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Research Article

Multiple Solutions for a Class of Differential Inclusion System Involving the $(p(x), q(x))$-Laplacian

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We consider a differential inclusion system involving the $(p(x), q(x))$-Laplacian with Dirichlet boundary condition on a bounded domain and obtain two nontrivial solutions under appropriate hypotheses. Our approach is variational and it is based on the nonsmooth critical point theory for locally Lipschitz functions.

1. Introduction

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions has been a new and interesting topic, which arises from nonlinear electrorheological fluids (see [1]) and elastic mechanics (see [2]). The study on variable exponent problems attracts more and more interest in recent years, and many results have been obtained on this kind of problems, for example [3–11].

Elliptic systems with standard growth conditions have been the subject of a sizeable literature. We refer to the excellent survey article of de Figueiredo [12].

In [11], the author obtained the existence and multiplicity of solutions for the following problem:

$$\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) &= F_u(x,u,v), \quad \text{in } \Omega,\\
-\operatorname{div}\left(|\nabla v|^{q(x)-2}\nabla v\right) &= F_v(x,u,v), \quad \text{in } \Omega,\\
u = v &= 0, \quad \text{on } \partial\Omega,
\end{aligned} \tag{P_1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $N \geq 2$, $(p,q) \in C(\overline{\Omega})^2$, $p(x) > 1$, $q(x) > 1$, for every $x \in \Omega$. The function F is assumed to be continuous in $x \in \Omega$ and of class C^1 in $u, v \in \mathbb{R}$. More precisely, the author was able to prove that, under suitable conditions, the system might have at least one solution or have infinite number of solutions.

Since many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities, now a question arises: whether there exist solutions for system (P_1) in the case where there is no continuously differentiable hypothesis required on the potential function F with respect to $t(s)$. See, for example, $F(x,t,s)$ is locally Lipschitz with respect to $t(s)$. That is the main problem which we want to solve in the present paper.

To this end, we mainly discuss the existence and multiplicity of solutions for the following nonlinear differential inclusion system involving the $(p(x), q(x))$-Laplacian:

$$\begin{gathered}
-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) \in \lambda\partial_u F(x,u,v), \quad \text{in } \Omega, \\
-\operatorname{div}\left(|\nabla v|^{q(x)-2}\nabla v\right) \in \lambda\partial_v F(x,u,v), \quad \text{in } \Omega, \\
u = v = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^1-boundary $\partial\Omega$, $\lambda > 0$ is the parameter, $p, q \in C(\overline{\Omega})$, $1 < p^- \leq p^+ < +\infty$, $1 < q^- \leq q^+ < +\infty$, $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(\cdot, t, s)$ is measurable in Ω for all $(t,s) \in \mathbb{R} \times \mathbb{R}$, and $F(x,t,s)$ is locally Lipschitz with respect to $t(s)$ (in general it can be nonsmooth), $\partial_t F(x,t,s)(\partial_s F(x,t,s))$ is the subdifferential with respect to the $t(s)$-variable in the sense of Clarke [13].

We emphasize that the operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous and, in general, it does not have the first eigenvalue. In other words, the infimum of the eigenvalues of $p(x)$-Laplacian equals 0 (see [14]).

Specially, if $F(x,\cdot,v), F(x,u,\cdot) \in C^1(\mathbb{R})$ for a.a. $x \in \Omega$, and $p(x) = p$, $q(x) = q$, then the problem (P) becomes the following problem:

$$\begin{gathered}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda F_u(x,u,v), \quad \text{in } \Omega, \\
-\operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right) = \lambda F_v(x,u,v), \quad \text{in } \Omega, \\
u = v = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{P_2}$$

There have been a large number of papers that study the existence of the solutions to (P_2). For instance, when $p > N$, $q > N$, Li and Tang [15] ensured the existence of three solutions to this problem. In [16], Kristály studied the multiplicity of solutions of the quasilinear elliptic systems (P_1), where Ω is a strip-like domain and $\lambda > 0$ is the parameter. Under some growth conditions on F, the author guaranteed the existence of an open interval $\Lambda \subset [0, +\infty)$, such that for each $\lambda \in \Lambda$ problem (P_2) has at least two distinct nontrivial solutions; when p and q are real numbers larger than 1, $\lambda = 1$, Boccardo and Guedes de Figueiredo [17] obtained the existence of solutions of the system (P_2).

But up to now, to the best of our knowledge, no paper discussing the solutions of problem (P) with nonsmooth potential via nonsmooth critical point theory can be found in the existing literature. In order to fill in this gap, we study problem (P) from a more extensive viewpoint. More precisely, we would study the existence of at least two nontrivial solutions for the problem (P) as the parameter $\lambda > \lambda_0$ for some constant λ_0.

This paper is divided into three sections: in the second section we introduce some necessary knowledge on the nonsmooth analysis, basic properties of the generalized Lebesgue-space $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$. In the third section, we give the assumptions on the nonsmooth potential $F(x,t,s)$ and prove the multiplicity results for problem (P).

2. Preliminary

2.1. Variable Exponent Sobolev Space

In order to discuss problem (P), we need some theories on $W_0^{k,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly we review some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For the details see [4, 18–20].

Firstly, we need to give some notations, which we shall use through this paper:

$$\begin{aligned} C_+\left(\overline{\Omega}\right) &= \left\{p \in C\left(\overline{\Omega}\right) : p(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}, \\ p^- &= \lim_{x\in\overline{\Omega}} p(x), p^+ = \max_{x\in\overline{\Omega}} p(x) \text{ for any } p \in C_+\left(\overline{\Omega}\right). \end{aligned} \tag{2.1}$$

Obviously, $1 < p^- \leq p^+ < +\infty$.

Denote by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathcal{U}(\Omega)$ are considered to be one element of $\mathcal{U}(\Omega)$, when they are equal almost everywhere.

For $p \in C_+(\overline{\Omega})$, define

$$L^{p(x)}(\Omega) = \left\{u \in \mathcal{U}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\}, \tag{2.2}$$

and with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)} dx \leq 1\}$, $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$, with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Denote $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Hereafter, let

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \geq N. \end{cases} \tag{2.3}$$

Lemma 2.1 (see [19]). *(1) The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces. Moreover, $L^{p(x)}(\Omega)$ is uniform convex.*

(2) Poincare inequality in $W_0^{1,p(x)}(\Omega)$ holds; that is, there exists a positive constant C such that

$$|u|_{L^{p(x)}(\Omega)} \leq C|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{2.4}$$

(3) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.

By (2) of Lemma 2.1, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

Lemma 2.2 (see [4]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $1/p(x)+1/q(x)=1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, one has*

$$\int_\Omega |uv|dx \leq \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{L^{p(x)}(\Omega)}|v|_{L^{q(x)}(\Omega)}. \tag{2.5}$$

Lemma 2.3 (see [4]). *Set $\rho(u)=\int_\Omega |u(x)|^{p(x)}dx$. For $u, u_k \in L^{p(x)}(\Omega)$, one has*

(1) *for $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(u/\lambda)=1$;*

(2) *$|u|_{p(x)} < 1(=1;>1) \Leftrightarrow \rho(u) < 1(=1;>1)$;*

(3) *if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;*

(4) *if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;*

(5) *$\lim_{k\to+\infty}|u_k|_{p(x)} = 0 \Leftrightarrow \mathrm{Lim}_{k\to+\infty}\rho(u_k)=0$;*

(6) *$|u_k|_{p(x)} \to +\infty \Leftrightarrow \rho(u_k) \to +\infty$.*

In this paper, the space $W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ will be endowed with the following equivalent norm:

$$\|(u,v)\| = \|u\| + \|v\|, \tag{2.6}$$

where

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u}{\lambda}\right|^{p(x)} dx \leq 1\right\}, \quad \|v\| = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla v}{\lambda}\right|^{q(x)} dx \leq 1\right\}. \tag{2.7}$$

Similar to Lemma 2.3, we have the following.

Lemma 2.4. *Set $\rho(u)=\int_\Omega |\nabla u(x)|^{p(x)}dx$. For $u, u_k \in W^{1,p(x)}(\Omega)$, one has*

(1) *for $u \neq 0$, $\|u\| = \lambda \Leftrightarrow \rho(u/\lambda)=1$;*

(2) *$\|u\| < 1(=1;>1) \Leftrightarrow \rho(u) < 1(=1;>1)$;*

(3) *if* $\|u\| > 1$, *then* $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(4) *if* $\|u\| < 1$, *then* $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(5) $\mathrm{Lim}_{k\to+\infty}\|u_k\| = 0 \Leftrightarrow \mathrm{Lim}_{k\to+\infty}\rho(u_k) = 0$;

(6) $\|u_k\| \to +\infty \Leftrightarrow \rho(u_k) \to +\infty$.

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx, \quad u \in W_0^{1,p(x)}(\Omega). \tag{2.8}$$

We know that (see [21]) $J \in C^1(W_0^{1,p(x)}(\Omega), R)$ *and* $p(x)$*-Laplacian operator* $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ *is the derivative operator of* J *in the weak sense. We denote* $\mathcal{L} = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, *then* $\langle \mathcal{L}(u), v\rangle = \int_\Omega(|\nabla u(x)|^{p(x)-2}\nabla u \cdot \nabla v dx$, *for all* $u, v \in W_0^{1,p(x)}(\Omega)$.

Lemma 2.5 (see [19]). *Set* $X = W_0^{1,p(x)}(\Omega)$, $\mathcal{L}$ *is as above, then*

(1) $\mathcal{L} : X \to X^*$ *is a continuous, bounded and strictly monotone operator;*

(2) $\mathcal{L}$ *is a mapping of type* (S_+), *if* $u_n \rightharpoonup u$*(weak) in* X *and* $\limsup_{n\to\infty}\langle \mathcal{L}(u_n), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* X*;*

(3) $\mathcal{L} : X \to X^*$ *is a homeomorphism.*

2.2. Generalized Gradient

Let X be a Banach space, X^* its topological dual space and we denote $\langle\cdot,\cdot\rangle$ as the duality bracket for pair (X^*, X). A function $\varphi : X \mapsto \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ we can find a neighbourhood U of x and a constant $k > 0$ (depending on U), such that $|\varphi(y) - \varphi(z)| \leq k\|y - z\|$, for all $y, z \in U$.

The generalized directional derivative of φ at the point $u \in X$ in the direction $h \in X$ is

$$\varphi^0(u; h) = \limsup_{u'\to u;\lambda\downarrow 0}\frac{\varphi(u' + \lambda h) - \varphi(u')}{\lambda}. \tag{2.9}$$

The generalized subdifferential of φ at the point $u \in X$ is defined by the

$$\partial\varphi(u) = \left\{u^* \in X^*; \langle u^*, h\rangle \leq \varphi^0(u; h),\ \forall h \in X\right\}, \tag{2.10}$$

which is a nonempty, convex and w^*-compact set of X. We say that $u \in X$ is a critical point of φ, if $0 \in \partial\varphi(u)$. For further details, we refer the reader to [12].

Finally we have the following Weierstrass Theorem and Mountain Pass Theorem.

Theorem 2.6. *If X is a reflexive Banach space and $\varphi : X \to \mathbb{R}$ satisfies*

(1) *φ is weak lower semicontinuous, that is,*

$$x_n \rightharpoonup x_0 \ (\text{weakly}) \ \text{in } X \longrightarrow \varphi(x_0) \leq \liminf_{n\to+\infty} \varphi(x_n); \tag{2.11}$$

(2) *φ is coercive, that is, $\lim_{\|x\|\to\infty}\varphi(x) = +\infty$,*

then there exists $x^* \in X$ such that $\varphi(x^*) = \min_{x\in X}\varphi(x)$.

Theorem 2.7 (see [22]). *Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz function and $x_0, x_1 \in X$. If there exists a bounded open neighbourhood U of x_0, such that $x_1 \in X \setminus U$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{\partial U}\varphi$ and φ satisfies the nonsmooth C-condition at level c, where $c = \inf_{\gamma\in\mathcal{T}}\max_{t\in[0,1]}\varphi(\gamma(t))$, $\mathcal{T} = \{\gamma \in C([0,1]; X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, then c is a critical value of φ and $c \geq \inf_{\partial U}\varphi$.*

3. Existence Results

For each $(u,v) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$, define

$$\begin{aligned} \Phi(u,v) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|\nabla v|^{q(x)}dx, \\ \Psi(u,v) &= \int_\Omega F(x,u,v)dx. \end{aligned} \tag{3.1}$$

By a solution of (2), we mean function $(u,v) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ to which there corresponds mapping $\Omega \ni x \to (w_1, w_2)$ with $w_1(x) \in \partial_u F(x,u,v)$, $w_2(x) \in \partial_v F(x,u,v)$ for almost every $x \in \Omega$ having the property that for every $(\xi,\eta) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$, the function $x \to (w_1(x)\xi(x), w_2(x)\eta(x)) \in L^1(\Omega) \times L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\xi dx + \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\eta dx = \lambda\int_\Omega w_1\xi dx + \lambda\int_\Omega w_2\eta dx. \tag{3.2}$$

Our hypotheses on nonsmooth potential $F(x,t,s)$ is as follows.

H(F): $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(x,0,0) = 0$ a.e. on Ω and satisfies the following facts:

(1) for all $t \in \mathbb{R}$, $s \in \mathbb{R}$, $x \mapsto F(x,t,s)$ is measurable;

(2) for almost all $x \in \Omega$, $t \mapsto F(x,t,s)$ and $s \mapsto F(x,t,s)$ are locally Lipschitz.

Lemma 3.1. *Suppose* H(F) *and the following conditions hold:*

(f_1) *there exists* $\alpha \in C_+(\overline{\Omega})$ *and* $\alpha(x) < p^*(x)$, *such that*

$$|w_1| \le c_1\left(1 + |t|^{\alpha(x)-1} + |s|^{\beta(x)(\alpha(x)-1)/\alpha(x)}\right), \tag{3.3}$$

for almost all $x \in \Omega$, *all* $t, s \in \mathbb{R}$ *and* $w_1 \in \partial_t F(x,t,s)$;

(f_2) *there exists* $\beta \in C_+(\overline{\Omega})$ *and* $\beta(x) < q^*(x)$, *such that*

$$|w_2| \le c_2\left(1 + |t|^{(\alpha(x)(\beta(x)-1))/(\beta(x))} + |s|^{\beta(x)-1}\right), \tag{3.4}$$

for almost all $x \in \Omega$, *all* $t, s \in \mathbb{R}$ *and* $w_2 \in \partial_s F(x,t,s)$;

then $\varphi(\cdot, v)(\varphi(u,\cdot))$ *is locally Lipschitz on* $W_0^{1,p(x)}(\Omega)(W_0^{1,q(x)}(\Omega))$.

Proof. We need only to prove that $\varphi(\cdot, v)$ is locally Lipschitz on $W_0^{1,p(x)}(\Omega)$.

By $\Phi(\cdot, v) \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, we have

$$\varphi(u_1, v) - \varphi(u_2, v) = J(u_1) - J(u_2) = J'(\overline{u}) \cdot (u_1 - u_2), \tag{3.5}$$

where $\overline{u} = tu_1 + (1-t)u_2,\ t \in (0,1)$.

Let $B_r = \{u \in W_0^{1,p(x)}(\Omega) : \|u - u_0\|_{W_0^{1,p(x)}(\Omega)} \le r\}$.

Note that B_r is w-compact. Then we obtain that there exists a positive constant M, such that $\|J'(\overline{u})\|_{W^{-1,p'(x)}(\Omega)} \le M$ with $1/p(x) + 1/p'(x) = 1$, for sufficiently small r.

Therefore, for any $u_1, u_2 \in B_r$, we have

$$\begin{aligned}
|\Phi(u_1, v) - \Phi(u_2, v)| &= \left|J'(\overline{u}) \cdot (u_1 - u_2)\right| \\
&\le \left\|J'(\overline{u})\right\|_{W^{-1,p'(x)(x)}(\Omega)} \|u_1 - u_2\|_{W_0^{1,p(x)}(\Omega)} \\
&\le M\|u_1 - u_2\|_{W_0^{1,p(x)}(\Omega)}, \quad \forall v \in W_0^{1,q(x)}(\Omega).
\end{aligned} \tag{3.6}$$

Fixing $v \in W_0^{1,q(x)}(\Omega)$, by ($f_1$) and Lebourg mean value theorem, we have

$$|F(x, u_1, v) - F(x, u_2, v)| \le c_1\left(1 + |\overline{u}|^{\alpha(x)-1} + |v|^{\beta(x)(\alpha(x)-1)/\alpha(x)}\right)|u_1 - u_2|. \tag{3.7}$$

Hence,

$$\begin{aligned}
&\left|\int_\Omega F(x, u_1, v)dx - \int_\Omega F(x, u_2, v)dx\right| \\
&\quad \le c_1 \int_\Omega |u_1 - u_2|dx + c_1 \int_\Omega |\overline{u}|^{\alpha(x)-1}|u_1 - u_2|dx + c_1 \int_\Omega |v|^{\beta(x)(\alpha(x)-1)/\alpha(x)}|u_1 - u_2|dx \\
&\quad \le \overline{c}_1|u_1 - u_2|_{\alpha(x)} + c_1\left||\overline{u}|^{\alpha(x)-1}\right|_{\alpha'(x)}|u_1 - u_2|_{\alpha(x)} + c_1\left||v|^{\beta(x)(\alpha(x)-1)/\alpha(x)}\right|_{\alpha'(x)}|u_1 - u_2|_{\alpha(x)},
\end{aligned} \tag{3.8}$$

where $1/\alpha'(x) + 1/\alpha(x) = 1$.

It is immediate that

$$\int_{\Omega}\left(|\overline{u}|^{\alpha(x)-1}\right)^{\alpha'(x)}=\int_{\Omega}|\overline{u}|^{\alpha(x)}dx\leq\begin{cases}|\overline{u}|_{\alpha(x)}^{\alpha^{+}}\leq c\|\overline{u}\|^{\alpha^{+}}, & |\overline{u}|_{\alpha(x)}>1,\\ |\overline{u}|_{\alpha(x)}^{\alpha^{+}}\leq c\|\overline{u}\|^{\alpha^{-}}, & |\overline{u}|_{\alpha(x)}<1,\end{cases}$$
$$\int_{\Omega}\left(|v|^{\beta(x)(\alpha(x)-1)/\alpha(x)}\right)^{\alpha'(x)}=\int_{\Omega}|v|^{\beta(x)}dx\leq\begin{cases}|v|_{\beta(x)}^{\beta^{+}}\leq c\|v\|^{\beta^{+}}, & |v|_{\beta(x)}>1,\\ |v|_{\beta(x)}^{\beta^{+}}\leq c\|v\|^{\beta^{-}}, & |v|_{\beta(x)}<1\end{cases}\tag{3.9}$$

are bounded.

So,

$$\left|\int_{\Omega}F(x,u_1,v)dx-\int_{\Omega}F(x,u_2,v)dx\right|\leq c\|u_1-u_2\|,\tag{3.10}$$

since $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$ is a compact embedding.

Therefore, $\varphi(\cdot,v)$ is locally Lipschitz. Similarly, we can prove that $\varphi(u,\cdot)$ is locally Lipschitz. □

Theorem 3.2. *Suppose that $H(F)$, (f_1) with $\alpha^+<p^-$, (f_2) with $\beta^+<q^-$ and the following conditions (f_3)-(f_4) hold:*

(f$_3$) *there exists $\gamma_i\in C(\overline{\Omega})$ $(i=1,2)$ with $p^+<\gamma_1(x)<p^*(x)$, $q^+<\gamma_2(x)<q^*(x)$ and $\mu_1,\mu_2\in L^\infty(\Omega)$, such that*

$$\limsup_{t\to0,s\to0}\frac{\langle w_1,t\rangle}{|t|^{\gamma_1(x)}}<\mu_1(x),\qquad\limsup_{t\to0,s\to0}\frac{\langle w_2,s\rangle}{|s|^{\gamma_2(x)}}<\mu_2(x)\tag{3.11}$$

uniformly for almost all $x\in\Omega$ and all $w_1\in\partial_tF(x,t,s)$, $w_2\in\partial_sF(x,t,s)$;

(f$_4$) *there exist $\xi_0\in R$, $\eta_0\in R$, $x_0\in\Omega$ and $r_0>0$, such that*

$$F(x,\xi_0,\eta_0)>\delta_0>0,\qquad\text{a.e. } x\in B_{r_0}(x_0),\tag{3.12}$$

where $B_{r_0}(x_0):=\{x\in\Omega:|x-x_0|\leq r_0\}\subset\Omega$.

Then there exists $\lambda_*>0$ such that, for each $\lambda>\lambda_*$, the problem (2) has at least two nontrivial solutions.

Proof. The proof is divided into four steps as follows. □

Step 1. We will show that φ is coercive in the step.

Firstly, for almost all $x\in\Omega$, by H(F)(2), $t\mapsto F(x,t,s)$ is differentiable almost everywhere on $\mathbb{R}$ and we have

$$\frac{d}{dt}F(x,t,s)\in\partial_tF(x,t,s).\tag{3.13}$$

Moreover, from (f_1), (f_2) and Young inequality, we can get that

$$\begin{aligned}
F(x,t,s) &= F(x,0,s) + \int_0^t \frac{d}{dy}F(x,y,s)dy \\
&= F(x,0,0) + \int_0^s \frac{d}{dz}F(x,0,z)dz + \int_0^t \frac{d}{dy}F(x,y,s)dy \\
&\le c_1\left[|t| + \frac{|t|^{\alpha(x)}}{\alpha(x)} + |s|^{\beta(x)(\alpha(x)-1)/\alpha(x)}|t|\right] \\
&\quad + c_2\left[|s| + |t|^{\alpha(x)(\beta(x)-1)/\beta(x)}|s| + \frac{1}{\beta(x)}|s|^{\beta(x)}\right] \\
&\le c_1\left[|t| + |t|^{\alpha(x)} + \frac{(\alpha(x)-1)|s|^{\beta(x)}}{\alpha(x)} + \frac{|t|^{\alpha(x)}}{\beta(x)}\right] \\
&\quad + c_2\left[|s| + |s|^{\beta(x)} + \frac{(\beta(x)-1)|t|^{\alpha(x)}}{\beta(x)} + \frac{|s|^{\beta(x)}}{\beta(x)}\right] \\
&\le c_1\left[|t| + 2|t|^{\alpha(x)} + |s|^{\beta(x)}\right] + c_2\left[|s| + 2|s|^{\beta(x)} + |t|^{\alpha(x)}\right],
\end{aligned} \tag{3.14}$$

for almost all $x \in \Omega$ and $t, s \in \mathbb{R}$.

Note that $1 < \alpha(x) \le \alpha^+ < p^- < p^*(x)$ and $1 < \beta(x) \le \beta^+ < q^- < q^*(x)$, then, by Lemma 2.1, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$ (compact embedding). Furthermore, there exists $c_3, c_4 > 0$ such that $|u|_{\alpha(x)} \le c_3\|u\|$ and $|v|_{\beta(x)} \le c_4\|v\|$.

So, for any $|u|_{\alpha(x)} > 1$ and $\|u\| > 1$,

$$\int_\Omega |u|^{\alpha(x)}dx \le |u|_{\alpha(x)}^{\alpha^+} \le c_3^{\alpha^+}\|u\|^{\alpha^+} \tag{3.15}$$

and, for any $|v|_{\beta(x)} > 1$ and $\|v\| > 1$,

$$\int_\Omega |v|^{\beta(x)}dx \le |v|_{\beta(x)}^{\beta^+} \le c_4^{\beta^+}\|v\|^{\beta^+}. \tag{3.16}$$

By (3.14), (3.15), (3.31), the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned}
\varphi(u,v) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - \lambda\int_\Omega F(x,u,v)dx \\
&\ge \frac{1}{p^+}\|u\|^{p^-} + \frac{1}{q^+}\|v\|^{q^-} - \lambda c_1\int_\Omega |u|dx - 2c_1c_3^{\alpha^+}\|u\|^{\alpha^+} - c_1c_4^{\beta^+}\|v\|^{\beta^+} \\
&\quad - \lambda c_2\int_\Omega |v|dx - 2c_2c_4^{\beta^+}\|v\|^{\beta^+} - c_2c_3^{\alpha^+}\|u\|^{\alpha^+}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{p^+}\|u\|^{p^-} + \frac{1}{q^+}\|v\|^{q^-} - 2\lambda c_1 |1|_{\alpha'(x)} |u|_{\alpha(x)} - 2\lambda c_1 |1|_{\beta'(x)} |v|_{\beta(x)} \\
&\quad - 2c_1 c_3^{\alpha^+}\|u\|^{\alpha^+} - c_1 c_4^{\beta^+}\|v\|^{\beta^+} - 2c_2 c_4^{\beta^+}\|v\|^{\beta^+} - c_2 c_3^{\alpha^+}\|u\|^{\alpha^+} \\
&\geq \frac{1}{p^+}\|u\|^{p^-} + \frac{1}{q^+}\|v\|^{q^-} - 2\lambda c_1 c |1|_{\alpha'(x)} \|u\| - 2\lambda c_1 c |1|_{\beta'(x)} \|v\| \\
&\quad - 2c_1 c_3^{\alpha^+}\|u\|^{\alpha^+} - c_1 c_4^{\beta^+}\|v\|^{\beta^+} - 2c_2 c_4^{\beta^+}\|v\|^{\beta^+} - c_2 c_3^{\alpha^+}\|u\|^{\alpha^+} \\
&\longrightarrow \infty, \text{ as } \|u,v\| \longrightarrow \infty.
\end{aligned}
\tag{3.17}
$$

Step 2. We will show that the φ is weakly lower semicontinuous.

Leting $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$, $v_n \rightharpoonup v$ weakly in $W_0^{1,q(x)}(\Omega)$ by Lemma 2.1(3), we obtain the following results:

$$
\begin{aligned}
&W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega);\, W_0^{1,q(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega); \\
&u_n \to u \text{ in } L^{p(x)}(\Omega);\ v_n \to v \text{ in } L^{q(x)}(\Omega); \\
&u_n \to u \text{ for a.a. } x \in \Omega;\ v_n \to v \text{ for a.a. } x \in \Omega; \\
&F(x, u_n(x), v_n(x)) \to F(x, u(x), v(x)) \text{ for a.a. } x \in \Omega.
\end{aligned}
$$

By Fatou's Lemma,

$$
\limsup_{n\to\infty} \int_\Omega F(x, u_n(x), v_n(x))dx \leq \int_\Omega F(x, u(x), v(x))dx. \tag{3.18}
$$

Thus,

$$
\begin{aligned}
\liminf_{n\to\infty} \varphi(u_n, v_v) &= \liminf_{n\to\infty}\left[\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}dx\right] \\
&\quad - \limsup_{n\to\infty}\int_\Omega F(x, u_n, v_n)dx \\
&\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - \lambda\int_\Omega F(x, u, v)dx. \\
&= \varphi(u)
\end{aligned}
\tag{3.19}
$$

Hence, by Theorem 2.6, we deduce that there exists a global minimizer $(u_0, v_0) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ such that

$$
\varphi(u_0, v_0) = \min_{(u,v)\in W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)} \varphi(u, v). \tag{3.20}
$$

Step 3. We will show that there exists $\lambda_* > 0$ such that for each $\lambda > \lambda_*$, $\varphi(u_0, v_0) < 0$.

By the condition (f_4), there exists $\xi_0, \eta_0 \in \mathbb{R}$ such that $F(x, \xi_0, \eta_0) > \delta_0 > 0$, a.e. $x \in B_{r_0}(x_0)$. It is clear that

$$\begin{aligned} 0 < M_1 := \max_{|t|\le|\xi_0|,|s|\le|\eta_0|} \Big\{ & c_1\Big[|t| + 2|t|^{\alpha^+} + |s|^{\beta^+}\Big] + c_2\Big[|s| + 2|s|^{\beta^+} + |t|^{\alpha^+}\Big], \\ & c_1\Big[|t| + 2|t|^{\alpha^-} + |s|^{\beta^-}\Big] + c_2\Big[|s| + 2|s|^{\beta^-} + |t|^{\alpha^-}\Big]\Big\} < +\infty. \end{aligned} \tag{3.21}$$

Now we denote

$$\begin{aligned} t_0 &= \left(\frac{M_1}{\delta_0 + M_1}\right)^{1/N}, \\ K(t) &:= \max\left\{\left(\frac{\xi_0}{r_0(1-t)}\right)^{p^-}, \left(\frac{\xi_0}{r_0(1-t)}\right)^{p^+}, \left(\frac{\xi_0}{r_0(1-t)}\right)^{q^-}, \left(\frac{\xi_0}{r_0(1-t)}\right)^{q^+}\right\}, \\ \lambda_* &= \max_{t\in[t_1,t_2]} \frac{K(t)(1-t^N)}{[\delta_0 t^N - M_1(1-t^N)]}, \end{aligned} \tag{3.22}$$

where $t_0 < t_1 < t_2 < 1$ and δ_0 is given in the condition (f_4). A simple calculation shows that the function $t \mapsto \delta_0 t^N - M_1(1-t^N)$ is positive whenever $t > t_0$ and $\delta_0 t_0^N - M_1(1-t_0^N) = 0$. Thus λ_* is well defined and $\lambda_* > 0$.

We will show that, for each $\lambda > \lambda_*$, the problem (P) has two nontrivial solutions. In order to do this, for $t \in [t_1, t_2]$, let us define

$$\eta_t(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus B_{r_0}(x_0), \\ \xi_0, & \text{if } x \in B_{tr_0}(x_0), \\ \dfrac{\xi_0}{r_0(1-t)}(r_0 - |x - x_0|), & \text{if } x \in B_{r_0}(x_0) \setminus B_{tr_0}(x_0). \end{cases} \tag{3.23}$$

By conditions (f_1) and (f_3) we have

$$\begin{aligned} \int_\Omega F(x, \eta_t(x), \eta_t(x))dx &= \int_{B_{tr_0}(x_0)} F(x, \eta_t(x), \eta_t(x))dx \\ &\quad + \int_{B_{r_0}(x_0)\setminus B_{tr_0}(x_0)} F(x, \eta_t(x), \eta_t(x))dx \\ &\ge w_N r_0^N t^N \delta_0 - M_1\left(1 - t^N\right) w_N r_0^N \\ &= w_N r_0^N \left(\delta_0 t^N - M_1\left(1 - t^N\right)\right). \end{aligned} \tag{3.24}$$

Hence, for $t \in [t_1, t_2]$,

$$
\begin{aligned}
\varphi(\eta_t, \eta_t) &= \int_\Omega \frac{1}{p(x)} |\nabla \eta_t|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla \eta_t|^{q(x)} dx - \lambda \int_\Omega F(x, \eta_t(x), \eta_t(x)) dx \\
&\le \frac{1}{p^-} \int_\Omega |\nabla \eta_t|^{p(x)} dx + \frac{1}{q^-} \int_\Omega |\nabla \eta_t|^{q(x)} dx - \lambda w_N r_0^N \left(\delta_0 t^N - M_1\left(1 - t^N\right)\right) \\
&\le \max\left\{ \left(\frac{\xi_0}{r_0(1-t)}\right)^{p^-}, \left(\frac{\xi_0}{r_0(1-t)}\right)^{p^+}, \left(\frac{\xi_0}{r_0(1-t)}\right)^{q^-}, \left(\frac{\xi_0}{r_0(1-t)}\right)^{q^+} \right\} \\
&\quad \times w_N r_0^N \left(1 - t^N\right) - \lambda w_N r_0^N \left(\delta_0 t^N - M_1\left(1 - t^N\right)\right) \\
&= w_N r_0^N \left[K(t)\left(1 - t^N\right) - \lambda\left(\delta_0 t^N - M_1\left(1 - t^N\right)\right)\right],
\end{aligned}
\tag{3.25}
$$

so that $\varphi(\eta_t, \eta_t) < 0$ whenever $\lambda > \lambda_*$.

Step 4. We will check the C-condition in the following.

Suppose $\{(u_n, v_n)\}_{n\ge1} \subseteq W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ such that $\varphi(u_n, v_n) \to c$ and $(1 + \|(u_n, v_n)\|) m(u_n, v_n) \to 0$. Let $(u_n^*, v_n^*) \in \partial\varphi(u_n, v_n)$ be such that $m(u_n, v_n) = \|(u_n^*, v_n^*)\|_{(W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega))^*}$. The interpretation of $(u_n^*, v_n^*) \in \partial\varphi(u_n, v_n)$ is that $u_n^* \in \partial_u\varphi(u_n, v_n)$ and $v_n^* \in \partial_v\varphi(u_n, v_n)$. We know that

$$
\begin{aligned}
u_n^* &= -\Delta_{p(x)} u_n - \lambda w_n^1, \\
v_n^* &= -\Delta_{q(x)} v_n - \lambda w_n^2
\end{aligned}
\tag{3.26}
$$

with $w_n^1 \in \partial_u \Psi(u_n, v_n)$ and $w_n^2 \in \partial_v \Psi(u_n, v_n)$. From Chang [23] we know that $w_n^1 \in L^{\alpha'(x)}(\Omega)$ and $w_n^2 \in L^{\beta'(x)}(\Omega)$, where $\alpha'(x) = \alpha(x)/(\alpha(x) - 1)$, $\beta'(x) = \beta(x)/(\beta(x) - 1)$.

Since φ is coercive, $\{u_n\}_{n\ge1}$, $\{v_n\}_{n\ge1}$ are bounded and passed to a subsequence, still denoting $\{u_n\}_{n\ge1}$ and $\{v_n\}_{n\ge1}$, we may assume that there exist $u \in W_0^{1,p(x)}(\Omega)$, $v \in W_0^{1,q(x)}(\Omega)$, such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,q(x)}(\Omega)$. Next we will prove that

$$
u_n \to u \text{ in } W_0^{1,p(x)}(\Omega), \qquad v_n \to v \text{ in } W_0^{1,q(x)}(\Omega).
\tag{3.27}
$$

By $W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$, $W_0^{1,q(x)}(\Omega) \to L^{q(x)}(\Omega)$, we have $u_n \to u$ in $L^{p(x)}(\Omega)$ and $v_n \to v$ in $L^{q(x)}(\Omega)$. Moreover, since $\|(u_n^*, v_n^*)\|_{(W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega))*} \to 0$, we get $\|u_n^*\|_{(W_0^{1,p(x)}(\Omega))*} \to 0$, $\|v_n^*\|_{(W_0^{1,q(x)}(\Omega))*} \to 0$, so $|\langle u_n^*, u_n\rangle| \le \varepsilon_n$, $|\langle v_n^*, v_n\rangle| \le \varepsilon_n$.

From (3.26), we have

$$
\begin{aligned}
&\left\langle -\Delta_{p(x)} u_n, u_n - u \right\rangle - \int_\Omega w_n^1 (u_n - u) dx \le \varepsilon_n, \quad \forall n \ge 1. \\
&\left\langle -\Delta_{q(x)} v_n, v_n - v \right\rangle - \int_\Omega w_n^2 (v_n - v) dx \le \varepsilon_n, \quad \forall n \ge 1.
\end{aligned}
\tag{3.28}
$$

Moreover, $\int_\Omega w_n^1(u_n - u)dx \to 0$ and $\int_\Omega w_n^2(v_n - v)dx \to 0$, since $u_n \to u$ in $L^{p(x)}(\Omega)$, $v_n \to v$ in $L^{q(x)}(\Omega)$, $\{w_n^1\}_{n\geq 1}$ in $L^{p'(x)}(\Omega)$ and $\{w_n^2\}_{n\geq 1}$ in $L^{q'(x)}(\Omega)$ are bounded, where $1/p(x) + 1/(p'(x)) = 1$, $1/(q(x)) + 1/(q'(x)) = 1$. Therefore,

$$\limsup_{n\to\infty}\langle -\Delta_{p(x)}u_n, u_n - u\rangle \leq 0, \qquad \limsup_{n\to\infty}\langle -\Delta_{q(x)}v_n, v_n - v\rangle \leq 0. \tag{3.29}$$

From Lemma 2.5, we have $u_n \to u, v_n \to v$ as $n \to \infty$. Thus φ satisfies the nonsmooth C-condition.

Step 5. We will show that there exists another nontrivial weak solution of problem (P).

From Lebourg Mean Value Theorem, we obtain

$$\begin{gathered} F(x,t,s) - F(x,0,s) = \langle w_1, t\rangle, \\ F(x,0,s) - F(x,0,0) = \langle w_2, s\rangle \end{gathered} \tag{3.30}$$

for some $w_1 \in \partial_t F(x, \vartheta t, s)$, $w_2 \in \partial_s F(x, 0, \tau s)$, and $0 < \vartheta,\ \tau < 1$. Thus by the condition (f_3), there exists $\beta \in (0,1)$ such that

$$\begin{aligned} |F(x,t,s)| &\leq |\langle w_1, t\rangle| + |\langle w_2, s\rangle| \\ &\leq \mu_1(x)|t|^{\gamma_1(x)} + \mu_2(x)|s|^{\gamma_2(x)}, \end{aligned} \tag{3.31}$$

for all $|t|, |s| < \beta$ and a.e. $x \in \Omega$.

It follows from the conditions (f_1), (f_2) $1 < \alpha^- \leq \alpha^+ < p^- \leq p^+ < \gamma_1(x) < p^*(x)$ and $1 < \beta^- \leq \beta^+ < q^- \leq q^+ < \gamma_2(x) < p^*(x)$ that for all $|t| > \beta$, $|s| > \beta$ and a.e. $x \in \Omega$,

$$\begin{aligned} |F(x,t,s)| &\leq c_1\left[|t| + 2|t|^{\alpha(x)} + |s|^{\beta(x)}\right] + c_2\left[|s| + 2|s|^{\beta(x)} + |t|^{\alpha(x)}\right] \\ &\leq \left(\frac{c_1}{\beta^{\gamma_1(x)-1}} + \frac{2c_2}{\beta^{\gamma_1(x)-\alpha(x)}} + \frac{c_2}{\beta^{\gamma_1(x)-\beta(x)}}\right)|t|^{\gamma_1(x)} \\ &\quad + \left(\frac{c_2}{\beta^{\gamma_2(x)-1}} + \frac{2c_2}{\beta^{\gamma_2(x)-\beta(x)}} + \frac{c_1}{\beta^{\gamma_1(x)-\alpha(x)}}\right)|t|^{\gamma_2(x)} \\ &\leq \left(\frac{c_1}{\beta^{\gamma_1^+-1}} + \frac{2c_2}{\beta^{\gamma_1^+-\alpha^-}} + \frac{c_2}{\beta^{\gamma_1^+-\beta^-}}\right)|t|^{\gamma_1(x)} \\ &\quad + \left(\frac{c_2}{\beta^{\gamma_2^+-1}} + \frac{2c_2}{\beta^{\gamma_2^+-\beta^-}} + \frac{c_1}{\beta^{\gamma_1^+-\alpha^-}}\right)|t|^{\gamma_2(x)}, \end{aligned} \tag{3.32}$$

this together with (3.31) yields that, for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$
\begin{aligned}
|F(x,t,s)| &\leq \left(\mu_1(x) + \frac{c_1}{\beta^{\gamma_1^+ - 1}} + \frac{2c_2}{\beta^{\gamma_1^+ - \alpha^-}} + \frac{c_2}{\beta^{\gamma_1^+ - \beta^-}}\right)|t|^{\gamma_1(x)} \\
&\quad + \left(\mu_2(x) + \frac{c_2}{\beta^{\gamma_2^+ - 1}} + \frac{2c_2}{\beta^{\gamma_2^+ - \beta^-}} + \frac{c_1}{\beta^{\gamma_1^+ - \alpha^-}}\right)|t|^{\gamma_2(x)} \\
&\leq c_3|\mathrm{t}|^{\gamma_1(x)} + c_4|s|^{\gamma_2(x)},
\end{aligned}
\tag{3.33}
$$

for positive constants c_3, c_4.

Note that $p^+ < \gamma_1(x) < p^*(x)$, $q^+ < \gamma_2(x) < q^*(x)$, then, by Lemma 2.1, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\gamma_1(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega) \hookrightarrow L^{\gamma_2(x)}(\Omega)$. Furthermore, there exist c_5, c_6 such that

$$
|u|_{\gamma_1(x)} \leq c_5\|u\|, \quad \forall u \in W_0^{1,p(x)}(\Omega), \qquad |v|_{\gamma_2(x)} \leq c_6\|u\|, \quad \forall v \in W_0^{1,q(x)}(\Omega). \tag{3.34}
$$

For all $\lambda > \lambda_*$, $\|(u,v)\| < 1$, $|u|_{\gamma_1(x)} < 1$ and $|v|_{\gamma_2(x)} < 1$, from (3.33) we have

$$
\begin{aligned}
\varphi(u,v) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - \lambda\int_\Omega F(x,u(x),v(x))dx \\
&\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - \lambda c_3\int_\Omega |u(x)|^{\gamma_1(x)}dx \\
&\quad - \lambda c_4\int_\Omega |v(x)|^{\gamma_2(x)}dx \\
&\geq \frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} - \lambda\left[c_3c_5^{\gamma_1^-}\|u\|^{\gamma_1^-} + c_4c_6^{\gamma_2^-}\|u\|^{\gamma_2^-}\right].
\end{aligned}
\tag{3.35}
$$

So, for $\rho > 0$ small enough, there exists a $\nu > 0$ such that

$$
\varphi(u,v) > \nu, \qquad \text{for } \|u,v\| = \rho \tag{3.36}
$$

and $\|(u_0,v_0)\| > \rho$. So by the Nonsmooth Mountain Pass Theorem (cf. Theorem 2.7), we can get $(u_1,v_1) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ which satisfies

$$
\varphi(u_1,v_1) = c > 0, \quad m(u_1,v_1) = 0. \tag{3.37}
$$

Therefore, (u_1,v_1) is another nontrivial solution of problem (P).

Remark 3.3. Let $p^- > \alpha^+$, $q^- > \beta^+$ and consider the following nonsmooth locally Lipschitz function:

$$
F(x,t,s) = \begin{cases}
t^{\gamma_1(x)} + s^{\gamma_2(x)}, & t \in (0,\delta) \text{ or } s \in (0,\delta), \\
\max\left\{|t-\delta|^{\theta(x)}, |t-\delta|^{\alpha(x)}\right\} + |\delta|^{\gamma_1(x)} & \\
+\max\left\{|s-\delta|^{\theta(x)}, |s-\delta|^{\beta(x)}\right\} + |\delta|^{\gamma_2(x)}, & t \in [\delta,+\infty) \text{ or } s \in [\delta,+\infty), \\
0, & t \in (-\infty,0] \text{ or } s \in (-\infty,0],
\end{cases}
\tag{3.38}
$$

where $0 < \delta < 1$, $\theta^- > 1$, $\theta^+ < \alpha^-$ and $\theta^+ < \beta^-$.

Obvious, $t \mapsto F(x,t,s)$ and $s \mapsto F(x,t,s)$ are locally Lipschitz. Then,

$$\partial_t F(x,t,s) = \begin{cases} \gamma_1(x) t^{\gamma_1(x)-1}, & t \in (0,\delta), \\ \theta(x)(t-\delta)^{\theta(x)-1}, & t \in (\delta, 1+\delta), \\ \alpha(x)(t-\delta)^{\alpha(x)-1}, & t \in (1+\delta, +\infty), \\ \left[0, \gamma_1(x)\delta^{\gamma_1(x)-1}\right], & t = \delta, \\ [\theta(x), \alpha(x)], & t = 1+\delta, \\ 0, & t \in (-\infty, 0], \end{cases}$$

$$\partial_s F(x,t,s) = \begin{cases} \gamma_2(x) s^{\gamma_2(x)-1}, & s \in (0,\delta), \\ \theta(x)(s-\delta)^{\theta(x)-1}, & s \in (\delta, 1+\delta), \\ \beta(x)(t-\delta)^{\beta(x)-1}, & s \in (1+\delta, +\infty), \\ \left[0, \gamma_1(x)\delta^{\gamma_1(x)-1}\right], & s = \delta, \\ \left[\theta(x), \beta(x)\right], & s = 1+\delta, \\ 0, & s \in (-\infty, 0]. \end{cases} \tag{3.39}$$

Hence, for any $w_1 \in \partial_t F(x,t,s)$ and $w_2 \in \partial_s F(x,t,s)$, we have

$$|w_1| \leq \begin{cases} \gamma_1(x) t^{\alpha(x)-1} t^{\gamma_1(x)-\alpha(x)} \leq \gamma_1^+ |t|^{\alpha(x)-1}, & t \in (0,\delta), \\ \theta(x)(t-\delta)^{\theta(x)-1} < \theta^+ < \theta^+ \left(\dfrac{1}{\delta}\right)^{\alpha^+-1} |t|^{\alpha(x)-1}, & t \in (\delta, 1+\delta), \\ \alpha(x)(t-\delta)^{\alpha(x)-1} \leq \alpha^+ |t|^{\alpha(x)-1}, & t \in (1+\delta, +\infty), \\ \gamma_1(x)\delta^{\gamma_1(x)-1} = \gamma_1(x)\delta^{\alpha(x)-1}\delta^{\gamma_1(x)-\alpha(x)} \leq \gamma^+ \delta^{\alpha(x)-1}, & t = \delta, \\ [\theta(x), \alpha(x)] \leq \alpha^+ (1+\delta)^{\alpha(x)-1}, & t = 1+\delta, \\ 0, & t \in (-\infty, 0], \end{cases}$$

$$|w_2| \leq \begin{cases} \gamma_2(x) s^{\beta(x)-1} s^{\gamma_2(x)-\beta(x)} \leq \gamma_2^+ |s|^{\beta(x)-1}, & s \in (0,\delta), \\ \theta(x)(s-\delta)^{\theta(x)-1} < \theta^+ < \theta^+ \left(\dfrac{1}{\delta}\right)^{\beta^+-1} |s|^{\beta(x)-1}, & s \in (\delta, 1+\delta), \\ \beta(x)(s-\delta)^{\beta(x)-1} \leq \beta^+ |s|^{\beta(x)-1}, & s \in (1+\delta, +\infty), \\ \gamma_2(x)\delta^{\gamma_2(x)-1} = \gamma_2(x)\delta^{\beta(x)-1}\delta^{\gamma_2(x)-\beta(x)} \leq \gamma^+ \delta^{\beta(x)-1}, & s = \delta, \\ \left[\theta(x), \beta(x)\right] \leq \beta^+ (1+\delta)^{\beta(x)-1}, & s = 1+\delta, \\ 0, & s \in (-\infty, 0]. \end{cases} \tag{3.40}$$

Therefore,

$$\begin{aligned}
|w_1| &\leq \left(\gamma_1^+ + \alpha^+ + \theta^+\left(\frac{1}{\delta}\right)^{(\alpha^+-1)}\right)|t|^{\alpha(x)-1}, \quad \forall w_1 \in \partial_t F(x,t,s),\\
|w_2| &\leq \left(\gamma_2^+ + \beta^+ + \theta^+\left(\frac{1}{\delta}\right)^{(\beta^+-1)}\right)|t|^{\beta(x)-1}, \quad \forall w_2 \in \partial_s F(x,t,s),\\
\limsup_{t\to 0, s\to 0}\frac{< w_1, t >}{|t|^{\gamma_1(x)}} &= \left\{\begin{array}{ll} \lim\limits_{t\to 0}\dfrac{\gamma_1(x)t^{\gamma_1(x)}}{t^{\gamma_1(x)}}, & t>0\\ 0, & t\leq 0\end{array}\right\} \leq \gamma_1(x),\\
\limsup_{t\to 0, s\to 0}\frac{< w_2, s >}{|s|^{\gamma_2(x)}} &= \left\{\begin{array}{ll} \lim\limits_{s\to 0}\dfrac{\gamma_2(x)s^{\gamma_2(x)}}{s^{\gamma_2(x)}}, & s>0\\ 0, & s\leq 0\end{array}\right\} \leq \gamma_2(x),
\end{aligned} \tag{3.41}$$

uniformly for almost all $x \in \Omega$, all $w_1 \in \partial_t F(x,t,s)$ and $w_2 \in \partial_s F(x,t,s)$.

Thus far the results involved potential functions exhibiting $p(x)$-sublinear. The next theorem concerns problems where the potential function is $p(x)$-superlinear.

Theorem 3.4. *Suppose that* H(F), (f_1) *with* $\alpha^- > p^+$, (f_2) *with* $\beta^- > q^+$, (f_3), (f_4) *and the following condition* (f_5) *hold.*

(f_5) *For almost all* $x \in \Omega$ *and all* $t, s \in \mathbb{R}$, *one has* $F(x,t,s) \leq \kappa(x)$ *with* $\kappa \in L^{\gamma_3(x)}(\Omega)$, $1 \leq \gamma_3(x) < \min\{p^-, q^-\}$.

Then there exists a $\lambda_* > 0$ *such that, for each* $\lambda > \lambda_*$, *the problem* (P) *has at least two nontrivial solutions.*

Proof. The steps are similar to those of Theorem 3.2. In fact,we only need to modify Step 1 and Step 5 as follows: Step 6 shows, that φ is coercive under the condition (f_5); Step 7 shows, that there exists second nontrivial solution under the conditions (f_1), (f_2), and (f_3). Then from Steps 6, 2, 3, 4, and 7 above, the problem (P) has at least two nontrivial solutions. □

Step 6. By (f_5), for all $(u,v) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$, $\|(u,v)\| > 1$, we have

$$\begin{aligned}
\varphi(u,v) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - \lambda\int_\Omega F(x,u(x),v(x))dx\\
&\geq \frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} - \lambda\int_\Omega \kappa(x)dx \longrightarrow \infty \longrightarrow \infty, \quad \text{as } \|(u,v)\| \to \infty.
\end{aligned} \tag{3.42}$$

Step 7. Because of hypothesis (f_1), (f_2) and mean value theorem for locally Lipschitz functions, we have

$$\begin{aligned}
F(x,t,s) &\le c_1\left[|t| + 2|t|^{\alpha(x)} + |s|^{\beta(x)}\right] + c_2\left[|s| + 2|s|^{\beta(x)} + |t|^{\alpha(x)}\right] \\
&\le c_1\left[\left|\frac{t}{\beta}\right|^{\alpha(x)-1}|t| + 2|t|^{\alpha(x)} + |s|^{\beta(x)}\right] \\
&\quad + c_2\left[\left|\frac{s}{\beta}\right|^{\beta(x)-1}|s| + 2|s|^{\beta(x)} + |t|^{\alpha(x)}\right] \\
&\le c_1\left[\left|\frac{1}{\beta}\right|^{\alpha^+-1}|t|^{\alpha(x)} + 2|t|^{\alpha(x)} + |s|^{\beta(x)}\right] \\
&\quad + c_2\left[\left|\frac{1}{\beta}\right|^{\beta^+-1}|t|^{\beta(x)} + 2|s|^{\beta(x)} + |t|^{\alpha(x)}\right] \\
&= c_7|t|^{\alpha(x)} + c_8|t|^{\beta(x)}
\end{aligned} \tag{3.43}$$

for a.e. $x \in \Omega$, all $|t| \ge \beta$, $|s| \ge \beta$ with $c_7, c_8 > 0$.

Combining (3.31) and (3.43), it follows that

$$|F(x,t,s)| \le \left[\mu_1(x)|t|^{\gamma_1(x)} + c_7|t|^{\alpha(x)}\right] + \left[\mu_2(x)|s|^{\gamma_2(x)} + c_8|s|^{\alpha(x)}\right] \tag{3.44}$$

for a.e. $x \in \Omega$ and all $t, s \in \mathbb{R}$.

Thus, for all $\lambda > \lambda_*$, $\|(u,v)\| < 1$, $|u|_{\gamma_1(x)} < 1$, $|u|_{\alpha(x)} < 1|v|_{\gamma_2(x)} < 1$ and $|v|_{\beta(x)} < 1$, we have

$$\begin{aligned}
\varphi(u,v) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - \lambda\int_\Omega F(x,u(x).v(x))dx \\
&\ge \frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} - \lambda\int_\Omega \mu_1(x)|u|^{\gamma_1(x)}dx - \lambda c_7\int_\Omega |u|^{\alpha(x)}dx \\
&\quad - \lambda\int_\Omega \mu_2(x)|v|^{\gamma_2(x)}dx - \lambda c_8\int_\Omega |v|^{\beta(x)}dx \\
&\ge \frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} - \lambda c_9\|u\|^{\gamma_1^-} - \lambda c_7\|u\|^{\alpha^-} - \lambda c_{10}\|v\|^{\gamma_2^-} - \lambda c_8\|v\|^{\beta^-}.
\end{aligned} \tag{3.45}$$

So, for $\rho > 0$ small enough, there exists a $\nu > 0$ such that

$$\varphi(u,v) > \nu, \qquad \text{for } \|u,v\| = \rho \tag{3.46}$$

and $\|(u_0,v_0)\| > \rho$. Arguing as in proof of Step 4 of Theorem 3.2, we conclude that φ satisfies the nonsmooth C-condition. So by the Nonsmooth Mountain Pass Theorem (cf. Theorem 2.7), we can get $(u_1,v_1) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ which satisfies

$$\varphi(u_1,v_1) = c > 0, \quad m(u_1,v_1) = 0. \tag{3.47}$$

Therefore, (u_1, v_1) is second nontrivial of problem (P).

Remark 3.5. Consider the following nonsmooth locally Lipschitz function:

$$F(x,t,s) = \begin{cases} -|t|^{\gamma_1(x)} - |s|^{\gamma_2(x)}, & |t| \le 1 \text{ or } |s| \le 1, \\ \cos(\pi|t|) + \cos(\pi|s|), & |t| > 1; \text{or } |s| > 1, \end{cases} \tag{3.48}$$

In the following, we will show that $F(x,t,s)$ satisfies hypotheses H(F) and (f_1)–(f_5).

It is clear that $F(x,0,0) = 0$ for a.e. $x \in \Omega$,then hypotheses H(F) is satisfied. A direct verification shows that conditions (f_4) and (f_5) are satisfied. Note that

$$\begin{gathered} \partial_t F(x,t,s) = \begin{cases} -\gamma_1(x) t^{\gamma_1(x)-1}, & 0 \le t < 1, \\ \gamma_1(x)(-t)^{\gamma_1(x)-1}, & -1 < t \le 0, \\ \left[-\gamma_1(x), 0\right], & t = 1, \\ \left[0, \gamma_1(x)\right], & t = -1, \\ \{-\pi \sin(\pi t)\}, & |t| > 1, \end{cases} \\ \partial_s F(x,t,s) = \begin{cases} -\gamma_2(x) s^{\gamma_2(x)-1}, & 0 \le s < 1, \\ \gamma_2(x)(-s)^{\gamma_1(x)-1}, & -1 < s \le 0, \\ \left[-\gamma_2(x), 0\right], & s = 1, \\ \left[0, \gamma_2(x)\right], & s = -1, \\ \{-\pi \sin(\pi s)\}, & |s| > 1. \end{cases} \end{gathered} \tag{3.49}$$

So,

$$\begin{gathered} |w_1| \le (\gamma_1(x) + \pi)|t|^{\gamma_1(x)-1}, \quad \forall w_1 \in \partial_t F(x,t,s), \\ |w_1| \le (\gamma_2(x) + \pi)|s|^{\gamma_2(x)-1}, \quad \forall w_2 \in \partial_s F(x,t,s), \\ \limsup_{t\to 0, s\to 0} \frac{< w_1, t >}{|t|^{\gamma_1(x)}} = \left\{ \begin{matrix} \lim\limits_{t\to 0} \dfrac{-\gamma_1(x) t^{\gamma_1(x)}}{t^{\gamma_1(x)}}, & t > 0 \\ \lim\limits_{t\to 0} \dfrac{-\gamma_1(x)(-t)^{\gamma_1(x)}}{(-t)^{\gamma_1(x)}}, & t \le 0 \end{matrix} \right\} = -\gamma_1(x) \le \gamma_1(x), \\ \limsup_{t\to 0, s\to 0} \frac{< w_2, s >}{|s|^{\gamma_2(x)}} = \left\{ \begin{matrix} \lim\limits_{t\to 0} \dfrac{-\gamma_2(x) s^{\gamma_2(x)}}{s^{\gamma_2(x)}}, & s > 0 \\ \lim\limits_{t\to 0} \dfrac{-\gamma_2(x)(-s)^{\gamma_2(x)}}{(-s)^{\gamma_2(x)}}, & s \le 0 \end{matrix} \right\} = -\gamma_2(x) \le \gamma_2(x) \end{gathered} \tag{3.50}$$

which shows that assumptions (f_1), (f_2), and (f_3) are fulfilled.

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