Research Article

f-Orthomorphisms and f-Linear Operators on the Order Dual of an f-Algebra

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We consider the f-orthomorphisms and f-linear operators on the order dual of an f-algebra. In particular, when the f-algebra has the factorization property (not necessarily unital), we prove that the orthomorphisms, f-orthomorphisms, and f-linear operators on the order dual are precisely the same class of operators.

1. Introduction

Let A be an f-algebra with °(A−) = {0}. Recall that we can define a multiplication on (A−)n, the order continuous part of the order bidual of A, with respect to which (A−)n can also be made an f-algebra. This is done in three steps:

1. \( A \times A \to A \)
   \( (a, f) \mapsto f \cdot a : (f \cdot a)(b) = f(ab) \) for \( b \in A \),

2. \( (A−)n × A− \to A− \)
   \( (F, f) \mapsto F \cdot f : (F \cdot f)(a) = F(f \cdot a) \) for \( a \in A \),

3. \( (A−)n × (A−)n \to (A−)n \)
   \( (F, G) \mapsto F \cdot G : (F \cdot G)(f) = F(G \cdot f) \) for \( f \in A− \).

With the so-called Arens multiplication defined in step (3), \( (A−)n \) is an Archimedean (and hence commutative) f-algebra. Moreover, if A has a multiplicative unit, then \( (A−)n = (A−)− \), the whole order bidual of A. The mapping \( V : (A−)n \to \text{Orth}(A−) \) defined by \( V(F) = V_F \) for all \( F \in (A−)n \), where \( V_f(f) = F \cdot f \) for every \( f \in A− \), is an algebra and Riesz isomorphism. See [1, 2] for details.
Let $A$ be an $f$-algebra. A Riesz space $L$ with $^\sim(L^-) = \{0\}$ is said to be an (left) $f$-module over $A$ (cf. [2, 3]) if $L$ is a left module over $A$ and satisfies the following two conditions:

(i) for each $a \in A^*$ and $x \in L^*$, we have $ax \in L^*$,

(ii) if $x \perp y$, then for each $a \in A$, we have $a \cdot x \perp y$.

When $A$ is an $f$-algebra with unit $e$, saying $L$ is a unital $f$-module over $A$ implies that the left multiplication satisfies $e \cdot x = x$ for all $x \in L$. From Corollary 2.3 in [2], we know that if $L$ is an $f$-module over $A$, then $L^-$ is an $f$-module over $A$ (and $(A^-)_n$). The $f$-module $L$ over $A$ with unit $e$ is said to be topologically full with respect to $A$ if for two arbitrary vectors $x, y$ satisfying $0 \leq y \leq x$ in $L$, there exists a net $0 \leq a_n \leq e$ in $A$ such that $a_n \cdot x \rightarrow y$ in $\sigma(L, L^-)$. If $L$ is topologically full with respect to $A$, then $L^-$ is topologically full with respect to $(A^-)_n$ [2, Proposition 3.12].

Let $A$ be a unital $f$-algebra, and, $L, M$ be $f$-modules over $A$. $T \in L_b(L, M)$ is called an $f$-linear operator if $T(a \cdot x) = a \cdot Tx$ for each $a \in A$ and $x \in L$. The collection of all $f$-linear operators will be denoted by $L_b(L, M; A)$. For each $x \in L$ and $f \in L^*$, we can define $\psi_{x,f} \in A^-$ by $\psi_{x,f}(a) = f(a \cdot x)$ for all $a \in A$. Let $S(x) := \{\psi_{x,f} : f \in L^-\}$. Then $S(x)$ is an order ideal in $A^-$ [2]. $T \in L_b(L, M)$ is said to be an $f$-orthomorphism if $S(Tx) \subseteq S(x)$ for each $x \in L$. The collection of all $f$-orthomorphisms will be denoted by Orth$(L, M; A)$. Turan [2] showed that Orth$(L, M; A) = L_b(L, M; A)$ whenever $M$ is topologically full with respect to $A$.

Clearly, $A^-$ is an $f$-module over the $f$-algebras $A$ and $(A^-)_n$, respectively. If $A$ is unital, then $A$ is topologically full with respect to itself ([2, Proposition 2.6]). From the above remarks we know that $A^-$ is topologically full with respect to $(A^-)_n$, and hence, the $f$-orthomorphisms and $f$-linear operators are precisely the same class of operators, that is,

$$\text{Orth}(A^-, A^-; (A^-)_n) = L_b(A^-, A^-; (A^-)_n).$$

An $f$-algebra $A$ is said to be square-root closed whenever for any $a \in A$ there exists $b \in A$ such that $|a| = b^2$. An immediate example is that a uniformly complete $f$-algebra with unit element is square-root closed [4]. However, a square-root closed $f$-algebra is not necessarily unital. For instance, $c_0$, with the familiar coordinatewise operations and ordering, is a square-root closed $f$-algebra without unit. We recall that an Archimedean $f$-algebra $A$ is said to have the factorization property if, given $a \in A$, there exist $b, c \in A$ such that $a = bc$. It should be noted that if $A$ is unital or square-root closed, then $A$ has the factorization property.

In this paper, we do not have to assume that the $f$-algebras are unital. We modify the definition of the $f$-orthomorphism introduced by Turan [2, Definition 3.7] and consider the $f$-orthomorphisms and $f$-linear operators on the order dual of an $f$-algebra. In particular, when the $f$-algebra with separating order dual has the factorization property, we prove that the orthomorphisms, $f$-orthomorphisms, and $f$-linear operators on the order dual are precisely the same class of operators, that is, the above equality (*) still holds.

Our notions are standard. For the theory of Riesz spaces, positive operators, and $f$-algebras, we refer the reader to the monographs [5–7].

2. $f$-Orthomorphisms on the Order Dual

Let $A$ be an $f$-algebra with separating order dual (and hence $A$ Archimedean!) and $f \in A^-$.

We consider the mapping $T_f : (A^-)_n \rightarrow A^-$ defined by $T_f(F) = F \cdot f$ for all $F \in (A^-)_n$. 

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It should be noted that the mapping \( V : (A^-)_n^\sim \rightarrow \text{Orth}(A^-) \) defined by \( V(F) = V_f \) for all \( F \in (A^-)_n^\sim \), where \( V_f(f) = F \cdot f \) for every \( f \in A^- \), is an algebra and Riesz isomorphism (cf. [2, Proposition 2.2]).

**Theorem 2.1.** For \( 0 \leq f \in A^- \), \( T_f \) is an interval preserving lattice homomorphism.

**Proof.** Clearly, \( T_f \) is linear and positive. Since the mapping \( V \) is a lattice homomorphism and \( V_f, V_G \in \text{Orth}(A^-) \) for \( F, G \in (A^-)_n^\sim \), we have

\[
T_f(F \vee G) = (F \vee G) \cdot f = V_{F \vee G}(f) \\
= (V(F \vee G))(f) \\
= (V(F) \vee V(G))(f) \\
= (V(F)(f)) \vee (V(G)(f)) \\
= F \cdot f \vee G \cdot f \\
= T_f(F) \vee T_f(G).
\]

Hence, \( T_f \) is a lattice homomorphism.

Next, we show that \( T_f \) is an interval preserving operator. We identify \( x \) with its canonical image \( x'' \) in \( (A^-)_n^\sim \) and denote the restriction of \( T_f \) to \( A \) by \( T_f|_A \). Then

\[
T_f|_A(x) = T_f(x'') = x'' \cdot f = f \cdot x.
\]

Thus, for each \( F \in (A^-)_n^\sim \) and \( x \in A \), we see that

\[
\left( (T_f|_A)' \right)(F)(x) = F((T_f|_A)(x)) = F(f \cdot x) = (F \cdot f)(x) = (T_f(F))(x),
\]

which implies that \( (T_f|_A)' \) is the same as \( T_f \) on \( (A^-)_n^\sim \). Since \( (T_f|_A)' \) is interval preserving (cf. [5, Theorem 7.8]), \( T_f \) is likewise an interval preserving operator. \( \square \)

**Corollary 2.2.** For \( f \in A^- \), \( F \in (A^-)_n^\sim \), one has \( |F \cdot f| = |F| \cdot |f| \). Furthermore, if \( f \perp g \) in \( A^- \), \( F \cdot f \perp G \cdot g \) holds for any \( F, G \in (A^-)_n^\sim \).

**Proof.** Since \( V_f \) is an orthomorphism on \( A^- \), we have \( V_f(f^+) \perp V_f(f^-) \) for each \( f \in A^- \), that is, \( F \cdot (f^+) \perp F \cdot (f^-) \). From Theorem 2.1, we know that

\[
|F \cdot f| = |F \cdot f^+| + |F \cdot f^-| \\
= |T_f \cdot (F)| + |T_f \cdot (F)| \\
= T_f \cdot (|F|) + T_f \cdot (|F|) \\
= |F| \cdot f^+ + |F| \cdot f^- = |F| \cdot |f|.
\]
Let \( f \perp g \) in \( A^- \). Then we have
\[
\|F \cdot f\| \wedge \|G \cdot g\| = \|F\| \cdot \|f\| \wedge \|G\| \cdot \|g\| \\
\leq (\|F\| + \|G\|) \cdot \|f\| \wedge (\|F\| + \|G\|) \cdot \|g\| = 0,
\]
which implies that \( F \cdot f \perp G \cdot g \) for all \( F,G \in (A^-)^{\tau} \).

Following the above discussion, we now consider \( R(f) = \{ F \cdot f : F \in (A^-)^{\tau} \} \), the image of \( (A^-)^{\tau} \) under \( T_f \).

**Corollary 2.3.** If \( A \) is an \( f \)-algebra and \( f \in (A^-) \), then \( R(f) = R(|f|) \), and \( R(f) \) is an order ideal in \( A^- \).

**Proof.** First, since \( T_{|f|} \) is an interval preserving lattice homomorphism, we can easily see that \( R(|f|) \) is an order ideal in \( A^- \). By Corollary 2.2 we conclude that \( R(f) \subseteq R(|f|) \).

Now, to complete the proof we only need to prove that \( R(|f|) \subseteq R(f) \). To this end, let \( P_1 : A^- \to B_f^{-} \), \( P_2 : A^- \to B_f^{-} \) be band projections, where \( B_f^{-} \) and \( B_f^{-} \) are the bands generated by \( f^+ \) and \( f^- \) in \( A^- \), respectively. If \( \pi = P_1 - P_2 \), we have
\[
\pi \in \text{Orth}(A^-), \quad \pi(f) = |f|, \quad \pi(|f|) = f.
\]

In addition, \( \pi(f) \cdot a = \pi(f \cdot a) \) for all \( a \in A \) (cf. Theorem 3.1). Since \( \pi \) is an orthomorphism on \( A^- \) and hence order continuous (cf. [5, Theorem 8.10]), we have \( \pi'((A^-)^{\tau}) \subseteq (A^-)^{\tau} \). For all \( a \in A \) and all \( F \in (A^-)^{\tau} \), from
\[
(F \cdot |f|)(a) = (F \cdot \pi(f))(a) \\
= F(\pi(f) \cdot a) \\
= F(\pi(f \cdot a)) \\
= (\pi(F \cdot f))(a),
\]
it follows that \( F \cdot |f| = \pi'(F) \cdot f \) for all \( F \in (A^-)^{\tau} \), which implies that \( R(|f|) \subseteq R(f) \), as desired.

Next, we give a necessary and sufficient condition for \( R(f) \perp R(g) \) when \( A \) has the factorization property. First, we need the following lemma.

**Lemma 2.4.** Let \( A \) be an \( f \)-algebra with the factorization property, and \( f \in A^- \). If \( f \cdot x = 0 \) for each \( x \in A \), then \( f = 0 \).

**Proof.** Since \( A \) has the factorization property, for each \( a \in A^+ \), there exist \( x,y \in A \) such that \( a = xy \). Hence, from
\[
f(a) = f(xy) = (f \cdot x)(y) = 0,
\]
it follows easily that \( f = 0 \) holds.
Theorem 2.5. Let $A$ be an $f$-algebra with the factorization property. If $f, g \in A^-$, then $f \perp g$ if and only if $R(f) \perp R(g)$.

Proof. If $f \perp g$ in $A^-$, then it follows from Corollary 2.2 that $F \cdot f \perp G \cdot g$ for all $F, G \in (A^-)_n^-$. This implies that $R(f) \perp R(g)$.

Conversely, if $R(f)$ and $R(g)$ are disjoint, then for each $F \in ((A^-)_n)^+$ we have

\[
F \cdot (|f| \land |g|) = V_F(|f| \land |g|) = F \cdot |f| \land F \cdot |g| = |F \cdot f| \land |F \cdot g| = 0.
\]

In particular, for any $x \in A$, its canonical image $x'' \in (A^-)_n^-$ also satisfies $x'' \cdot (|f| \land |g|) = (|f| \land |g|) \cdot x = 0$. By the preceding lemma, we have $|f| \land |g| = 0$, that is, $f \perp g$, as desired. \hfill \Box

Now, we give the definition of the so-called $f$-orthomorphism.

Definition 2.6. Let $A$ be an $f$-algebra and $T \in L_b(A^-)$. $T$ is called an $f$-orthomorphism on $A^-$ if $R(Tf) \subseteq R(f)$ for each $f \in A^-$. The collection of all $f$-orthomorphisms on $A^-$ will be denoted by $\text{Orth}(A^-, A^-; (A^-)_n^-)$.

The next result deals with the relationship between the $f$-orthomorphisms and the orthomorphisms on the order dual of an $f$-algebra with the factorization property. Note that $\text{Orth}(A^-)$ is a band in $L_b(A^-)$.

Theorem 2.7. Let $A$ be an $f$-algebra. Then $\text{Orth}(A^-, A^-; (A^-)_n^-)$ is a linear subspace of $L_b(A^-)$ and $\text{Orth}(A^-) \subseteq \text{Orth}(A^-, A^-; (A^-)_n^-)$.

If $A$, in addition, has the factorization property, then $\text{Orth}(A^-, A^-; (A^-)_n^-) = \text{Orth}(A^-)$.

Proof. First, we can easily see that $\text{Orth}(A^-, A^-; (A^-)_n^-)$ is a linear subspace of $L_b(A^-)$. To prove $\text{Orth}(A^-) \subseteq \text{Orth}(A^-, A^-; (A^-)_n^-)$, let $\pi \in \text{Orth}(A^-)$. We claim that $F \cdot \pi(f) = \pi'(F) \cdot f$ for all $F \in (A^-)_n^-$ and all $f \in A^-$. To this end, let $F \in (A^-)_n^-$, $f \in A^-$, and $x \in A$ be arbitrary. Since $(A^-)_n^-$ is a commutative $f$-algebra, by Theorem 3.1, we have

\[
(\pi'(F) \cdot f)(x) = \pi'(F)(f \cdot x) = F(\pi(f \cdot x)) = F(\pi(x'' \cdot f)) = F(x'' \cdot (\pi(f))) = (F \cdot x'')(\pi(f)) = (x'' \cdot F)(\pi(f)) = x''(F \cdot \pi(f)) = (F \cdot \pi(f))(x).
\]

Thus, $F \cdot \pi(f) = \pi'(F) \cdot f$. This implies that $R(\pi(f)) \subseteq R(f)$ for each $f \in A^-$, that is, $\text{Orth}(A^-) \subseteq \text{Orth}(A^-, A^-; (A^-)_n^-)$.

If $A$ has the factorization property, we prove that $\text{Orth}(A^-, A^-; (A^-)_n^-) \subseteq \text{Orth}(A^-)$ holds. To this end, take $T \in \text{Orth}(A^-, A^-; (A^-)_n^-)$ and $f, g \in A^-$ satisfying $f \perp g$ in $A^-$. Then, it follows from Theorem 2.5 that $R(f) \perp R(g)$. Since $T \in \text{Orth}(A^-, A^-; (A^-)_n^-)$, we have
orthomorphism on $A$. Therefore, $R(T(f)) \perp R(g)$, which implies that $T(f) \perp g$, and hence $T$ is an orthomorphism on $A^-$, as desired.

\[\square\]

3. $f$-Linear Operators on the Order Dual

Let $A$ be an $f$-algebra with separating order dual and $T \in L_b(A^-)$. Recall that $T$ is called to be $f$-linear with respect to $(A^-)_n$ if $T(G \cdot f) = G \cdot T(f)$ for all $f \in A^-$ and $G \in (A^-)_n$. The set of all $f$-linear operators on $A^-$ will be denoted by $L_b(A^-, A^-; (A^-)_n)$. It follows from [3, Lemma 4.4] that $L_b(A^-, A^-; (A^-)_n)$ is a band in $L_b(A^-)$.

**Theorem 3.1.** Let $A$ be an $f$-algebra with separating order dual. Then $\text{Orth}(A^-) \subseteq L_b(A^-, A^-; (A^-)_n)$.

**Proof.** Clearly $\text{Orth}(A^-)$ is commutative since $\text{Orth}(A^-)$ is an Archimedean $f$-algebra. To complete the proof, let $\pi \in \text{Orth}(A^-)$. We have

$$\pi(G \cdot f) = \pi(V_G(f)) = V_G(\pi(f)) = G \cdot (\pi(f)),$$

for all $f \in A^-$ and $G \in (A^-)_n$. Hence, $\pi \in L_b(A^-, A^-; (A^-)_n)$.

The following result deals with the order adjoint of an $f$-linear operator on the order dual of an $f$-algebra. It should be noted that the order adjoint of an order-bounded operator is order continuous (cf. [5, Theorem 5.8]).

**Lemma 3.2.** Let $T \in L_b(A^-, A^-; (A^-)_n)$. Then the order adjoint $T'$ of $T$ satisfies $T'(F) \cdot f = F \cdot T(f)$ for all $F \in (A^-)_n$ and $f \in A^-$. In particular, $G \cdot T'(F) = T'(G \cdot F)$ for all $F, G \in (A^-)_n$.

**Proof.** Since $T \in L_b(A^-, A^-; (A^-)_n)$, and $(A^-)_n$ is a commutative $f$-algebra, we have

$$\begin{align*}
(T'(F) \cdot f)(x) &= T'(F)(f \cdot x) = F(T(f \cdot x)) \\
&= F(T(x'' \cdot f)) \\
&= F(x'' \cdot (T(f))) \\
&= (F \cdot x'')(T(f)) \\
&= (x'' \cdot F)(T(f)) \\
&= x''(F \cdot T(F)) = (F \cdot T(F))(x),
\end{align*}$$

for all $F \in (A^-)_n$, $f \in A^-$, and $x \in A$, which implies that $T'(F) \cdot f = F \cdot T(f)$.

Let $F, G \in (A^-)_n$ be given. Then for $f \in A^-$, from

$$\begin{align*}
(G \cdot T'(F))(f) &= G(T'(F) \cdot f) = G(F \cdot T(f)) \\
&= (G \cdot F)(T(f)) \\
&= (T'(G \cdot F))(f),
\end{align*}$$

it follows that $G \cdot T'(F) = T'(G \cdot F)$. This completes the proof.

\[\square\]
Theorem 3.3.

\[ L_b(A^-, A^-; (A^-)_n) \subseteq \text{Orth}(A^-, A^-; (A^-)_n). \]  

Proof. For \( T \in L_b(A^-, A^-; (A^-)_n) \), we know that \( |T| \) is also \( f \)-linear with respect to \( (A^-)_n \). Assume that \( 0 \leq G \in (A^-)_n \) and \( f \in A^- \). So by Lemma 3.2, we have

\[ 0 \leq G \cdot (|T(f)|) \leq G \cdot (|T||f|) = (|T'||(G)) \cdot |f| = T_{f|f'}(|T'||(G)). \]  

Since \( T_{f|f'} \) is interval preserving, there exists \( F \in (A^-)_n \) such that \( 0 \leq F \leq |T'||(G)\) and \( G \cdot (|T(f)|) = F \cdot |f| \). It is now immediate that \( R(|T(f)|) \subseteq R(|f|) \), and hence, \( L_b(A^-, A^-; (A^-)_n) \subseteq \text{Orth}(A^-, A^-; (A^-)_n) \), as desired. \( \square \)

Combining Theorems 3.1, 3.3, and 2.7, we have the following result.

Theorem 3.4. If \( A \) is an \( f \)-algebra with separating order dual, then

\[ \text{Orth}(A^-) \subseteq L_b(A^-, A^-; (A^-)_n) \subseteq \text{Orth}(A^-, A^-; (A^-)_n). \]  

In particular, if, in addition, \( A \) has the factorization property, then

\[ \text{Orth}(A^-) = L_b(A^-, A^-; (A^-)_n) = \text{Orth}(A^-, A^-; (A^-)_n). \]

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