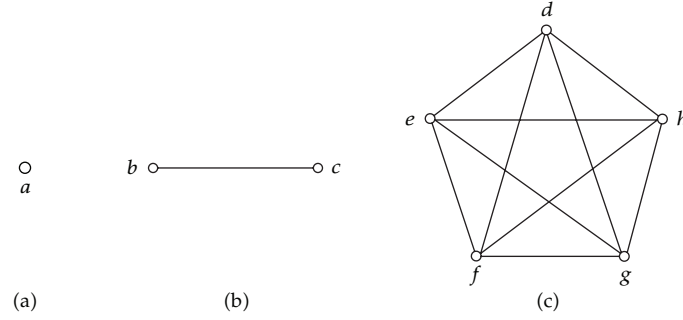


Figure 1: The complete graphs of equivalence classes.

Example 3.2. Let $U = \{a, b, c, d, e, f, g, h\}$ be the universe, R an equivalence relation, and $U/R = \{T_1, T_2, T_3\} = \{\{a\}, \{b, c\}, \{d, e, f, g, h\}\}$. Then each equivalence class can be transformed to a complete graph showed in Figure 1.

Figure 1(a) represents the complete graph of the equivalence class T_1 . Because T_1 just includes one element a , there is only one vertex and no edge in the complete graph Figure 1(a). Figure 1(b) represents the complete graph of T_2 , which includes two vertices and only one edge connecting the two vertices. And Figure 1(c) represents the complete graph of T_3 . We can find that there are five vertices in Figure 1(c) and each pair of vertices are connected by an edge. Here, we denote the complete graphs Figures 1(a), 1(b), and 1(c) as $K_{|T_1|}$, $K_{|T_2|}$, and $K_{|T_3|}$, respectively.

From the above example, for any two elements in the universe, if they are indiscernible then there is one edge between them. Then the partition is transformed to be a graph $G = (U, E)$, where

$$E_P = \{xy : x \in T \in P \wedge y \in T - \{x\}\}. \quad (3.1)$$

It is obvious that if x and y belong to the same equivalence class, then there is an edge xy in E . So, we can formulate the equivalence class as follows: for all $x \in U$,

$$[x]_R = \{x\} \cup \{y : xy \in E\}. \quad (3.2)$$

Furthermore, for any subset $X \subseteq U$, the lower and the upper approximations of X can be formulated as follows:

$$\begin{aligned} \underline{R}(X) &= \cup \{T \in U/R : G[T] \subset G[X]\}, \\ \overline{R}(X) &= \cup \{T \in U/R : \exists Y \subseteq T \text{ s.t. } K_{|T|}[Y] \subset G[X]\}. \end{aligned} \quad (3.3)$$

3.2. The Cycle

A *walk* [37] W in a graph is an alternating sequence of vertices and edges, say $x_0, e_1, x_1, e_2, \dots, e_l, x_l$ where $e_i = x_{(i-1)}x_i$, $0 < i \leq l$. For simplicity, the walk W can also be denoted by $x_0x_1 \cdots x_l$; the *length* of W is l , that is, the number of its edges. A walk that starts and ends

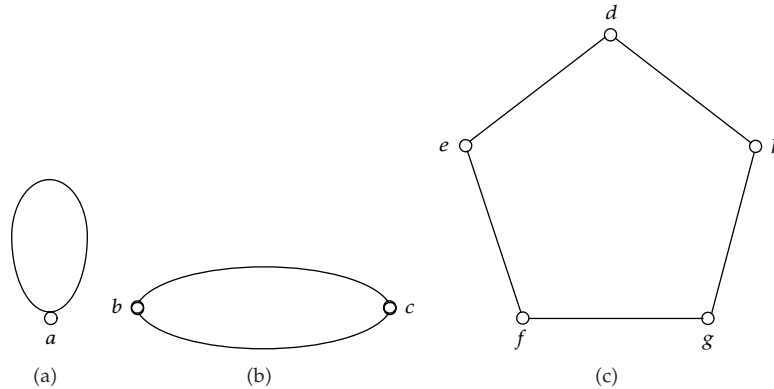


Figure 2: The cycles of equivalence classes.

at the same vertex but otherwise has no repeated vertices is called a *cycle* [41]. A cycle on one vertex consists of a single vertex with a loop, and a cycle on two vertices consists of two vertices joined by a pair of parallel edges [42].

Then, how to build a bridge between rough sets and cycle? In rough sets, elements contained in the same equivalence class are indiscernible, and any proper subset of an equivalence class is no longer an equivalence class. So, we can convert an equivalence class to a cycle whose vertices set is the equivalence class. Therefore, each vertex is connected with all vertices in the cycle [42]. This reflects the indiscernible relationship among the elements of an equivalence class. Furthermore, any subgraph of the cycle does not contain a cycle. That is, any subgraph of the cycle is no longer a cycle. This can be illustrated in the following example.

Example 3.3 (Continued from Example 3.2). For any equivalence class in U/R , it can be represented by a cycle. As shown in Figure 2, the equivalence class T_1 , T_2 , and T_3 are represented by Figures 2(a), 2(b), and 2(c), respectively.

Figure 2(a) is a cycle with only one vertex and one edge. It is also called a loop. That means the vertex a is connected with itself. Figure 2(b) is a cycle with two vertices and two edges. And it is generally regarded as a parallel edges. Figure 2(c) is not only a cycle but also a simple graph. Obviously, any subgraph of each cycle in Figure 2 is no longer a cycle. And, for any two different elements of the universe, they belong to the same equivalence class if and only if they are connected to each other.

It is worth noting that the sequence of vertices in a cycle is not emphasized here. We only care that the vertices, namely, elements of some equivalence class, can form a cycle. So, to Figure 2(c), the cycle $defghd$ and cycle $dfhegd$ can be treated as the same cycle.

For convenience, to an equivalence class T in U/R , the cycle whose vertices set is equal to T is denoted by C_T , that is, $C_T = (T, E_T)$ is a graph (cycle) where E_T is the set of edges of C_T . Then the partition U/R can be transformed to be a graph $G = (U, E)$ where

$$E = \cup\{E_T : T \in U/R\}. \quad (3.4)$$

One may ask which edges belong to E_T exactly? In fact, it is nonnecessary to define the edges of E_T exactly. Here, we just need to form a cycle with the vertex set T . That is, each pair of vertices of T are connected and the degree of each vertex is equal to 2. In other words, we simply need to know that each vertex of C_T is adjacent with two other vertices (except the loop and parallel edges) and do not need to care which two vertices they are.

We can find from the Example 3.3 that, for any two elements in the universe, they belong to the same equivalence class if and only if they are connected with each other. Therefore, we can formulate the equivalence class as follows: for all $x \in U$,

$$[x]_R = \{y \in U : y \text{ is connected to } x\}. \quad (3.5)$$

Likewise, for any subset $X \subseteq U$, the lower and the upper approximations of X can be formulated as follows:

$$\begin{aligned} \underline{R}(X) &= \cup \{T \in U/R : C_T \subset G[X]\}, \\ \overline{R}(X) &= \cup \{T \in U/R : \exists Y \subseteq T \text{ s.t. } C_T[Y] \subset G[X]\}. \end{aligned} \quad (3.6)$$

So far, rough sets are interpreted from the viewpoints of complete graph and cycle, respectively. The above analysis shows that there are some similarities, and also some differences, between the two ways to illustrate rough sets. Because there are closed connections between graph theory and matroid theory, we will study the matroidal structure of rough sets through the two kinds of graphs.

4. Matroidal Structure of Rough Sets Constructed from the Viewpoint of Complete Graph

In Section 3, we discussed rough sets from the viewpoint of graph theory. Two graphic ways are provided to describe rough sets. In this section, we will construct two types of matroidal structures of rough sets. One of them is established by using the principle of complete graph and the other of cycle.

For convenience, in this section, we suppose that U is the universe, R an equivalence relation over U and $P = U/R$ the partition. And $G_P = (U, E_P)$ is the graph induced by P , where $E_P = \{xy : x \in T \in P \wedge y \in T - \{x\}\}$.

4.1. The First Type of Matroidal Structure of Rough Sets

We know that a complete graph is a simple graph. Then, for any vertex v of a complete graph, there is not a loop whose vertex is v . That is to say there is not an edge between a vertex and itself. Furthermore, for any two vertices coming from different complete graphs, there is not an edge between them as well. Because an equivalence class can be represented by a complete graph, we can construct the matroidal structure of rough sets from this perspective.

In this subsection, the first type of matroid induced by a partition will be established and defined. And then some characteristics of it such as the base, circuit, rank function, and closure are studied.

Proposition 4.1. *Let $\mathcal{O}_P = \{X \subseteq U : G_P[X] \text{ is an empty graph}\}$. Then there is a matroid M on U such that $\mathcal{O}(M) = \mathcal{O}_P$.*

Proof. According to Definition 2.3, we just need to prove that \mathcal{O}_P satisfies axioms (I1)~(I3). It is obvious that (I1) and (I2) hold. Suppose that $X, Y \in \mathcal{O}_P$ and $|X| < |Y|$. Because $G_P[X]$ and $G_P[Y]$ are empty graphs, according to the definition of E_P , each $x \in X$ belongs to a different equivalence class with the others of X and the same to each $y \in Y$. Since $|X| < |Y|$, thus there must be at least one element $y_0 \in Y$ such that y_0 belongs to some equivalence class which does not include any element of X . Therefore, $X \cup \{y_0\} \in \mathcal{O}_P$. As a result, \mathcal{O}_P satisfies (I3). That is, there exists a matroid M on U such that $\mathcal{O}(M) = \mathcal{O}_P$. \square

If $G[X]$ is an empty graph, then it means that any two different vertices of $G[X]$ are nonadjacent. That is, each vertex of $G[X]$ comes from a different complete graph with others. For instance, in Example 3.2, we can get \mathcal{O}_P as follows:

$$\begin{aligned} \mathcal{O}_P = \{ & \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,f\}, \{a,g\}, \\ & \{a,h\}, \{b,d\}, \{b,e\}, \{b,f\}, \{b,g\}, \{b,h\}, \{c,d\}, \{c,e\}, \{c,f\}, \{c,g\}, \{c,h\}, \{a,b,d\}, \\ & \{a,b,e\}, \{a,b,f\}, \{a,b,g\}, \{a,b,h\}, \{a,c,d\}, \{a,c,e\}, \{a,c,f\}, \{a,c,g\}, \{a,c,h\} \}. \end{aligned} \quad (4.1)$$

Definition 4.2 (The first type of matroid induced by a partition). The first type of matroid induced by a partition P over U , denoted by I -MIP, is such a matroid whose ground set $E = U$ and independent sets $\mathcal{O} = \{X \subseteq U : G_P[X] \text{ is an empty graph}\}$.

Obviously, matroids proposed in Proposition 4.1 is a I -MIP. From the above result of \mathcal{O}_P , we can find that any two elements of X come from different equivalence class. Therefore, we can get the following proposition.

Proposition 4.3. *Let $M_P = (U, \mathcal{O}_P)$ be an I -MIP. Then for all $X \subseteq U$, $X \in \mathcal{O}_P$ if and only if for all $T \in P$ such that $|X \cap T| \leq 1$.*

Proof. (\Rightarrow): If $X \in \mathcal{O}_P$, then $G_P[X]$ is an empty graph. According to the definition of E_P , each element of X comes from a different equivalence class with the others of X . That is, for all $T \in P$, $|X \cap T| \leq 1$.

(\Leftarrow): Let $X \subseteq U$. If for all $T \in P$, $|X \cap T| \leq 1$, then $G_P[X]$ is an empty graph. Therefore, $X \in \mathcal{O}_P$. \square

A matroid can be determined by its base, its rank function, or its circuit. So it is possible to axiomatize matroids in terms of their sets of bases, their rank functions, or their sets of circuits [40]. Here we will axiomatize the I -MIP in terms of its circuit.

Theorem 4.4. *Let M be a matroid induced by P . Then M is an I -MIP if and only if for all $C \in \mathcal{C}(M)$, $|C| = 2$.*

Proof. According to Definition 2.3, we know that $\mathcal{O}(M) \subseteq 2^U$. If $\mathcal{O}(M) = 2^U$, then M is a I -MIP induced by $P = \{\{x\} : x \in U\}$. In this case, according to (2.2), $\mathfrak{D}(M) = \emptyset$ and $\mathcal{C}(M) = \emptyset$. It indicates that there is not any circuit in $\mathcal{C}(M)$, and, therefore, we do not need to care whether the cardinality of each circuit of $\mathcal{C}(M)$ is equal to 2. That is, $\mathcal{C}(M) = \emptyset$ is compatible with

the description that for all $C \in \mathcal{C}(M)$, $|C| = 2$. Similarly, if $\mathcal{C}(M) = \emptyset$, then M is a I -MIP induced by $P = \{\{x\} : x \in U\}$. So, Theorem 4.4 is true when the set of circuits of M is empty.

Next, we prove that Theorem 4.4 is true when the set of circuits of M is nonempty.

(\Rightarrow): According to (2.2), $\mathfrak{D}(M) = 2^U - \mathcal{O}_P$. Therefore, in terms of Definition 4.2 and Proposition 4.1, for all $X \subseteq U$, $X \in \mathfrak{D}(M)$ if and only if $G_P[X]$ is not an empty graph. That is, there is at least one edge in $G_P[X]$. Obviously, the set of endpoints of each edge of $G_P[X]$ is a dependent set. So, for all $X \in \mathfrak{D}(M)$, there is a set Y composed of the endpoints of some edge of $G_P[X]$ such that $Y \subseteq X$. According to Definition 2.5, $Y \in \mathcal{C}(M)$ and $X \notin \mathcal{C}(M)$. That is, for all $C \in \mathcal{C}(M)$, $|C| = 2$.

(\Leftarrow): $\mathfrak{D}(M) = \{X \subseteq U : \exists C \in \mathcal{C}(M) \text{ s.t. } C \subseteq X\}$. According to (2.2), $\mathcal{O}(M) = 2^U - \mathfrak{D}(M)$. Therefore, for all $I \in \mathcal{O}(M)$, $\nexists C \in \mathcal{C}(M)$ such that $C \subseteq I$, that is, for all $C \in \mathcal{C}(M)$, $|C \cap I| \leq 1$. So, for all $C_1, C_2 \in \mathcal{C}(M)$; if $I \cap C_1 \neq \emptyset$, then $I \cap C_2 = C_1 \cap C_2$. According to Theorem 2.6, if $C_1 \cap C_2 \neq \emptyset$, then $\exists C_3 \in \mathcal{C}(M)$ such that $C_3 = C_1 \cup C_2 - C_1 \cap C_2$. For all $C_i \in \mathcal{C}(M)$, let $T_{C_i} = \{C \in \mathcal{C}(M) : C \cap C_i \neq \emptyset\}$. Then $|I \cap T_{C_i}| = 1$. Furthermore, if $C_1 \cap C_2 = \emptyset$, then $T_{C_1} \cap T_{C_2} = \emptyset$. If for all $C \in \mathcal{C}(M)$, $I \cap C = \emptyset$, then for all $y \in I$, $\nexists C \in \mathcal{C}(M)$ such that $y \in C$. Thus, $P_C(M) = \{T_{C_i} : C_i \in \mathcal{C}(M)\} \cup \{\{y\} : y \in U - \cup \mathcal{C}(M)\}$ is a partition over U . So $\mathcal{O}(M) = \{I \subseteq U : \text{for all } X \in P_{\mathcal{C}(M)}, |I \cap X| \leq 1\}$. According to Definition 4.2, $M = (U, \mathcal{O})$ is a I -MIP.

Summing up, Theorem 4.4 is true. \square

In terms of Proposition 4.1, we can get a matroid induced by a partition. Then one may ask whether there is a bijection between a partition and the I -MIP induced by the partition. This question will be answered by the following theorem.

Theorem 4.5. *Let \mathcal{P} be the collection of all partitions over U , \mathcal{M} the set of all I -MIP induced by partitions of \mathcal{P} , $f : \mathcal{P} \rightarrow \mathcal{M}$, that is, for all $P \in \mathcal{P}$, $f(P) = M_P$, where M_P is the I -MIP induced by P . Then f satisfies the following conditions:*

- (1) for all $P_1, P_2 \in \mathcal{P}$, and if $P_1 \neq P_2$ then $f(P_1) \neq f(P_2)$,
- (2) for all $M \in \mathcal{M}$, $\exists P_M \in \mathcal{P}$ s.t. $f(P_M) = M$.

Proof. (1) Let $P_1, P_2 \in \mathcal{P}$, $P_1 \neq P_2$, and $M_{P_1} = (U, \mathcal{O}_{P_1})$, $M_{P_2} = (U, \mathcal{O}_{P_2})$ are two I -MIP induced by P_1 and P_2 , respectively. We need to prove that there is an $I_1 \in \mathcal{O}_{P_1}$ such that $I_1 \notin \mathcal{O}_{P_2}$, or there is an $I_2 \in \mathcal{O}_{P_2}$ such that $I_2 \notin \mathcal{O}_{P_1}$. Because $P_1 \neq P_2$, there is at least one equivalence class $T_1 \in P_1$ such that $T_1 \notin P_2$. If $\exists T_2 \in P_2$ such that $T_1 \subset T_2$, then $\exists X \subseteq U$ and $X \in \mathcal{O}_{P_1}$ such that $X \cap (T_2 - T_1) \neq \emptyset$. That means $X \notin \mathcal{O}_{P_2}$. Else, there at least two equivalence classes $T_{2i}, T_{2j} \in P_2$ such that $T_{2i} \cap T_1 \neq \emptyset$ and $T_{2j} \cap T_1 \neq \emptyset$. That is, there is a set $Y \in \mathcal{O}_{P_2}$ such that $Y \cap T_{2i} \cap T_1 \neq \emptyset$ and $Y \cap T_{2j} \cap T_1 \neq \emptyset$. Obviously, according to Proposition 4.3, $Y \notin \mathcal{O}_{P_1}$.

(2) Let $M = (U, \mathcal{O})$ be a I -MIP, for all $x \in U$, $C_x = \{x\} \cup (\cup \{C \in \mathcal{C}(M) : C \cap \{x\} \neq \emptyset\})$. According to Theorem 4.4, for all $y \in U$ and $y \notin C_x$, then $C_x \cap C_y = \emptyset$. Therefore, we can get a family $C_U = \{C_x : x \in U\}$. It is obvious that $\cup C_U = U$. Therefore, C_U is a partition of U . That is, $C_U \in \mathcal{P}$ and $f(C_U) = M$. \square

Theorem 4.5 shows that there is one-to-one correspondence between a partition and the I -MIP induced by the partition.

4.2. Characteristics of I -MIP

The characteristics of a matroid are very important to describe the matroid from different aspects. In this subsection, we will study the characteristics of I -MIP such as the base, circuit, rank function, and closure.

The set of bases of a matroid is the collection of all maximal independent sets. Observing from the result of \mathcal{O}_P in Section 4.1, the maximal independent set is the vertex set whose cardinality is equal to the cardinality of P . Then the following proposition can be obtained.

Proposition 4.6. *Let M_P be the I-MIP induced by P , $Y \subseteq U$, and $G_P[Y] = (Y, E_Y)$ a subgraph of G_P . Then $\mathcal{B}_P = \{X \subseteq U : |X| = |P| \wedge E_X = \emptyset\}$ is the set of bases of M_P .*

Proof. According to Definition 2.7, we need to prove that $\mathcal{B}(M_P) = \mathcal{B}_P$, namely, $\text{Max}(\mathcal{O}_P) = \{X \subseteq U : |X| = |P| \wedge E_X = \emptyset\}$. In terms of Proposition 4.3, for all $I \in \mathcal{O}_P$, $|I \cap T| \leq 1$ for all $T \in P$. So, for all $I \in \mathcal{B}(M_P)$, $|I| = |P|$. According to Proposition 4.1 and Definition 4.2, for all $I \in \mathcal{B}(M_P)$, $G_P[I]$ is an empty graph, that is, $I \in \mathcal{B}_P$. Similarly, we can prove in the same way that for all $X \in \mathcal{B}_P$, $X \in \mathcal{B}(M_P)$. That is, $\mathcal{B}(M_P) = \mathcal{B}_P$. \square

For a base B in \mathcal{B}_P , we can say that B is such a set including one and only one element of every equivalence class of P . Then we can get the following corollary.

Corollary 4.7. *Let $X \subseteq U$. Then $X \in \mathcal{B}_P$ if and only if for all $T \in P$, $|X \cap T| = 1$.*

Proof. According to Proposition 4.6, it is straightforward. \square

Corollary 4.8. $\cup \mathcal{B}_P = U$.

Proof. According to Proposition 4.6, it is straightforward. \square

For a subset X of U , X is either an independent set or a dependent set of M_P . And so the opposition to the Proposition 4.1, X is a dependent set if and only if there is at least one pair of vertices of the vertex set of $G_P[X]$, which is adjacent. Furthermore, a minimal dependent set of M_P is the vertex set of an edge of G_P . Then we can get the following proposition.

Proposition 4.9. *Let $M_P = (U, \mathcal{O}_P)$ be the I-MIP induced by P . Then $\mathcal{C}_P = \{\{x, y\} : xy \in E_P\}$ is the set of circuits of M_P .*

Proof. According to Definition 2.5, we need to prove $\mathcal{C}(M_P) = \mathcal{C}_P$, that is, $\text{Min}(\text{Opp}(\mathcal{O}_P)) = \{\{x, y\} : xy \in E_P\}$. $\mathfrak{D}(M) = \text{Opp}(\mathcal{O}_P)$, for all $I \in \mathfrak{D}(M)$, $\exists x, y \in I$ such that $xy \in E(G_P[I])$. Furthermore, $xy \in E_P$. If $\{x, y\} \subset I$, then $\{x, y\} \in \mathcal{C}(M_P)$ and $I \notin \mathcal{C}(M_P)$; else, $\{x, y\} = I \in \mathcal{C}(M_P)$. So, for all $I \in \mathcal{C}(M_P)$, $I \in \mathcal{C}_P$. Similarly, for all $\{x, y\} \in \mathcal{C}_P$, $\exists I \in \mathfrak{D}(M_P)$ such that $\{x, y\} \subseteq I$. Therefore, $\{x, y\} \in \mathcal{C}(M_P)$. As a result, $\mathcal{C}_P = \{\{x, y\} : xy \in E_P\}$. \square

Likewise, Proposition 4.3 provides a necessary and sufficient condition to decide whether a set is an independent set of M_P . In this way, we can get the family of dependent sets of M_P as follows:

$$\mathfrak{D}(M_P) = \{X \subseteq U : \exists T \in P \text{ s.t. } |X \cap T| > 1\}. \quad (4.2)$$

Moreover, in terms of the Definition 2.5, we can get the set of circuits of M_P as follows:

$$\mathcal{C}(M_P) = \text{Min}(\mathfrak{D}(M_P)). \quad (4.3)$$

According to Proposition 4.1, it can be found that each subset of U which contains exactly one element is an independent set. So, for any dependent set X of M_P , if $|X| > 2$ then there must exist a subset Y of X such that $|Y| = 2$ and Y is a dependent set. Thus, we can get the following proposition.

Proposition 4.10. *Let M_P be the I-MIP induced by P . Then $C_+ = \{X \subseteq U : \exists T \in P \text{ s.t. } X \subseteq T \wedge |X| = 2\}$ is the set of circuits of M_P .*

Proof. According to Proposition 4.3, for all $X \in C_+$, X is a dependent set, that is, $X \in \mathfrak{D}(M_P)$. In terms of Proposition 4.9, $X \in \mathcal{C}(M_P)$. Similarly, for all $I \in \mathcal{C}(M_P)$, according to Proposition 4.9, $I \in C_+$. That is, C_+ is the set of circuits of M_P . \square

From Propositions 4.9 and 4.10, we can find that C_P and C_+ are the set of circuits of M_P . Therefore, we can get the following corollary.

Corollary 4.11. $C_P = C_+$.

Propositions 4.1 and 4.3 provide two ways to transform an partition to a matroid. Then, how to convert an I-MIP to a partition? In the following proposition, this question is answered through the set of circuits of the I-MIP.

Proposition 4.12. *Let M_P be the I-MIP induced by P . Then for all $x \in U$,*

$$[x]_R = \{x\} \cup \{y \in U : \{x, y\} \in \mathcal{C}(M_P)\}. \quad (4.4)$$

Proof. According to Proposition 4.10, for all $T \in P$, if $x \in T$ then $\{x, y\} \in \mathcal{C}(M_P)$ for each $y \in T - \{x\}$. And for any $y \in U$ and $y \notin T$, $\{x, y\} \notin \mathcal{C}(M_P)$. Therefore, $T = [x]_R = \{x\} \cup \{y \in U : \{x, y\} \in \mathcal{C}(M_P)\}$. \square

Proposition 4.12 shows that if two different elements form a circuit, then they belong to the same equivalence class. In terms of (3.2), there is an edge in E_P whose vertex set just contains the two elements. For a subset $X \subseteq U$, if X does not contain a circuit, then X is an independent set and the rank of it is equal to $|X|$. In other words, if $G_P[X]$ is an empty graph, that is, each pair of vertexes of $G_P[X]$ is nonadjacent, then the rank of X is equal to $|X|$. According to Definition 2.8, for any subset of the universe, the rank of the subset is the number of the maximal independent set contained in the subset. Therefore, we can get the following proposition.

Proposition 4.13. *Let M_P be the I-MIP induced by P . Then for all $X \subseteq U$, $r_P(X) = \max\{|Y| : Y \subseteq X, G_P[Y] \text{ is an empty graph}\}$ is the rank of X in M_P .*

If the set of bases \mathcal{B}_{M_P} of M_P has been obtained, then for all $X \subseteq U$; we can get the rank of X as follows:

$$r_+(X) = \max\{|B \cap X| : B \in \mathcal{B}_{M_P}\}. \quad (4.5)$$

It can be proved easily that $r_P(X) = r_+(X)$. So, r_+ is also the rank function of M_P .

Different from the rank of X in M_P , the closure of X is the maximal subset of U , which contains X and its rank is equal to X . For an element $y \in U - X$, if there is an element $x \in X$

such that $\{x, y\}$ form a circuit, then the rank of X is equal to it of $X \cup \{y\}$. That is, y belongs to the closure of X . Therefore, we can get the following proposition.

Proposition 4.14. *Let M_P be the I -MIP induced by P . Then for all $X \subseteq U$, $\text{cl}_P(X) = X \cup \{y \in U - X : x \in X \wedge xy \in E_P\}$ is the closure of X in M_P .*

Proof. According to Definition 2.9, we need to prove that $\text{cl}_P(X) = \text{cl}_{M_P}(X)$, that is, $X \cup \{y \in U - X : x \in X \wedge xy \in E_P\} = \{x \in U : r_{M_P}(X) = r_{M_P}(X \cup \{x\})\}$. It is obvious that, for all $X \subseteq U$, $X \subseteq \text{cl}_P(X)$. So, we just need to prove that for all $y \in U - X$ if there is an element $x \in X$ such that $xy \in E_P$ if and if only $y \in \{x \in U : r_{M_P}(X) = r_{M_P}(X \cup \{x\})\}$. According to (3.1), $xy \in E_P$ if and if only x and y belong to the same equivalence. According to Definition 2.8, for all $X \subseteq U$, $r_{M_P}(X)$ is equal to the number of the maximal independent set contained in X . According to Proposition 4.3, for all $X' \subseteq X$; if X' is a maximal independent set contained in X , then $X' \cup \{y\}$ is not an independent set. That is, $r_{M_P}(X) = |X'| = r_{M_P}(X' \cup \{y\}) = r_{M_P}(X \cup \{y\})$. Thus, for all $x \in U$, $x \in \text{cl}_P(X)$ if and if only $x \in \{x \in U : r_{M_P}(X) = r_{M_P}(X \cup \{x\})\}$. \square

From Proposition 4.14, it can be found that, for any element $y \in U - X$, if $xy \in E(G_P[X])$, then $y \in \text{cl}_P(X)$. Therefore, the closure of X can be equivalently represented as

$$\text{cl}_P(X) = X \cup \{y \in U - X : xy \in E(G_P[X])\}. \quad (4.6)$$

For any element $x \in U$, according to Figure 1, it can be found that if there is an element $y \in U - \{x\}$ such that $xy \in E_P$, then x and y must belong to the same equivalence class. Then we can get the following corollary.

Corollary 4.15. *Let $x \in U$. For all $T \in P$; if $x \in T$ then $\text{cl}_P(\{x\}) = \text{cl}_P(T)$.*

Next, we will discuss the hyperplane of the I -MIP. From the Definition 2.10, we know that a hyperplane of a matroid is a closed set and the rank of it is one less than the rank of the matroid. Because the rank of the I -MIP induced by P is equal to the cardinality of P , we can get the following proposition.

Proposition 4.16. *Let M_P be the I -MIP induced by P . Then $\mathcal{H}_P = \{U - T : T \in P\}$ is the hyperplane of M_P .*

Proof. According to (4.5) and Proposition 4.6, we know that the rank of the I -MIP induced by P is equal to $|P|$. Furthermore, in terms of Proposition 4.14 and Corollary 4.15, for all $T \in P$, $U - T$ is a closed set and $r_{M_P}(U - T) = |P| - 1$. So $U - T \in \mathcal{H}(M_P)$, that is, for all $X \in \mathcal{H}_P$, $X \in \mathcal{H}(M_P)$. Similarly, we can get that for all $X \in \mathcal{H}(M_P)$, $X \in \mathcal{H}_P$. \square

4.3. Approximations Established through I -MIP

So far, the base, circuit, rank function, closure, and hyperplane of a I -MIP are established. Next, we will further study the approximations in rough sets in this subsection through these characteristics.

Proposition 4.17. Let M_P be the I-MIP induced by P , $\mathcal{B}(M_P) = \mathcal{B}_P$. Then for all $X \subseteq U$,

$$\overline{R}(X) = \cup\{B - (B_X - X) : B \in \mathcal{B}_P \wedge (B_X - X) \subseteq B\}, \quad (4.7)$$

where $B_X \in \mathcal{B}_P$ a base having the maximal intersection with X .

Proof. For all $B \in \mathcal{B}_P$, $B \cap X$ is an independent set contained in X . Because B_X is a base having the maximal intersection with X , $r_{M_P}(X) = |B_X \cap X|$. Furthermore, for all $y \in B_X - X$, $[y]_R \cap X = \emptyset$. Let $S_1 = \{[y]_R : y \in B_X - X\}$ and $S_2 = P - S_1$. Then $\overline{R}(X) = U - \cup S_1 = \cup S_2$. If $B_X - X \subseteq B$, then $B - (B_X - X) \subseteq \cup S_2$. According to Corollary 4.7, for all $Y \subseteq \cup S_2$, if for all $T \in (P - S_2)$ and $|Y \cap T| = 1$, then $Y \cup (B_X - X) \in \mathcal{B}_P$. And $\cup\{Y \subseteq \cup S_2 : \text{for all } T \in (P - S_2), |Y \cap T| = 1\} = \cup S_2$. Therefore, $\cup S_2 = \cup\{B - (B_X - X) : B \in \mathcal{B}_P \wedge (B_X - X) \subseteq B\}$. That is, $\overline{R}(X) = \cup\{B - (B_X - X) : B \in \mathcal{B}_P \wedge (B_X - X) \subseteq B\}$. \square

In rough sets, an element in the lower approximation certainly belongs to X , while an element in the upper approximation possibly belongs to X [43]. And the boundary region of X is the set of elements in which each element does not certainly belong to either X or $\sim X$. In general, we can get the boundary region of X by the difference set of the lower and upper approximation of X . But here, we can provide a matroidal approach to obtain the boundary region of X firstly, and then the lower and the upper approximations should be established.

Proposition 4.18. Let M_P be the I-MIP induced by P , $\mathcal{C}(M_P) = \mathcal{C}_P$. Then for all $X \subseteq U$,

$$BN_R(X) = \cup\{C \in \mathcal{C}_P : |C \cap X| = 1\}. \quad (4.8)$$

Proof. According to Proposition 4.10 and Corollary 4.11, for all $C \in \mathcal{C}_P$, $\exists T \in P$ such that $C \subseteq T$. $|C \cap X| = 1$ means that each element of C does not certainly belong either to X or to $\sim X$. And then $\cup\{C \in \mathcal{C}_P : |C \cap X| = 1\}$ is the collection of all elements, which do not certainly belong either to X or to $\sim X$. So $BN_R(X) = \cup\{C \in \mathcal{C}_P : |C \cap X| = 1\}$. \square

Proposition 4.19. Let M_P be the I-MIP induced by P , $\mathcal{C}(M_P) = \mathcal{C}_P$. Then for all $X \subseteq U$,

$$\begin{aligned} \underline{R}(X) &= X - BN_R(X), \\ \overline{R}(X) &= X \cup BN_R(X), \end{aligned} \quad (4.9)$$

where $BN_R(X) = \cup\{C \in \mathcal{C}_P : |C \cap X| = 1\}$.

Proof. According to the definition of the boundary region and Proposition 4.18, it is straightforward. \square

Corollary 4.20. Let M_P be the I-MIP induced by P , $\mathcal{C}(M_P) = \mathcal{C}_P$ and $X \subseteq U$. Then for all $C \in \mathcal{C}_P$, $C \not\subseteq \overline{R}(X)$ if and only if for all $x \in X$, $\{x\} \in P$.

Proof. (\Rightarrow): According to Proposition 4.10 and Corollary 4.11, if for all $C \in \mathcal{C}_P$, $C \not\subseteq \overline{R}(X)$, then for all $T \in P$ and $|T| \geq 2$, $T \cap X = \emptyset$. That is, for all $x \in X$, $\{x\} \in P$.

(\Leftarrow): It is straightforward. \square

Proposition 4.21. *Let M_P be the I-MIP induced by P , $r_{M_P} = r_P$. Then for all $X \subseteq U$, the following equations hold:*

- (1) $\overline{R}(X) = \{x \in U : r_P(X) = r_P(X \cup \{x\})\}$,
- (2) $\overline{R}(X) = \cup \{T \in P : r_P(X) = r_P(X \cup T)\}$,
- (3) $\overline{R}(X) = \text{Max}(\{A \subseteq U : r_P(X) = r_P(A)\})$.

Proof. (1) According to Proposition 4.13, $r_P(X) = |Y|$ where $Y \subseteq X$ and for all $T \in P$, $|Y \cap T| \leq 1$. Let $T \in P$. If $|Y \cap T| = 0$, then $T \cap X = \emptyset$ and for all $t \in T$, $r_P(X) = r_P(X \cup \{t\}) - 1$, that is, $T \not\subseteq \overline{R}(X)$ and for all $t \in T$, $t \notin \{x \in U : r_P(X) = r_P(X \cup \{x\})\}$; else, if $|Y \cap T| = 1$, then $X \cap T \neq \emptyset$ and for all $x \in T$, $r_P(X) = r_P(X \cup \{x\})$, that is, $T \subseteq \overline{R}(X)$ and for all $t \in T$, $t \in \{x \in U : r_P(X) = r_P(X \cup \{x\})\}$. That is, $\overline{R}(X) = \{x \in U : r_P(X) = r_P(X \cup \{x\})\}$.

Similarly, we can prove that (2) and (3) are true. \square

Proposition 4.21 provides three ways to get the upper approximation of X with rank function. This intensifies our understanding to rank function of M_P .

Proposition 4.22. *Let M_P be the I-MIP induced by P , $\text{cl}_{M_P} = \text{cl}_P$. Then for all $X \subseteq U$,*

$$\overline{R}(X) = \text{cl}_P(X). \quad (4.10)$$

Proof. According to Proposition 4.14 and Corollary 4.15, we can get that $\text{cl}_P(X) = \cup \{T \in P : x \in X \wedge x \in T\}$. Therefore, according to the definition of the upper approximation, it is obvious that $\overline{R}(X) = \text{cl}_P(X)$. \square

The compact formulation of the upper approximation in Proposition 4.22 indicates that the closure is an efficient way to get the approximations in rough sets.

Proposition 4.23. *Let M_P be the I-MIP induced by P , $\mathcal{H}(M_P) = \mathcal{H}_P$. Then for all $X \subseteq U$,*

$$\overline{R}(X) = \cup \{\sim H : H \in \mathcal{H}_P \wedge X - H \neq \emptyset\}. \quad (4.11)$$

Proof. According to Proposition 4.16, for all $H \in \mathcal{H}_P$, $U - H \in P$, that is, $\sim H \in P$. And if $X - H \neq \emptyset$, then $X \cap (\sim H) \neq \emptyset$. Therefore, in terms of (2.1), $\sim H \subseteq \overline{R}(X)$. Thus, $\overline{R}(X) = \cup \{\sim H : H \in \mathcal{H}_P \wedge X - H \neq \emptyset\}$. \square

5. Matroidal Structure of Rough Sets Constructed from the Viewpoint of Cycle

In Section 3.2, the relationships between a cycle and an equivalence class are analyzed in detail. And a partition over the universe is transformed to a graph composed of some cycles. So, inspired by the cycle matroid introduced in [39], we will construct the matroidal structure of rough sets from this viewpoint. A new matroid will be established and the characteristics of it are studied. Then the approximations in rough sets are investigated via these characteristics.

For convenience, in this section, we suppose that U is a universe, R an equivalence relation over U , and $P = U/R$ the partition. And $G'_p = (U, E'_p)$ is the graph induced by P , where $E'_p = \cup\{E_{T_i} : T_i \in P\}$.

5.1. The Second Type of Matroidal Structure of Rough Sets

In this subsection, the second type of matroid induced by a partition is defined. Similar to the discussion of I -MIP, the base, circuit, rank function, and closure of the second type of matroid are investigated.

From the analysis in Section 3.2, we know that for all $T \in P$ and for all $K \subset T$, C_K is not a cycle. Obviously, any subgraph of C_K is also not a cycle. Therefore, we can get the following proposition.

Proposition 5.1. *Let $\mathcal{O}'_p = \{X \subseteq U : \text{for all } T \in P, C_T \not\subseteq G'_p[X]\}$. Then there exists a matroid M' on U such that $\mathcal{O}(M') = \mathcal{O}'_p$.*

Proof. According to Definition 2.3, we need to prove that \mathcal{O}'_p satisfies (I1)~(I3). In terms of Section 3.2, for all $T \in P$, C_T is the cycle whose vertices set is T , that is, $C_T = (T, E_T)$ where E_T is the set of edges of C_T . So, it is obvious that \mathcal{O}'_p satisfies (I1) and (I2). Here, we just need to prove \mathcal{O}'_p satisfies (I3).

Let $I_1, I_2 \in \mathcal{O}'_p$, $|I_1| < |I_2|$ and $I_2 - I_1 = \{e_1, e_2, \dots, e_m\}$ ($1 \leq m \leq |U|$). Suppose that for all $e_i \in I_2 - I_1$ for $1 \leq i \leq m$, $\exists T_{e_i} \in P$ such that $T_{e_i} \subseteq I_1 \cup \{e_i\}$, that is, $T_{e_i} - \{e_i\} \subseteq I_1$. Because $|T_{e_i} - \{e_i\}| \geq 1$, $|T_{e_1} - \{e_1\}| + |T_{e_2} - \{e_2\}| + \dots + |T_{e_m} - \{e_m\}| \geq m = |I_2 - I_1|$. It is obvious that $T_{e_i} \not\subseteq (I_1 \cap I_2) \cup \{e_i\}$. So $|I_1| = |T_{e_1} - \{e_1\}| + |T_{e_2} - \{e_2\}| + \dots + |T_{e_m} - \{e_m\}| + |I_1 \cap I_2|$. Since $|I_2| = |I_2 - I_1| + |I_2 \cap I_1|$, we can get that $|I_1| \geq |I_2|$. It is contradictory with the known conditions that $|I_1| < |I_2|$. So $\exists e_i \in I_2 - I_1$ such that for all $T \in P$, $T \not\subseteq I_1 \cup \{e_i\}$, namely, $C_T \not\subseteq G'_p[I_1 \cup \{e_i\}]$. That is $I_1 \cup \{e_i\} \in \mathcal{O}'_p$. As a result, \mathcal{O}'_p satisfies (I1)~(I3). That is, there exists a matroid M' on U such that $\mathcal{O}(M') = \mathcal{O}'_p$. \square

The matroid mentioned in Proposition 5.1 is established from the viewpoint of cycle. It is a new type of matroid induced by a partition and is defined as follows.

Definition 5.2 (The second type of matroid induced by a partition). The second type of matroid induced by a partition P over U , denoted by II -MIP, is such a matroid whose ground set $E = U$ and independent sets $\mathcal{O} = \{X \subseteq U : \text{for all } T \in P, C_T \not\subseteq G[X]\}$.

Because of the intuition of a graph, it is easy to understand the matroid established in Definition 5.2. In fact, we can formulate a II -MIP as follows.

Proposition 5.3. *Let $\mathcal{O}'_+ = \{\cup_{i=1}^n S_i : S_i \subset T_i \wedge T_i \in P\}$ where $n = |P|$. Then $M'_+ = (U, \mathcal{O}'_+)$ is a II -MIP.*

Proof. Because $S_i \subset T_i$, for all $X \in \mathcal{O}'_+$, $X \cap T \subset T$ for each $T \in P$. That means for all $T \in P$, $C_T \not\subseteq G'_p[X]$, that is, $X \in \mathcal{O}'_p$. Conversely, for all $X \in \mathcal{O}'_p$, since for all $T \in P$, $C_T \not\subseteq G'_p[X]$, we can get that $T \not\subseteq X$, that is, $T \cap X \subset T$. So $X \in \mathcal{O}'_+$. As a result, $M'_+ = (U, \mathcal{O}'_+)$ is a II -MIP. \square

In terms of Propositions 5.1 and 5.3, we can find that the two matroids established in them are equivalent. That is to say for any matroid $M = (U, \mathcal{O})$, if $\mathcal{O} = \mathcal{O}'_+$, then M is a II -MIP. So, the following corollary can be obtained.

Corollary 5.4. $\mathcal{O}'_P = \mathcal{O}'_+$.

Similar to the axiomatization of *I-MIP*, we will axiomatize *II-MIP* with the set of circuits of it in the following.

Theorem 5.5. Let $M = (U, \mathcal{O})$ be a matroid. Then M is a *II-MIP* if and only if for all $C_1, C_2 \in \mathcal{C}(M)$, $C_1 \cap C_2 = \emptyset$ and $\cup \mathcal{C}(M) = U$.

Proof. (\Rightarrow): Let $M = (U, \mathcal{O})$ be the *II-MIP* induced by P . Therefore, for all $T \in P$, $T \notin \mathcal{O}$, and for all $X \subset T$, $X \in \mathcal{O}$. And then for all $T \in P$, $T \in \mathfrak{D}(M)$. So, according to (2.2) and Definition 2.5, $\mathcal{C}(M) = P$. Thus, for all $C_1, C_2 \in \mathcal{C}(M)$, $C_1 \cap C_2 = \emptyset$, and $\cup \mathcal{C}(M) = U$.

(\Leftarrow): Since for all $C_1, C_2 \in \mathcal{C}(M)$, $C_1 \cap C_2 = \emptyset$, and $\cup \mathcal{C}(M) = U$, we can regard the $\mathcal{C}(M)$ as a partition over U . Furthermore, $\mathfrak{D}(M) = \{X \subseteq U : \exists C \in \mathcal{C}(M) \text{ s.t. } C \subseteq X\}$. Because $\mathcal{O} = 2^U - \mathfrak{D}(M)$, for all $X \in \mathcal{O}$, $\nexists C \in \mathcal{C}(M)$ such that $C \subseteq X$. According to Proposition 5.1 and Definition 5.2, we can get that $\mathcal{O} = \mathcal{O}'_+ = \mathcal{O}'_P$. That is, $M = (U, \mathcal{O})$ is a *II-MIP*. \square

Theorem 4.5 shows there is one-to-one correspondence between a partition and the *I-MIP* induced by the partition. In the following theorem, we will discuss the relationship between a partition and the *II-MIP* induced by the partition.

Theorem 5.6. Let \mathcal{P} be the collection of all partitions over U , \mathcal{M}' the set of all *II-MIP* induced by partitions of \mathcal{P} , $g : \mathcal{P} \rightarrow \mathcal{M}'$, that is, for all $P \in \mathcal{P}$, $g(P) = M'_P$ where M'_P is the *II-MIP* induced by P . Then g satisfy the following conditions:

- (1) for all $P_1, P_2 \in \mathcal{P}$, if $P_1 \neq P_2$ then $g(P_1) \neq g(P_2)$,
- (2) for all $M' \in \mathcal{M}'$, $\exists P_{M'} \in \mathcal{P}$ s.t. $(P_{M'}) = M'$.

Proof. (1) Let P_1 and P_2 are two different partitions over U , and $M'_{P_1} = (U, \mathcal{O}'_{P_1})$, $M'_{P_2} = (U, \mathcal{O}'_{P_2})$ are two *II-MIP* induced by P_1 and P_2 , respectively. We need to prove that there is an $I_1 \in \mathcal{O}'_{P_1}$ such that $I_1 \notin \mathcal{O}'_{P_2}$, or there is an $I_2 \in \mathcal{O}'_{P_2}$ such that $I_2 \notin \mathcal{O}'_{P_1}$. Because $P_1 \neq P_2$, there must be an equivalence class $T_1 \in P_1$ such that $T_1 \notin P_2$. Suppose that for all $T \in P_2$, $T_1 \not\subseteq T$. Then $\exists X \subset T_1$ such that $X \in \{S : S \subset T_1\}$ and $X \notin \{S : S \subset T, \text{ for all } T \in P_2\}$. According to Proposition 5.3, $X \in \mathcal{O}'_{P_1}$ and $X \notin \mathcal{O}'_{P_2}$. Conversely, if $\exists T_2 \in P_2$ such that $T_1 \subseteq T_2$, then $\exists X \subset T_2$ such that $X \in \{S : S \subset T_2\}$ and $X \notin \{S : S \subset T, \text{ for all } T \in P_1\}$. Thus, according to Proposition 5.3, $X \in \mathcal{O}'_{P_2}$ and $X \notin \mathcal{O}'_{P_1}$.

(2) Let $M' = (U, \mathcal{O}')$ be a *II-MIP* matroid. According to Theorem 5.5, $\mathcal{C}(M')$ is a partition on U . That is, $g(\mathcal{C}(M')) = M'$. \square

Theorem 5.6 shows that there is also a bijection between a partition and the *II-MIP* induced by the partition.

5.2. Characteristics of *II-MIP*

The *II-MIP* is established from the viewpoint of cycle. There is a big difference between the formulations of *I-MIP* and *II-MIP* and the same as the characteristics between them. In this subsection, we will formulate the characteristics of *II-MIP*.

A base of a matroid is one of the maximal independent sets of the matroid. From Propositions 5.1 and 5.3, the cardinality of one of the maximal independent sets is equal to

$|U| - |P|$ and just one element of each equivalence class does not belong to the independent set.

Proposition 5.7. *Let M'_P be the II-MIP induced by P , $n = |P|$. Then $\mathcal{B}'_P = \{\bigcup_{i=1}^n (T_i - x_i) : T_i \in P \wedge x_i \in T_i\}$ is the set of bases of M'_P .*

Proof. According to Definition 2.7, Proposition 5.3, and Corollary 5.4, it is straightforward. \square

So any equivalence class is not contained in some base of a II-MIP. And there will be a cycle if a new element is put in the base. Specifically, if for all $T \in P$, $|T| = 1$, then $\mathcal{B}'_P = \{\emptyset\}$. Next, we can also formulate the set of bases of a II-MIP from the viewpoint of graph as follows.

Corollary 5.8. $\mathcal{B}'_P = \text{Max}(\{B \subseteq U : G'_P[B] \text{ does not contain a cycle}\})$.

Proof. According to Propositions 5.1 and 5.7, it is straightforward. \square

From Proposition 5.7 and Corollary 5.8, we can find that for all $T \in P$, if $|T| = 1$, then T is not contained in any base of the II-MIP induced by P . Therefore, we can get the following corollary.

Corollary 5.9. *Let $\mathcal{B}(M'_P) = \mathcal{B}'_P$. Then $\mathcal{B} \cap \mathcal{B}'_P = \emptyset$.*

As the analysis in Section 3.2, any equivalence class of a partition can be converted to a cycle. And any proper subset of an equivalence class does not form a cycle. So we can formulate the set of circuits of a II-MIP as follows.

Proposition 5.10. *Let M'_P be the II-MIP induced by P . Then $\mathcal{C}'_P = P$ is the set of circuits of M'_P .*

Proof. According to Theorem 5.5, it is straightforward. \square

Likewise, from the viewpoint of graph, we can get another formulation of the set of circuits of a II-MIP.

Corollary 5.11. $\mathcal{C}'_P = \{C \subseteq U : G'_P[C] \text{ is a cycle}\}$.

Proof. According to the definition of G'_P and Proposition 5.10, it is straightforward. \square

Next, we will formulate the rank function of II-MIP.

Proposition 5.12. *Let M'_P be the II-MIP induced by P . Then for all $X \subseteq U$, $r'_P(X) = \max\{|Y| : Y \subseteq X \wedge (\text{for all } T \in P, T \not\subseteq Y)\}$ is the rank of X in M'_P .*

Proof. According to Definition 2.8 and Proposition 5.3, it is straightforward. \square

In all the characteristics of a matroid introduced in this paper, the rank function of a matroid is the one and only one numeric characteristic. In the following content, we will further study some properties of the rank function of II-MIP.

Theorem 5.13. *Let M'_P be the II-MIP induced by P , $r'_P = r_{M'_P}$, and $X \subseteq U$ an R-definable set. For all $Y \subseteq U$, if $X \cap Y = \emptyset$, then $r'_P(X \cup Y) = r'_P(X) + r'_P(Y)$.*

Proof. Because X is an R -definable set and $X \cap Y = \emptyset$, according to Proposition 5.3 and Corollary 5.4, for all $I_1 \in \text{Max}(\{I \in \mathcal{D}'_P : I \subseteq X\})$ and for all $I_2 \in \text{Max}(\{I \in \mathcal{D}'_P : I \subseteq Y\})$, $I_1 \cap I_2 = \emptyset$. Furthermore, $I_1 \cup I_2 \in \mathcal{D}'_P$ and $I_1 \cup I_2 \in \text{Max}(\{Z \subseteq X \cup Y : \text{for all } T \in P, T \not\subseteq Z\})$. According to Definition 2.8, $r'_P(X) = |I_1|$ and $r'_P(Y) = |I_2|$. Because $I_1 \cap I_2 = \emptyset$, $|I_1 \cup I_2| = |I_1| + |I_2|$. As a result, $r'_P(X \cup Y) = r'_P(X) + r'_P(Y)$. \square

Theorem 5.13 provides a way to separate the rank of a subset into the sum of ranks of two disjoint sets contained in the subset. And one of the two disjoint sets is a definable set. It is obvious that any subset of the universe can be represented as the union of a definable set, and an indefinable set which are disjoint and contained in the subset. This will be very helpful to some proofs in the following. In order to study the properties of the rank function of II -MIP conveniently, the concept of lower approximation number will be defined as follows.

Definition 5.14 (Lower approximation number). Let R be a relation on U . For all $X \subseteq U$, $f_{*R}(X) = |\{RN_R(x) : x \in U \wedge RN_R(x) \subseteq X\}|$ is called the lower approximation number of X with respect to R . When there is no confusion, we omit the subscript R .

In classical rough sets, R usually refers to the equivalence relation. For a better understanding to the lower approximation number, an example is served.

Example 5.15. Let $U = \{a, b, c, d, e, f, g, h\}$ be the universe, R an equivalence relation over U , and $U/R = \{T_1, T_2, T_3, T_4\} = \{\{a, b\}, \{c\}, \{d, e, f\}, \{g, h\}\}$. Compute the lower approximation numbers of X_1, X_2 , and X_3 where $X_1 = \{a, b, c\}$, $X_2 = \{a, d, g\}$, and $X_3 = \{a, c, g, h\}$.

Because R is an equivalence relation, according to Definition 2.1, for all $x \in U$, $RN_R(x) = [x]_R$. Therefore, we can get that $f_*(X_1) = |\{T_1, T_2\}| = 2$, $f_*(X_2) = |\emptyset| = 0$, and $f_*(X_3) = |\{T_2, T_4\}| = 2$.

Similarly, according to Definition 2.2, we can get that $f^*(X_1) = |\{T_1, T_2\}| = 2$, $f^*(X_2) = |\{T_1, T_3, T_4\}| = 3$, and $f^*(X_3) = |\{T_1, T_2, T_4\}| = 3$.

Theorem 5.16. Let M'_P be the II -MIP induced by P , $r'_P = r_{M'_P}$. Then for all $X \subseteq U$, $r'_P(X) = |X| - f_*(X)$.

Proof. Let $A \subseteq X$ be the largest R -definable set contained in X . Then $X = A \cup (X - A)$. Thus $X - A$ is an R -indefinable set and for all $T \in P$, $T \not\subseteq X - A$. According to Proposition 5.12 and Definition 5.14, $r'_P(A) = |A| - f_*(X)$ and $r'_P(X - A) = |X - A| = |X| - |A|$. Therefore, in terms of Theorem 5.13, $r'_P(X) = r'_P(A \cup (X - A)) = r'_P(A) + r'_P(X - A) = |X| - f_*(X)$. \square

Theorem 5.16 combines the lower approximation number and the rank function of II -MIP closely. And the formulation is very simple. This is useful to study rough sets with matroidal approaches and vice versa.

Lemma 5.17. Let M'_P be the II -MIP induced by P , $r_{M'_P} = r'_P$. Then for all $X \subseteq U$, $r'_P(X) + r'_P(\sim X) = |U| - (f_*(X) + f_*(\sim X))$.

Proof. Let $A \subseteq X$ be the largest R -definable set contained in X and $B \subseteq \sim X$ the largest R -definable set contained in $\sim X$. Then, in terms of Theorem 5.16, $r'_P(X) = r'_P(A \cup (X - A)) = |X| - f_*(X)$ and $r'_P(\sim X) = r'_P(B \cup (\sim X - B)) = |\sim X| - f_*(\sim X)$. So $r'_P(X) + r'_P(\sim X) = |X| + |\sim X| - f_*(X) - f_*(\sim X) = |U| - (f_*(X) + f_*(\sim X))$. \square

The last lemma discusses the relationship between the *II-MIP* ranks of a subset and its complementary set. For a subset X of U , if X is a definable set then $\sim X$ is also a definable set. So, we can get the following lemma.

Lemma 5.18. *Let M'_P be the *II-MIP* induced by P , $r'_P = r_{M'_P}$. Then for all $X \subseteq U$, X is an R -definable set if and only if $r'_P(X) + r'_P(\sim X) = |U| - |P|$.*

Proof. (\Rightarrow): According to Theorems 5.13 and 5.16 and Lemma 5.17, it is straightforward.

(\Leftarrow): According to Lemma 5.17, $|P| = f_*(X) + f_*(\sim X)$. Then, according to Definition 5.14, X is an R -definable set. \square

From Definition 2.9, we know that, for any subset X of U , if $x \in U - X$ such that the rank of $X \cup \{x\}$ is equal to the rank of X , then x belongs to the closure of X . In *II-MIP*, we can say that if $X \cup \{x\}$ contains one cycle more than X , then x belongs to the closure of X .

Proposition 5.19. *Let M'_P be the *II-MIP* induced by P . Then for all $X \subseteq U$, $\text{cl}'_P(X) = X \cup \{x \in U - X : \exists Y \subseteq X \text{ s.t. } Y \cup \{x\} \in P\}$ is the closure of X in M'_P .*

Proof. According to Proposition 5.12, for all $x \in X$, $r'_P(X) = r'_P(X \cup \{x\})$. Then, in terms of Definition 2.9, $X \subseteq \text{cl}'_{M'_P}(X)$. Let $Y_X \subseteq X$ and $|Y_X| = r'_P(X)$. For all $x \in U - X$, if $x \in \text{cl}'_{M'_P}(X)$ then $r'_P(X) = r'_P(X \cup \{x\})$. According to Proposition 5.12, for all $T \in P$, $T \not\subseteq Y_X$. That means $\exists T \in P$ such that $T \subseteq Y_X \cup \{x\}$, that is, $\exists Y \subseteq Y_X$ such that $Y \cup \{x\} \in P$. As a result $\text{cl}'_{M'_P}(X) = \text{cl}'_P(X)$. \square

The hyperplane of *II-MIP* can be formulated as follows.

Proposition 5.20. *Let M'_P be the *II-MIP* induced by P . Then $\mathcal{H}'_P = \{U - X : T \in P \wedge X \subseteq T \wedge |X| = 2\}$ is the hyperplane of M'_P .*

Proof. According to Definition 2.10, we need to prove that for all $H \in \mathcal{H}'_P$, H is a close set of M'_P , and $r_{M'_P}(H) = r_{M'_P}(U) - 1$. And more, we need to prove that for all $Y \subseteq U$, if $Y \notin \mathcal{H}'_P$, then Y is not a hyperplane of M'_P .

(1) Is a close set of M'_P .

For all $H \in \mathcal{H}'_P$, there is an equivalence class $T \in P$ and a subset $X \subseteq T$ such that $|X| = 2$ and $H = U - X$. Therefore, for all $Y \subseteq H$ and for all $x \in X$, $Y \cup \{x\}$ is not an equivalence class, that is, $Y \cup \{x\} \notin P$. That is, the set $\{x \in X : \exists Y \subseteq H \text{ s.t. } Y \cup \{x\} \in P\} = \emptyset$. Thus, according to Proposition 5.19, $\text{cl}'_{M'_P}(H) = H$. That is to say H is a close set of M'_P .

(2) For all $H \in \mathcal{H}'_P$, $r_{M'_P}(H) = r_{M'_P}(U) - 1$.

For any $T \in P$, since T is R -definable, by Theorem 5.13

$$\begin{aligned} r_{M'_P}(U) &= r_{M'_P}(T \cup \sim T) \\ &= r_{M'_P}(T) + r_{M'_P}(\sim T) \\ &= |T| - 1 + r_{M'_P}(\sim T). \end{aligned} \tag{5.1}$$

That is,

$$r_{M'_p}(\sim T) = r_{M'_p}(U) - |T| + 1. \quad (5.2)$$

Furthermore, since $\sim T$ is also R -definable, by Theorem 5.13

$$\begin{aligned} r_{M'_p}(H) &= r_{M'_p}((T - X) \cup \sim T) \\ &= r_{M'_p}(T - X) + r_{M'_p}(\sim T) \\ &= |T| - 2 + r_{M'_p}(\sim T). \end{aligned} \quad (5.3)$$

That is,

$$r_{M'_p}(H) = r_{M'_p}(\sim T) + |T| - 2. \quad (5.4)$$

Therefore, from (5.2) and (5.4), $r_{M'_p}(H) = r_{M'_p}(U) - 1$.

(3) For all $Y \subseteq U$, if $Y \notin \mathcal{H}'_p$, then Y is not a hyperplane of M'_p .

If $Y \notin \mathcal{H}'_p$, then there are two cases.

(1) $|U - Y| = 2$ and $U - Y \not\subseteq T$, for all $T \in P$.

(2) $|U - Y| \neq 2$.

Next, we discuss these two cases, respectively. □

Case 1. By Proposition 5.19, $\text{cl}'_p(Y) = U$ so Y is not a close set.

Case 2. if $|U - Y| = 1$, then $\text{cl}'_p(Y) = U$ so Y is not a close set. If $|U - Y| > 2$, then suppose that $\text{cl}'_p(Y) = Y$. In that case, for $T \in P$ such that $(U - Y) \cap T \neq \emptyset$, $|(U - Y) \cap T| \geq 2$. So

$$r_{M'_p}(Y) = \sum_{\substack{T \in P \\ (U-Y) \cap T \neq \emptyset}} (|T| - |(U - Y) \cap T|) + \sum_{\substack{T \in P \\ (U-Y) \cap T = \emptyset}} (|T| - 1). \quad (5.5)$$

In that case, $r_{M'_p}(Y)$ is not equal to $r_{M'_p}(U) - 1 = \sum_{T \in P} (|T| - 1) - 1$.

Summing up, $\mathcal{H}'_p = \{U - X : T \in P \wedge X \subseteq T \wedge |X| = 2\}$ is the hyperplane of M'_p .

5.3. Approximations Established through II-MIP

In this subsection, we will redefine the lower and the upper approximations with some characteristics of II-MIP. Three pairs of approximations are established.

Proposition 5.10 shows that the set of circuits of a II-MIP is equal to the partition which induces the II-MIP. So we can get the following proposition.

Proposition 5.21. Let M'_p be the II-MIP induced by P , $\mathcal{C}(M'_p) = \mathcal{C}'_p$. Then for all $X \subseteq U$,

$$\begin{aligned}\underline{R}(X) &= \cup\{C \in \mathcal{C}'_p : C \subseteq X\}, \\ \overline{R}(X) &= \cup\{C \in \mathcal{C}'_p : C \cap X \neq \emptyset\}.\end{aligned}\tag{5.6}$$

Proof. According to Proposition 5.10, (2.1), it is straightforward. \square

We know that the lower and the upper approximations of a subset are all definable sets. The lower approximation is the largest definable set contained in the subset. And the upper approximation is the smallest definable set, which contains the subset. Since Lemma 5.18 provides a way to decide whether a subset is a definable set, we can define the lower and the upper approximations as follows.

Proposition 5.22. Let M'_p be the II-MIP induced by P , $r_{M'_p} = r'_p$. Then for all $X \subseteq U$,

$$\begin{aligned}\underline{R}(X) &= \text{Max}(\{Y \subseteq X : r'_p(Y) + r'_p(\sim Y) = |U| - |P|\}), \\ \overline{R}(X) &= \text{Min}(\{X \subseteq Y : r'_p(Y) + r'_p(\sim Y) = |U| - |P|\}).\end{aligned}\tag{5.7}$$

Proof. According to Lemma 5.18, for all $X \subseteq U$, if $r'_p(X) + r'_p(\sim X) = |U| - |P|$, then X is an R -definable set. So $\text{Max}(\{Y \subseteq X : r'_p(Y) + r'_p(\sim Y) = |U| - |P|\})$ is the largest R -definable set contained in X . And $\text{Min}(\{X \subseteq Y : r'_p(Y) + r'_p(\sim Y) = |U| - |P|\})$ is the smallest R -definable set contained X . And then, according to (2.1), it is straightforward. \square

Next, we will establish the lower approximation via the closure of II-MIP.

Proposition 5.23. Let M'_p be the II-MIP induced by P , $\text{cl}_{M'_p} = \text{cl}'_p$. Then for all $X \subseteq U$,

$$\underline{R}(X) = \{x \in X : \text{cl}'_p(X) = \text{cl}'_p(X - \{x\})\}.\tag{5.8}$$

Proof. For all $T \in P$, if $T \subseteq X$, then for all $x \in T$, $T - \{x\} \subseteq X$. According to Proposition 5.19, $\text{cl}'_p(X) = \text{cl}'_p(X - \{x\})$. That is, $x \in \{x \in X : \text{cl}'_p(X) = \text{cl}'_p(X - \{x\})\}$. In terms of (2.1), $\underline{R}(X) = \{x \in X : \text{cl}'_p(X) = \text{cl}'_p(X - \{x\})\}$. \square

In terms of (P2), we can get the upper approximation by using the duality. That is, $\overline{R}(X) = \sim \underline{R}(\sim X) = \sim \text{cl}'_p(\sim X - \{x\})$.

6. Relationship between I-MIP and II-MIP

In the previous two sections, we have got two types of matroids induced by a partition from the viewpoints of complete graph and cycle, respectively. And it can be found that the matroidal characteristics of them are very different. But, if the two types of matroid are induced by the same partition, what are the relationships between them? In this section, we will study this issue.

Definition 6.1 (Dual matroid see [39]). Let $M = (E, \mathcal{O})$ be a matroid, and \mathcal{B} the set of bases of M . The dual matroid M^* is the matroid on the set E whose bases $\mathcal{B}^* = \text{Com}(\mathcal{B})$. If $\mathcal{O}(M) = \mathcal{O}(M^*)$, then M is called an identically self-dual matroid.

For convenience, in the following content, we suppose that U is the universe, R is an equivalence relation over U , $P = U/R$ is the partition on U , and M_P is the I -MIP induced by P and M'_P the II -MIP induced by P .

Proposition 6.2. *Let M_P^* be the dual matroid of M_P . Then $M_P^* = M'_P$.*

Proof. For all $B \in \mathcal{B}(M_P)$, according to Definition 6.1, $U - B \in \mathcal{B}(M_P^*)$. For all $T \in P$ such that $|T \cap B| = 1, \exists x \in T$ such that $T - \{x\} \subseteq U - B$. Therefore, $\mathcal{B}(M_P^*) = \{\bigcup_{i=1}^n (T_i - x_i) : T_i \in P \wedge x_i \in T_i\}$ where $n = |P|$. In terms of Proposition 5.7, $\mathcal{B}(M_P^*) = \mathcal{B}(M'_P)$. As a result, $M_P^* = M'_P$. \square

Proposition 6.2 shows that M_P and M'_P are dual matroids. This result is very interesting and helpful to study rough sets. Maybe people want to ask some questions about the relationships between M_P and M'_P as follows: whether a I -MIP and a II -MIP which induced by different partitions could be dual matroids? And whether two different I -MIP (or II -MIP) could be dual matroids? Next, we will answer them.

Proposition 6.3. *Let P_1 and P_2 be two partitions over U , M_{P_1} the I -MIP induced by P_1 , and M_{P_2} the II -MIP induced by P_2 . Then $M_{P_1}^* = M_{P_2}$ if and only if $P_1 = P_2$.*

Proof. According to Theorems 4.5 and 5.6 and Proposition 6.2, it is straightforward. \square

Proposition 6.3 shows that a I -MIP and a II -MIP induced by different partitions are not dual matroids.

Proposition 6.4. *Let P_1 and P_2 be two different partitions over U and M_{P_1} and M_{P_2} are the I -MIP induced by P_1 and P_2 , respectively. Then $M_{P_1}^* \neq M_{P_2}$.*

Proof. Suppose that $M_{P_1}^* = M_{P_2}$. From Proposition 6.2, $M_{P_1}^*$ is a II -MIP. In terms of Proposition 5.10, we can get that $\mathcal{C}(M_{P_1}^*) = P_1 = \mathcal{C}(M_{P_2})$. Therefore, according to Theorem 4.4, for all $T \in P_1, |T| = 2$. Thus $P_1 = P_2$. This is contrary to the known condition that P_1 and P_2 are two different partitions over U . As a result, $M_{P_1}^* \neq M_{P_2}$. \square

Proposition 6.5. *Let P_1 and P_2 be two different partitions over U and M_{P_1} and M_{P_2} are the II -MIP induced by P_1 and P_2 , respectively. Then $M_{P_1}^* \neq M_{P_2}$.*

Proof. Similar to the proof of Proposition 6.4, it is straightforward. \square

Propositions 6.4 and 6.5 show that two different I -MIP are not dual matroids. And the same as two different II -MIP. One can find that we emphasize in Propositions 6.4 and 6.5 that P_1 and P_2 are two different partitions over U . So, in Propositions 6.4 and 6.5, could $M_{P_1}^*$ be equal to M_{P_2} when $P_1 = P_2$? We will discuss this problem in the following proposition.

Proposition 6.6. *$M_P = M'_P$ if and only if for all $T \in P, |T| = 2$.*

Proof. (\Rightarrow): According to Definition 6.1, if $M_P = M'_P$, then $\mathcal{O}(M_P) = \mathcal{O}(M'_P)$. According to Propositions 4.1, 4.3, and 5.1, for all $T \in P, |T| = 2$.

(\Leftarrow): It is straightforward. \square

Proposition 6.6 provides a necessary and sufficient condition to decide whether a *I-MIP* or a *II-MIP* is a self-dual matroid. This gives a good answer to the previous question that whether $M_{P_1}^*$ could be equal to M_{P_2} when $P_1 = P_2$.

The set of circuits of a matroid is complementary to the set of hyperplanes of its dual matroid [39]. That is,

$$\begin{aligned}\mathcal{H}(M_P) &= \{U - C : C \in \mathcal{C}'_P\}, \\ \mathcal{H}(M'_P) &= \{U - C : C \in \mathcal{C}_P\}.\end{aligned}\tag{6.1}$$

Next, we will study when a hyperplane of M_P is also a hyperplane of M'_P .

Proposition 6.7. *Let $H \in \mathcal{H}(M_P)$. Then $H \in \mathcal{H}(M'_P)$ if and only if $U - H \in \mathcal{C}(M_P)$.*

Proof. (\Rightarrow): According to Proposition 4.16, $U - H \in P$, that is, $\exists T \in P$ such that $U - H = T$. If $H \in \mathcal{H}(M'_P)$, according to Proposition 5.20, then $|U - H| = |T| = 2$. According to Proposition 4.10 and Corollary 4.11, $T \in \mathcal{C}(M_P)$, that is, $U - H \in \mathcal{C}(M_P)$.

(\Leftarrow): If $U - H \in \mathcal{C}(M_P)$, then $|U - H| = 2$ and $\exists T \in P$ such that $T = U - H$. Therefore, according to Propositions 4.16 and 5.20, $H \in \mathcal{H}(M'_P)$. \square

From the Propositions 4.14 and 5.19, we can find, for any subset $X \subseteq U$, that the $\text{cl}'_P(X)$ is generally the subset of $\text{cl}_P(X)$. Now, we will study under what conditions that the $\text{cl}'_P(X)$ is certain the subset of $\text{cl}_P(X)$.

Proposition 6.8. *Let $X \subseteq U$, $\text{cl}_{M_P}(X) = \text{cl}_P(X)$, and $\text{cl}_{M'_P}(X) = \text{cl}'_P(X)$. Then $\text{cl}'_P(X) \subseteq \text{cl}_P(X)$ if and only if for all $x \in U - X$, $r'_P(\{x\}) \neq 0$.*

Proof. (\Rightarrow): Let $\text{cl}'_P(X) \subseteq \text{cl}_P(X)$. According to Propositions 4.14 and 5.19, we can get that $\{y \in U - X : \exists Y \subseteq X \text{ s.t. } Y \cup \{y\} \in P\} \subseteq \{y \in U - X : x \in X \wedge xy \in E_P\}$. That is, for all $y \in \{y \in U - X : \exists Y \subseteq X \text{ s.t. } Y \cup \{y\} \in P\}$, $y \in \{y \in U - X : x \in X \wedge xy \in E_P\}$. In terms of Section 4, we know that $E_P = \{xy : x \neq y \wedge (\exists T \in P \text{ s.t. } \{x, y\} \subseteq T)\}$. Therefore, for all $y \in \{y \in U - X : x \in X \wedge xy \in E_P\}$, $\{y\} \notin P$. And for all $y \in \{y \in U - X : \exists Y \subseteq X \text{ s.t. } Y \cup \{y\} \in P\}$, $\{y\} \notin P$. That is, for all $x \in U - X$, $r'_P(\{x\}) \neq 0$.

(\Leftarrow): According to Propositions 4.14 and 5.19, it is straightforward. \square

The formulation of rank functions of M_P and M'_P are very different. In consideration of that M_P and M'_P are dual matroids, it is meaningful to discuss the relationship between the rank functions of them.

Theorem 6.9. *Let $X \subseteq U$, $r_{M_P} = r_P$, and $r_{M'_P} = r'_P$. Then $r_P(X) = r'_P(X)$ if and only if $|X| = f^*(X) + f_*(X)$.*

Proof. (\Rightarrow): If $r_P(X) = r'_P(X)$, according to Theorem 5.16, $r_P(X) = |X| - f_*(X)$. Then, in terms of Definition 2.2 and Proposition 4.13, we can get that $r_P(X) = f^*(X)$. Therefore, $f^*(X) = |X| - f_*(X)$. And then $|X| = f_*(X) + f^*(X)$.

(\Leftarrow): With the same way used in the above proof, it is straightforward. \square

In Theorem 6.9, we make use of the lower and the upper approximations numbers to discuss the relationship between the rank functions of M_P and M'_P . It adequately reflects the close relation between rough sets and matroids.

7. Conclusions

We make a further study to the combination of rough sets and matroids. Two graphical ways are provided to establish and understand the matroidal structure of rough sets intuitively. For a better research on the relationships between rough sets and matroids, the concept of lower approximation number is proposed. And then, some meaningful results are obtained. For example, Theorem 5.16 shows that, for any subset $X \subseteq U$, the rank of X within the context of *II-MIP* is equal to the difference of the cardinality and the lower approximation number of X . And Theorem 6.9 indicates that the rank of X within the context of *I-MIP* is equal to it within the context of *II-MIP* if and only if the cardinality of X is equal to the difference of its lower and upper approximation numbers. Furthermore, the relationships between the two kinds of matroids established in Sections 4 and 5 are discussed. It is so exciting that the two kinds of matroids are dual matroids. And this is meaningful to the study of the combination of rough sets and matroids.

Matroids possess a sophisticated mathematical structure. And it has been widely used in real world. So we hope our work in this paper could be contributive to the theoretical development and applications of rough sets. In future works, we will study the axiomatization of rough sets with matroidal approaches and explore the wider applications of rough sets with matroids.

Acknowledgments

This work is in part supported by National Science Foundation of China under Grant nos. 60873077, 61170128 and the Natural Science Foundation of Fujian Province, China under Grant no. 2011J01374.

References

- [1] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning About Data*, Kluwer Academic, Boston, Mass, USA, 1991.
- [2] Z. Pawlak, "Rough sets," *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.
- [3] J. Pomykala, *Approximation, Similarity and Rough Constructions*, vol. CT-93-07 of *ILLC Prepublication*, University of Amsterdam, 1993.
- [4] R. Slowinski and D. Vanderpooten, "Similarity relation as a basis for rough approximations," ICS Research Report 53/95, Warsaw University of Technology, 1995, Also in *Advances in Machine Intelligence and Soft-Computing*, P. P. Wang, Ed., pp. 17–33, Raleigh, NC, USA: Bookwrights, 1997.
- [5] A. Skowron and J. Stepaniuk, "Tolerance approximation spaces," *Fundamenta Informaticae*, vol. 27, no. 2-3, pp. 245–253, 1996.
- [6] E. Bryniarski, "A calculus of rough sets of the first order," *Bulletin of the Polish Academy of Sciences*, vol. 36, no. 16, pp. 71–77, 1989.
- [7] J. A. Pomykala, "Approximation operations in approximation space," *Bulletin of the Polish Academy of Sciences*, vol. 35, no. 9-10, pp. 653–662, 1987.
- [8] W. Zhu, "Generalized rough sets based on relations," *Information Sciences*, vol. 177, no. 22, pp. 4997–5011, 2007.
- [9] W. Zhu and F. Y. Wang, "On three types of covering-based rough sets," *IEEE Transactions on Knowledge and Data Engineering*, vol. 19, no. 8, pp. 1131–1144, 2007.
- [10] W. Zhu and F. Y. Wang, "Reduction and axiomization of covering generalized rough sets," *Information Sciences*, vol. 152, pp. 217–230, 2003.
- [11] Q. Hu, D. Yu, J. Liu, and C. Wu, "Neighborhood rough set based heterogeneous feature subset selection," *Information Sciences*, vol. 178, no. 18, pp. 3577–3594, 2008.

- [12] Y. Y. Yao, "Neighborhood systems and approximate retrieval," *Information Sciences*, vol. 176, no. 23, pp. 3431–3452, 2006.
- [13] F. Min, H. He, Y. Qian, and W. Zhu, "Test-cost-sensitive attribute reduction," *Information Sciences*, vol. 181, no. 22, pp. 4928–4942, 2011.
- [14] Z. Wang, L. Shu, and X. Ding, "Reduction of neighborhood-based generalized rough sets," *Journal of Applied Mathematics*, vol. 2011, Article ID 409181, 22 pages, 2011.
- [15] D. Dubois and H. Prade, "Rough fuzzy sets and fuzzy rough sets," *International Journal of General Systems*, vol. 17, no. 2-3, pp. 191–209, 1990.
- [16] G. L. Liu, "Generalized rough sets over fuzzy lattices," *Information Sciences*, vol. 178, no. 6, pp. 1651–1662, 2008.
- [17] J. S. Mi and W. X. Zhang, "An axiomatic characterization of a fuzzy generalization of rough sets," *Information Sciences*, vol. 160, no. 1–4, pp. 235–249, 2004.
- [18] D. Pei, "A generalized model of fuzzy rough sets," *International Journal of General Systems*, vol. 34, no. 5, pp. 603–613, 2005.
- [19] W. Zhu, "A Class of covering-based fuzzy rough sets," in *Proceedings of the 4th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD '07)*, vol. 1, pp. 7–11, Haikou, China, August 2007.
- [20] T. J. Li, W. X. Zhang, and J. M. Ma, "Rough sets on atomic boolean lattices," in *Proceedings of IEEE International Conference on Granular Computing*, vol. 1, pp. 176–179, July 2005.
- [21] Z. Pawlak and A. Skowron, "Rough sets and Boolean reasoning," *Information Sciences*, vol. 177, no. 1, pp. 41–73, 2007.
- [22] Y. Y. Yao, "A comparative study of fuzzy sets and rough sets," *Information Sciences*, vol. 109, no. 1–4, pp. 227–242, 1998.
- [23] G. Qi and W. Liu, "Rough operations on Boolean algebras," *Information Sciences*, vol. 173, no. 1–3, pp. 49–63, 2005.
- [24] E. F. Lashin, A. M. Kozae, A. A. Abo Khadra, and T. Medhat, "Rough set theory for topological spaces," *International Journal of Approximate Reasoning*, vol. 40, no. 1-2, pp. 35–43, 2005.
- [25] W. J. Liu, "Topological space properties of rough sets," in *Proceedings of The 3rd International Conference on Machine Learning and Cybernetics*, vol. 4, pp. 2353–2355, August 2004.
- [26] A. Wasilewska, "Topological rough algebras," in *Rough Sets and Data Mining*, T. Y. Lin and N. Cercone, Eds., pp. 411–425, Kluwer Academic, Boston, Mass, USA, 1997.
- [27] W. Zhu, "Topological approaches to covering rough sets," *Information Sciences*, vol. 177, no. 6, pp. 1499–1508, 2007.
- [28] D. G. Chen, W. X. Zhang, D. Yeung, and E. Tsang, "Rough approximations on a complete completely distributive lattice with applications to generalized rough sets," *Information Sciences*, vol. 176, no. 13, pp. 1829–1848, 2006.
- [29] H. R. Li, W. X. Zhang, and H. Wang, "Classification and reduction of attributes in concept lattices," in *Proceedings of IEEE International Conference on Granular Computing*, pp. 142–147, May 2006.
- [30] W. X. Zhang, Y. Y. Yao, and Y. Liang, *Rough Set and Concept Lattice*, Xi'an Jiaotong University Press, 2006.
- [31] Y. Y. Yao and T. Y. Lin, "Generalization of rough sets using modal logics," *Intelligent Automation and Soft Computing*, vol. 2, pp. 103–120, 1996.
- [32] W. Zhu and S. Wang, "Matroidal approaches to generalized rough sets based on relations," *International Journal of Machine Learning and Cybernetics*, vol. 2, no. 4, pp. 273–279, 2011.
- [33] S. P. Wang, W. Zhu, and F. Min, "Transversal and function matroidal structures of covering-based rough sets," in *Rough Sets and Knowledge Technology*, pp. 146–155, 2011.
- [34] H. Whitney, "On the abstract properties of linear dependence," *American Journal of Mathematics*, vol. 57, no. 3, pp. 509–533, 1935.
- [35] R. J. Wilson, "An introduction to matroid theory," *The American Mathematical Monthly*, vol. 80, pp. 500–525, 1973.
- [36] W. Zhu and F. Y. Wang, "Topological properties in covering-based rough sets," in *Proceedings of the 4th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD '07)*, vol. 1, pp. 289–293, Haikou, China, August 2007.
- [37] B. Bollobás, *Modern Graph Theory*, vol. 184 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1998.
- [38] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 2nd edition, 2000.
- [39] H. J. Lai, *Matroid Theory*, New Frontiers of Science, Higher Education Press, 2002.
- [40] P. J. Cameron, "Notes on matroids and codes," 1998.
- [41] Wikipedia: Glossary of graph theory—wikipedia, the free encyclopedia, 2011.

- [42] J. A. Bondy and U. S. R. Murty, *Graph Theory*, vol. 244 of *Graduate Texts in Mathematics*, Springer, 2008.
- [43] Y. Y. Yao, S. K. M. Wong, and T. Y. Lin, "A review of rough set models," in *Rough Sets and Data Mining*, Kluwer Academic, Boston, Mass, USA, 1997.