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Research Article

Dynamic Analysis of a Predator-Prey (Pest) Model with Disease in Prey and Involving an Impulsive Control Strategy

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The dynamic behaviors of a predator-prey (pest) model with disease in prey and involving an impulsive control strategy to release infected prey at fixed times are investigated for the purpose of integrated pest management. Mathematical theoretical works have been pursuing the investigation of the local asymptotical stability and global attractivity for the semitrivial periodic solution and population persistent, which depicts the threshold expression of some critical parameters for carrying out integrated pest management. Numerical analysis indicates that the impulsive control strategy has a strong effect on the dynamical complexity and population persistent using bifurcation diagrams and power spectra diagrams. These results show that if the release amount of infective prey can satisfy some critical conditions, then all biological populations will coexist. All these results are expected to be of use in the study of the dynamic complexity of ecosystems.

1. Introduction

Predator-prey models with disease are a major concern and are now becoming a new field of study known as ecoepidemiology. The disease factor in predator-prey systems has been firstly considered by Anderson and May [1]. In subsequent years, many authors studied the dynamics of ecological models with infected prey, and their papers mainly focused on this issue [2–8]. The infection rate and the predation rate are the two primary factors, which can control the chaotic dynamics of an ecoepidemiological system [9]. Das et al. [10] studied the HP model [11] by introducing disease in prey populations, which can be described as follows:

$$\begin{aligned}
\frac{ds}{dt} &= rs\left(1-\frac{s}{k}\right) - \alpha is - c_1 a_1 \frac{p_1 s}{b_1 + s},\\
\frac{di}{dt} &= \alpha is - a_2 ip_1 - d_1 i,\\
\frac{dp_1}{dt} &= a_1 \frac{p_1 s}{b_1 + s} + c_2 ip_1 - a_3 \frac{p_1 p_2}{b_2 + p_1} - d_2 p_1,\\
\frac{dp_2}{dt} &= c_3 a_3 \frac{p_1 p_2}{b_2 + p_1} - d_3 p_2,
\end{aligned} \tag{1.1}$$

where s, i, p_1, and p_2 are respectively the susceptible prey population, infected prey population, the intermediate predator population, and the top-predator population, a_1 and a_2 are the maximal predation rate of intermediate predator for susceptible and infected prey, respectively, a_3 is the maximal predation rate of top-predator for intermediate predator, b_1 and b_2 are the half saturation constant for functional response of intermediate and the top-predator respectively, c_1 is the conversion rate of susceptible prey to intermediate predator, c_2 is the conversion rate of infected prey to intermediate predator, and c_3 is the conversion rate of intermediate predator to top predator.

Through the dimensionless transformation (seeing [10]), the system can change into the following form:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) - ais - b\frac{p_1 s}{1 + cs},\\
\frac{di}{dt} &= ais - dip_1 - ei,\\
\frac{dp_1}{dt} &= f\frac{p_1 s}{1 + cs} + gip_1 - h\frac{p_1 p_2}{1 + mp_1} - jp_1,\\
\frac{dp_2}{dt} &= k\frac{p_1 p_2}{1 + mp_1} - lp_2.
\end{aligned} \tag{1.2}$$

In recent decades, technological revolutions have recently hit the industrial world; thus, infected population can now be controlled by many methods such as spraying pesticides and vaccination. It is well known that pest management involves using pesticides and releasing natural enemies, which have been focused by many researchers [12–14]. Control of an infected population can be achieved by chemical or biological control or both, which is called an impulsive control strategy in biomathematics. Systems with impulsive control strategies to describe time-varying processes are characterized by the fact that at certain moments, their states undergo abrupt change. Recently, impulsive control strategies have been recently introduced into population ecology [15–18], chemotherapeutic approaches to treat disease [19], and food webs [20–25].

Based on the two aspects discussed, the authors constructed a predator-prey model with disease in prey (a pest) and involving an impulsive control strategy for the purpose of integrated pest management. The impulsive control strategy was used to introduce infected prey (a pest) at a fixed time on the basis of system (1.2). The predator-prey model with disease in prey and involving an impulsive control strategy can be described by the following differential equations:

$$\left.\begin{aligned}\frac{dx(t)}{dt} &= x(t)(1-x(t)) - ax(t)y(t) - b\frac{x(t)z(t)}{1+cx(t)},\\ \frac{dy(t)}{dt} &= ax(t)y(t) - dy(t)z(t) - ey(t),\\ \frac{dz(t)}{dt} &= f\frac{x(t)z(t)}{1+cx(t)} + gy(t)z(t) - h\frac{q(t)z(t)}{1+mz(t)} - jz(t),\\ \frac{dq(t)}{dt} &= K\frac{q(t)z(t)}{1+mz(t)} - lq(t),\end{aligned}\right\} t \neq nT,$$
$$\left.\begin{aligned}\Delta x(t) &= 0,\\ \Delta y(t) &= p,\\ \Delta z(t) &= 0,\\ \Delta q(t) &= 0,\end{aligned}\right\} t = nT, \tag{1.3}$$

where $x(t), y(t), z(t)$, and $q(t)$ are respectively the densities of susceptible prey (a pest), infected prey (a pest), the intermediate predator (natural enemy), and the top predator at time t. Then, $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$, $\Delta z(t) = z(t^+) - z(t)$, and $\Delta q(t) = q(t^+) - q(t)$. We have

$$\begin{aligned} &a = \frac{\alpha k}{r}, \quad b = \frac{c_1 a_1 k}{rb_1}, \quad c = \frac{k}{b_1}, \quad d = \frac{a_2 k}{r}, \quad e = \frac{d_1}{r}, \quad f = \frac{a_1 k}{b_1 r},\\ &g = \frac{c_2 a_2 k}{r}, \quad h = \frac{a_3 k}{b_2 r}, \quad m = \frac{k}{b_1}, \quad j = \frac{d_2}{r}, \quad K = \frac{c_3 a_3 k}{b_2 r}, \quad l = \frac{d_3}{r},\end{aligned} \tag{1.4}$$

where a_1 and a_2 are the maximal predation rates of the intermediate predator on susceptible and infected prey respectively; a_3 is the maximal predation rate of the top predator on the intermediate predator; b_1 and b_2 are the half-saturation constants for functional response of the intermediate prey and the top predator respectively; c_1 is the conversion rate of susceptible prey to intermediate predators; c_2 and c_3 are, respectively, the conversion rate of infected prey to intermediate predators and the conversion rate of the intermediate predator to the top predator; d_1, d_2, d_3 are the death rates of infected prey, the intermediate predator, and the top predator, respectively; α is the incidence rate; r is the intrinsic growth rate; k is the carrying capacity (see [10]); $p > 0$ is the introduced amount of infective prey population at $t = nT$, $n \in N$, $N = \{0, 1, 2 \ldots\}$, where T is the period of the impulsive control. It is known that pest outbreak will cause some serious ecological and economic problems, and we can directly gather infected prey to increase the amount of infected prey and indirectly carry out integrated pest management.

The paper is organized as follows: in the next section, a mathematical analysis of the model is carried out. Section 3 describes some numerical simulations, and the last section contains a brief discussion.

2. Mathematical Analysis

Some important notations, lemmas, and definitions will be provided, which are frequently used in subsequent proofs.

Let $R_+ = [0, +\infty), R_+^4 = \{X = (x(t), y(t), z(t), q(t)) \in R^4 | X \geq 0\}$. Denote $f = (f_1, f_2, f_3, f_4)$ as the map defined by the right-hand side of the first, second, third, and fourth equations of system (1.3). Let$V_0 = \{V : R_+ \times R_+^4 \to R_+\}$, then V is said to belong to class V_0 if

(1) V is continuous on $(nT, (n+1)T] \times R_+^4, n \in N$, and for each $X \in R^4$ $\lim_{(t,\mu)\to(nT^+,X)} V(t,\mu) = V(nT^+, X)$ exists;

(2) V is locally Lipschitzian in X.

Definition 2.1 (see [26]). Let $V \in V_0$, and then, for $(nT, (n+1)T] \times R_+^4$, the upper right derivative of $V(t, X)$ with respect to the impulsive differential system (1.3) can be defined as

$$D^+V(t, X) = \lim_{h\to 0^+} \sup \frac{1}{h}\left[V(t+h, X+hf(t, X)) - V(t, X)\right]. \tag{2.1}$$

The solution of system (1.3) is a piecewise continuous function $X : R_+ \times R_+^4$, where $X(t)$ is continuous on$(nT, (n+1)T], n \in N$, and $X(nT^+) = \lim_{t\to nT^+} X(T)$ exists. Obviously the smoothness properties of f can guarantee the global existence and uniqueness of the solution of system (1.3); for details see [26–28].

Definition 2.2 (see [21]). system (1.3) is said to be uniformly persistent if there is an $\omega > 0$ (independent of the initial conditions) such that every solution $(x(t), y(t), z(t), q(t))$ of system (1.3) satisfies the following:

$$\lim_{t\to\infty} \inf x(t) \geq \omega, \quad \lim_{t\to\infty} \inf y(t) \geq \omega, \quad \lim_{t\to\infty} \inf z(t) \geq \omega, \quad \lim_{t\to\infty} \inf q(t) \geq \omega. \tag{2.2}$$

Definition 2.3 (see [24]). System (1.3) is said to be permanent if there exists a compact region $\Omega_0 \subset \operatorname{int} R_+^4$ such that every solution $(x(t), y(t), z(t), q(t))$ of system (1.3) will eventually enter and remain in the region Ω_0.

Lemma 2.4 (see [24]). *Suppose that $X(t)$ is a solution of system* (1.3) *with $X(0^+) \geq 0$; then $X(t) \geq 0$ for all $t \geq 0$. Furthermore, $X(t) > 0$, $t > 0$ if $X(0^+) > 0$.*

Lemma 2.5. *There exists a constant M such that $x(t) \leq M$, $y(t) \leq M$, $z(t) \leq M$, and $q(t) \leq M$ for each solution $X = (x(t), y(t), z(t), q(t))$ of system* (1.3) *for all sufficiently large t. Details can be found in Theorem 2.2 of [29].*

Lemma 2.6 (see [26]). *Let $V \in V_0$, and assume that*

$$\begin{aligned} D^+V(t, X) &\leq g(t, V(t, X)), \quad t \neq nT, \\ V(t, X(t^+)) &\leq \Phi_n(V(t, X(t))), \quad t = nT, \end{aligned} \tag{2.3}$$

where $g : R_+ \times R_+ \rightarrow R$ *is continuous in* $(nT, (n+1)T]$ *for* $u \in R_+^2, n \in N$, $\lim_{(t,y)\rightarrow(nT^+)} g(t,v) = g(nT^+, u)$ *existing, and* ϕ_n^i $(i = 1,2) : R_+ \rightarrow R_+$ *nondecreasing. Let* $r(t)$ *be a maximal solution of the scalar impulsive differential equation as follows:*

$$\begin{aligned} \frac{du(t)}{dt} &= g(t, u(t)), \quad t \neq nT, \\ u(t^+) &= \Phi_n(u(t)), \quad t = nT, \\ u(0^+) &= u_0, \end{aligned} \tag{2.4}$$

existing on $(0, +\infty]$. *Then* $V(0^+, X_0) \leq u_0$, *implying that* $V(t, X(t)) \leq r(t)$, $t \geq 0$, *where* $X(t)$ *is any solution of system* (1.3). *Note that if certain smoothness conditions on* g *exist to guarantee the existence and uniqueness of solutions for* (2.4), *then* $r(t)$ *is the unique solution of* (2.4).

For convenience, some basic properties of certain subsystems of system (1.3) are now provided as follows:

$$\begin{aligned} \frac{dy(t)}{dt} &= -ey(t), \quad t \neq nT, \\ y(t^+) &= y(t) + p, \quad t = nT, \\ y(0^+) &= y_0. \end{aligned} \tag{2.5}$$

Therefore, the following lemma holds.

Lemma 2.7 (see [26]). *For a positive periodic solution* $y^*(t)$ *of system* (2.5) *and the solution* $y(t)$ *of system* (2.5) *with initial value* $y_0 = y(0^+) \geq 0$, $|y(t) - y^*(t)| \rightarrow 0$, $t \rightarrow \infty$, *where*

$$\begin{aligned} y^*(t) &= \left(\frac{p\exp(-e(t-nT))}{1-\exp(-eT)}\right), \quad t \in (nT, (n+1)T], n \in N, \\ y^*(0^+) &= \left(\frac{p}{1-\exp(-eT)}\right), \\ y(t) &= \left(y(0^+) - \left(\frac{p}{1-\exp(-eT)}\right)\right)\exp(-eT) + y^*(t). \end{aligned} \tag{2.6}$$

Next, the stability of susceptible prey and of predator-eradication periodic solutions will be studied.

Theorem 2.8. *The solution* $(0, y^*(t), 0, 0)$ *is said to be locally asymptotically stable if* $T < (p/e)$.

Proof. The local stability of periodic solution $(0, y^*(t), 0, 0)$ may be determined by considering the behavior of small-amplitude perturbations of the solution. Define

$$x(t) = u(t), \qquad y(t) = v(t) + y^*(t), \qquad z(t) = w(t), \qquad q(t) = h(t). \tag{2.7}$$

Substituting (2.7) into (1.3), a linearization of the system can be obtained as follows:

$$\begin{aligned}
&\left.\begin{aligned}
\frac{du(t)}{dt} &= (1 - ay^*(t))u(t),\\
\frac{dv(t)}{dt} &= ay^*(t)u(t) - ev(t) - dy^*(t)w(t),\\
\frac{dw(t)}{dt} &= (gy^*(t) - j)w(t),\\
\frac{dh(t)}{dt} &= -lh(t),
\end{aligned}\right\} \quad t \neq nT,\\
&\left.\begin{aligned}
\Delta u(t) &= 0,\\
\Delta v(t) &= p,\\
\Delta w(t) &= 0,\\
\Delta h(t) &= 0,
\end{aligned}\right\} \quad t = nT.
\end{aligned} \tag{2.8}$$

This can be rewritten as

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \\ h(t) \end{pmatrix} = \phi(t) \begin{pmatrix} u(0) \\ v(0) \\ w(0) \\ h(0) \end{pmatrix}, \quad 0 \leq t \leq T, \tag{2.9}$$

where $\phi(t)$ satisfies

$$\frac{d\phi(t)}{dt} = \begin{pmatrix} 1 - ay^*(t) & 0 & 0 & 0 \\ ay^*(t) & -e & -dy^*(t) & 0 \\ 0 & 0 & gy^*(t) - j & 0 \\ 0 & 0 & 0 & -l \end{pmatrix}, \tag{2.10}$$

with $\phi(0) = I$, where I is the identity matrix, and

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \\ w(nT^+) \\ h(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \\ w(nT) \\ h(nT) \end{pmatrix}. \tag{2.11}$$

Hence, the stability of the periodic solution $(0, y^*(t), 0, 0)$ is determined by the eigenvalues of

$$\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \phi(t). \tag{2.12}$$

If the absolute values of all eigenvalues are less than one, the periodic solution $(0, y^*(t), 0, 0)$ is locally stable. Then all eigenvalues of ϕ can be denoted by $\lambda_1, \lambda_2, \lambda_3$, and λ_4, where $\lambda_1 = \exp \int_0^T (1 - ay^*(t))dt$, $\lambda_2 = \exp(-eT) < 1$, $\lambda_3 = \exp \int_0^T (gy^*(t) - j)dt$, $\lambda_4 = \exp(-lT) < 1$.

Clearly, $|\lambda_3| = \exp(-gp) < 1$ with $|\lambda_1| < 1$ only if $T < (p/e)$ according to the Floquet theory of impulsive differential equations, and the periodic solution $(0, y^*(t), 0, 0)$ is locally stable. This completes the proof. □

Theorem 2.9. *The solution* $(0, y^*(t), 0, 0)$ *is said to be globally attractive if* $gM < j$ *and*

$$\frac{1}{a} < \frac{p\exp(-(dM+e)T)}{1-\exp(-(dM+e)T)}. \tag{2.13}$$

Proof. Let $V(t) = fKx(t) + bKz(t) + bhq(t)$; then

$$V'\big|_{(1.1)} = fK\big(1 - ay(t)\big)x(t) + bK\big(gy(t) - j\big)z(t) - fKx^2(t) - bhlq(t). \tag{2.14}$$

By Lemma 2.5, there exists a constant $M > 0$ such that $x(t) \le M$, $y(t) \le M$, $z(t) \le M$, $q(t) \le M$ for each solution $X = (x(t), y(t), z(t), q(t))$ of system (1.3) with sufficiently large t.

Then,

$$\begin{gathered}\frac{dy(t)}{dt} = ay(t)x(t) - dy(t)z(t) - ey(t) \ge -(dM+e)y(t), \quad t \ne nT \\ \Delta y = p, \quad t = nT\end{gathered} \tag{2.15}$$

$$\begin{aligned}V|_{(1.1)} &= fK\big(1 - ay(t)\big)x(t) + bK\big(gy(t) - j\big)z(t) - fKx^2(t) - bhlq(t) \\ &\le fK\big(1 - ay(t)\big)x(t) + bK\big(gM - j\big)z(t) - fKx(t) - bhlq(t).\end{aligned} \tag{2.16}$$

By Lemmas 2.6 and 2.7, there exists a $t_1 > 0$, and an $\varepsilon > 0$ can be selected to be small enough so that $y(t) \ge y_1^*(t) - \varepsilon$ for all $t \ge t_1$. By (2.15),

$$\begin{gathered}y(t) \ge y_1^*(t) - \varepsilon = \frac{p\exp(-(dM+e)T)}{1-\exp(-(dM+e)T)} - \varepsilon, \\ \lambda \triangleq \frac{p\exp(-(dM+e)T)}{1-\exp(-(dM+e)T)} - \varepsilon.\end{gathered} \tag{2.17}$$

Let $1 - a\lambda < 0$ and $gM - j < 0$. Therefore, when $t \ge t_1$, by (2.16), $V'|_{(1.1)} < 0$. So $V(t) \to 0$ and $x(t) \to 0$, $z(t) \to 0$, $q(t) \to 0$ as $t \to \infty$. It is known from the fact that the limiting state of system (1.3) is exactly system (2.5) and from Lemma 2.7 that $(0, y^*(t), 0, 0)$ is globally attractive. This completes the proof. □

Theorem 2.10. *System* (1.3) *is permanent if* $T > p/e$, $gM > j$,

$$\frac{1}{a} > \frac{p\exp(-(dM+e)T)}{1-\exp(-(dM+e)T)}, \qquad bhlM > \frac{p\exp((aM-e)T)}{1-\exp((aM-e)T)}. \tag{2.18}$$

Proof. From Lemma 2.5, there exists a constant $M > 0$ such that $x(t) \le M$, $y(t) \le M$, $z(t) \le M$, $q(t) \le M$ for each solution $X = (x(t), y(t), z(t), q(t))$ of system (1.3) with t sufficiently large.

From (2.15), it is known that $y(t) \geq y_1^*(t) - \varepsilon = (p\exp(-(dM+e)T))/(1-\exp(-(dM+e)T)) - \varepsilon \triangleq \delta_1$ for large enough t.

Therefore, it is only necessary to find a δ_2 that satisfies $x(t) > \delta_2$, $z(t) > \delta_2$, $q(t) > \delta_2$. This will be achieved in the following two steps.

Let $\delta_3 > 0, \delta_4 > 0, \gamma = e - a\delta_3$, and $V(t) = fKx(t) + bKz(t) + bhq(t)$.

Then

$$\begin{aligned} V|_{(1.1)} &= fK(1 - ay(t))x(t) + bK(gy(t) - j)z(t) - fKx^2(t) - bhlq(t) \\ &\geq fK(1 - ay(t) - M)x(t) + bK(gy(t) - j)z(t) - fKx(t) - bhlq(t). \end{aligned} \tag{2.19}$$

First, it will be proved that there exists a $t_2 \in (0, +\infty)$ such that $x(t_2) > \delta_4$, $z(t_2) > \delta_4$, and $q(t_2) > \delta_4$ because $V(t)$ is ultimately bounded.

Next, it will be proved that $x(t) < \delta_3$, $z(t) < \delta_3$, $q(t) < \delta_3$ cannot hold for all $t \in (0, +\infty)$. Otherwise,

$$\begin{gathered} \frac{dy(t)}{dt} = ay(t)x(t) - dy(t)z(t) - ey(t) \leq (a\delta_3 - e)y(t), \quad t \neq nT \\ \Delta y = p, \quad t = nT. \end{gathered} \tag{2.20}$$

Then let $v_1(t)$ be the solution of

$$\begin{gathered} \frac{dv_1(t)}{dt} = (a\delta_3 - e)v_1(t), \quad t \neq nT \\ \Delta v_1(t) = p, \quad t = nT. \end{gathered} \tag{2.21}$$

It follows that $y(t) < v_1(t)$ and $v_1(t) \to v_1^*(t) (t \to \infty)$ where $v_1^*(t) = (p\exp(-\gamma(t-n)T))/(1-\exp(-\gamma T))$.

So there exists a $t_3 > 0$ such that

$$y(t) < v_1(t) < v_1^*(t) + \varepsilon_1 = \frac{p\exp(-\gamma(t-n)T)}{1-\exp(-\gamma T)} + \varepsilon_1 < \frac{p}{1-\exp(-\gamma T)} + \varepsilon_1. \tag{2.22}$$

Then

$$\begin{aligned} V|_{(1.1)} &= fK(1 - ay(t))x(t) + bK(gy(t) - j)z(t) - fKx^2(t) - bhlq(t) \\ &\geq fK(1 - ay(t) - M)x(t) + bK(gy(t) - j)z(t) - fKx(t) - bhlq(t) \\ &\geq fK\left(1 - a\frac{p}{1-\exp(-\gamma T)} + \varepsilon_1 - M\right)x(t) + bK\left(g\frac{p}{1-\exp(-\gamma T)} + \varepsilon_1 - j\right)z(t). \end{aligned} \tag{2.23}$$

According to the above conditions, $V'|_{(1.1)} > 0$; then $V(t) \to \infty$ and $x(t) \to \infty$, $z(t) \to \infty$, $q(t) \to \infty$ as $t \to \infty$; however, this is a contradiction. Therefore, $V(t)$ is ultimately bounded.

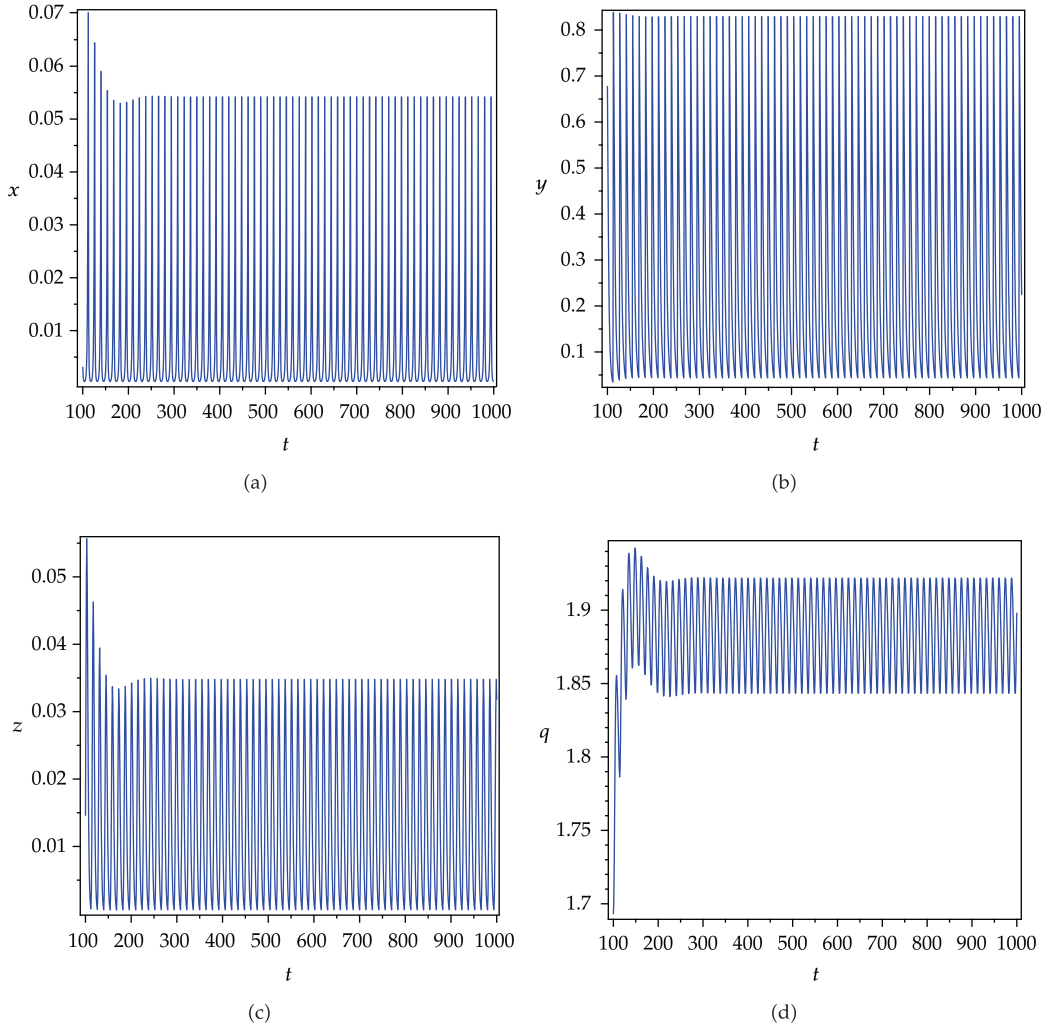

Figure 1: Dynamics of system (1.3) with $b = 8$, $d = 3$, $c = 3$, $e = 0.2$, $f = 3$, $g = 2.5$, $h = 0.3$, $m = 1.2$, $j = 0.2$, $k = 0.6$, $l := 0.008$, $a := 3$, $p = 1.03$, $T = 14$. Time series of (a) susceptible prey, (b) infected prey, (c) the intermediate predator, and (d) the top predator.

Second, if $x(t) > \delta_3$, $z(t) > \delta_3$, $q(t) > \delta_3$ for all $t \geq t_2$, then the objective has been attained. To show this, let $t^* = \inf_{t \geq t_2}\{V(t) < \delta_5\}$, and it follows that $V(t) \geq \delta_5$ for $t \in [t_2, t^*)$ and that $V(t^*) = \delta_5$. Suppose that $t^* \in (n_1 T, (n_1 + 1)T], n_1 \in N$. Select $n_2, n_3 \in N$ such that $n_2 T > (\ln(\varepsilon_1/(M + p))/ - \gamma)$, $\exp(n_3 \alpha_1 T)\exp(\alpha_2(n_2 + 1)T) > 1$, where

$$\alpha_1 = fK\left(1 - a\frac{p}{1 - \exp(-\gamma T)} + \varepsilon_1 - M\right)x(t) + bK\left(g - \frac{p}{1 - \exp(-\gamma T)} + \varepsilon_1 - j\right)z(t) > 0,$$
$$\alpha_2 = -\frac{ap\exp((aM - e)T)}{1 - \exp((aM - e)T)}x(t) - jz(t) < 0. \tag{2.24}$$

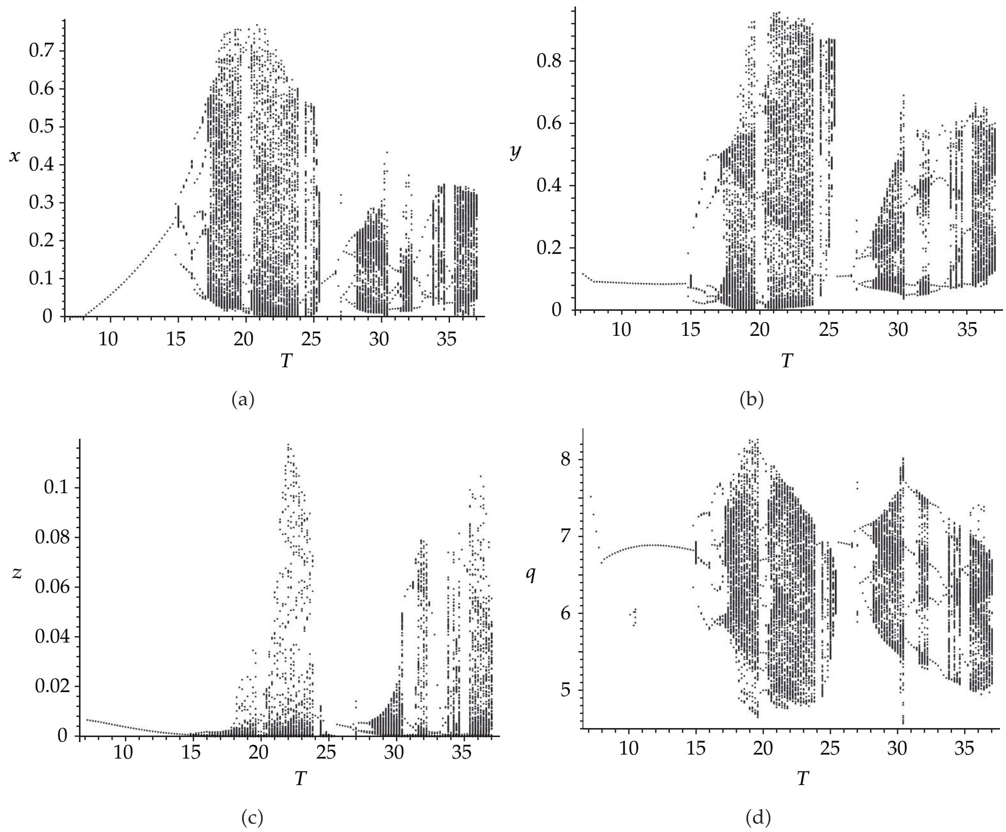

Figure 2: Bifurcation diagram of system (1.3) with initial conditions $x(0) = 0.1, y(0) = 0.2, z(0) = 0.3, q(0) = 0.4$, $7 \le T \le 37$, $b = 6$, $d = 3$, $c = 3$, $e = 0.2$, $f = 2$, $g = 2.5$, $h = 0.1$, $m = 2$; $j = 0.1$, $k = 0.6$, $l = 0.01$, $a = 3$, and $p = 0.6$.

Let $T_2 = n_2T + n_3T$. It is claimed that there must be a $t_3 \in ((n_1+1)T, (n_1+1)T + T_2]$ such that $V(t) \ge \delta_5$. Otherwise, consider (2.21) with $v_1(t^{*+}) = y(t^{*+})$. Then

$$v_1(t) = \left(v_1((n_1+1)T^+) - \frac{p}{1-\exp(-\gamma T)}\right)\exp(-\gamma(t-(n_1+1)T)) + v_1^*(t). \tag{2.25}$$

For $t \in ((n+1)T, (n+1)T]$, $n_1 + 1 < n < n_1 + n_2 + n_3 + 1$, it can be shown that $|v_1(t) - v_1^*(t)| < (M+p)\exp(-\gamma n_1 T) < \varepsilon_1$, $y(t) \le v_1(t) \le v_1^*(t) + \varepsilon_1$ for $t \in ((n_1+n_2+1)T, (n_1+1)T + T_2]$,

$$\begin{aligned} V'\big|_{(1.1)} &\ge fK\left(1 - a\frac{p}{1-\exp(-\gamma T)} + \varepsilon_1 - M\right)x(t) + bK\left(g\frac{p}{1-\exp(-\gamma T)} + \varepsilon_1 - j\right)z(t) \\ &= \alpha_1 > 0, \end{aligned} \tag{2.26}$$

and $V((n_1+1)T + T_2) \ge V((n_1+n_2+1)T)\exp(\alpha_1 n_3 T)$. For $t \in [t^*, (n_1+n_2+1)T]$, it can be shown that $V|_{(1.1)} \ge -\alpha_2 V(t) > 0$; then, $V((n_1+n_2+1)T) \ge V^*(t)\exp(-\alpha_2(n_2+1)T)$, so

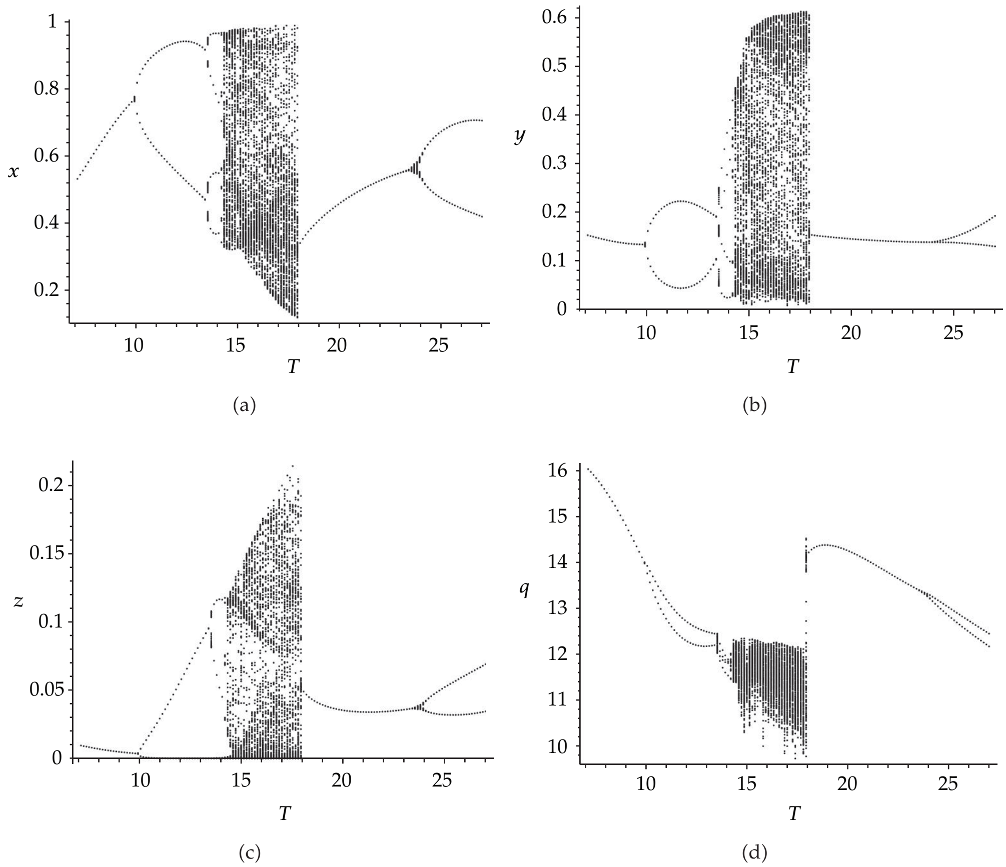

Figure 3: Bifurcation diagram of system (1.3) with initial conditions $x(0) = 0.1, y(0) = 0.2, z(0) = 0.3, q(0) = 0.4$, $7 \leq T \leq 27$, $b = 10$, $d = 3$, $c = 3$, $e = 0.5$, $f = 5$, $g = 2.5$, $h = 0.1$, $m = 2$; $j = 0.2$, $k = 0.3$, $l = 0.01$, $a = 1.3$, and $p = 0.6$.

$V((n_1+1)T+T_2) \geq V^*(t)\exp(-\alpha_2(n_2+1)T+\alpha_1 n_3 T) > \delta_5$, which is a contradiction. Therefore, there exists a $t_3 \in ((n_1+1)T, (n_1+1)T+T_2]$ such that $V(t) \geq \delta_5$, resulting in $V(t) \geq V^*(t)\exp(-\alpha_2(n_1+n_2+n_3+1)T) \triangleq \delta_6$.

When $t \geq t_3$, the same procedure can be performed. According to the above discussion, if $\Omega_0 = \{(x(t), y(t), z(t), q(t)) : V(t) = fKx(t) + bKz(t) + bhlq(t), \delta \leq V(t) \leq M_1\} \subset \operatorname{int} R^3_+$, every solution of system (1.3) will eventually enter and remain in the region Ω_0. This completes the proof. □

3. Numerical Analysis

3.1. Bifurcation

To study the dynamics of system (1.3), the period T and the impulsive control parameter p are used as the bifurcation parameter. The bifurcation diagram provides a summary of the basic dynamic behavior of the system [30, 31].

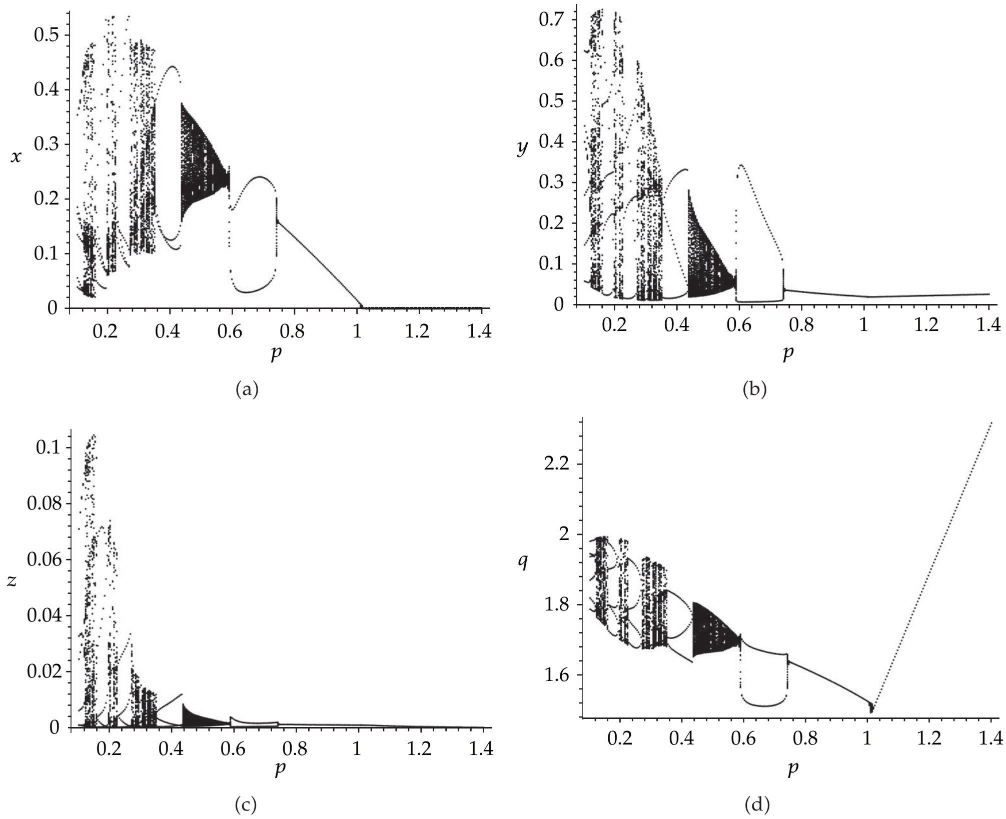

Figure 4: Bifurcation diagram of system (1.3) with initial conditions $x(0) = 0.1$, $y(0) = 0.2$, $z(0) = 0.3$, $q(0) = 0.4$, $0.1 \leq p \leq 2.4$, $b = 8$, $d = 3$, $c = 3$, $e = 0.2$, and $f = 3$ s, $g = 2.5$, $h = 0.1$, $m = 0.2$, $j = 0.1$, $k = 0.6$, $l = 0.008$, $a = 3$, $p = 0.6$.

First, the influence of the period T is studied using the time series shown in Figure 1. The bifurcation diagrams are shown in Figures 2 and 3. Next, the influence of the impulsive control parameter p is investigated. The bifurcation diagrams for this are shown in Figure 4.

To clearly see the dynamics of system (1.3), it is necessary to examine the phase diagrams at different value of the period T and parameter p corresponding to the bifurcation diagrams in Figures 2 and 4; the results of this analysis are shown in Figures 5 and 6.

Figures 2, 3, and 4 reveal the complex dynamics of system (1.3), including period-doubling cascades, symmetry-breaking pitchfork bifurcation, chaos, and nonunique dynamics. Because every bifurcation diagram is similar, only one needs to be explained. Take Figure 4(a) as an example. When $p \in [0, 0.124]$, the dynamics of the system are not obvious, but with increasing p, the dynamics become more obvious. The system enters into a chaotic band with periodic windows. When p is between 0.124 and 0.153, the chaotic behavior is intense, as can be seen in Figure 6(a). When p moves beyond 0.153, the chaotic behavior disappears. When $p \in [0.203, 0.219]$, the chaotic attractor gains in strength, and the chaotic behavior appears again. When p becomes greater than 0.219, periodic windows appear, as can be seen in Figures 6(b) and 6(c). When p is in the interval between 0.328 and 0.35, chaotic behavior ensues, as can be seen in Figure 6(d). As the value of p increases further, the system enters a stable state, as is shown in Figures 6(e), 6(f), and 6(g). When p moves

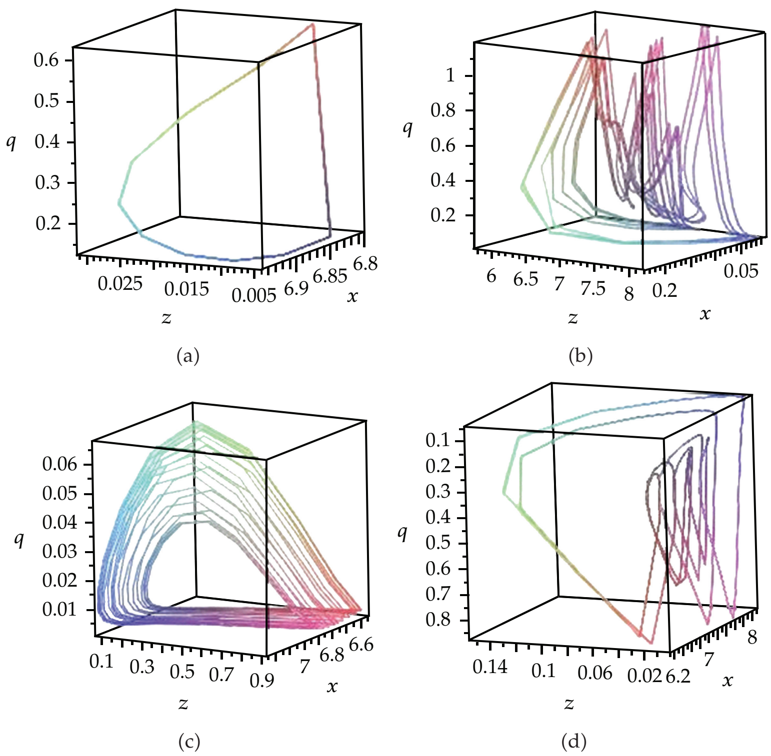

Figure 5: Periodic and chaotic behavior corresponding to Figure 2 as shown by phase diagrams: (a) $T = 10$, (b) $T = 17$, (c) $T = 15$, and (d) $T = 27.7$.

beyond 1.001, an unexpectedly chaotic phase appears, as is shown in Figure 6(h). It is clear that seasonal disturbances have little effect on the maximum density of all species; however, serious periodic oscillations are generated, and weak periodic solutions lose their stability and move into chaos. In summary, the key factor in the long-term dynamic behavior of system (1.3) is impulse perturbations, but seasonal disturbances can aggravate periodic oscillations and promote the emergence of chaos. Based on the above numerical simulation analysis, it is clear that impulsive control strategy has an important effect on the dynamical behaviors of the system, and weak periodic solutions lose their stability and move into chaos. In summary, the key factor in the long-term dynamic behavior of system (1.3) is impulsive control strategy, but disease disturbances can aggravate periodic oscillations and promote the emergence of chaos.

3.2. The largest Lyapunov Exponent

To detect whether the system exhibits chaotic behavior, one of the commonest methods is to calculate the largest Lyapunov exponent. The largest Lyapunov exponent takes into account the average exponential rates of divergence or convergence of nearby orbits in phase space [32]. A positive largest Lyapunov exponent indicates that the system is chaotic. If the largest Lyapunov exponent is negative, there must be periodic windows or a stable state. Through the largest Lyapunov exponent, it is possible to judge that at what time the system is chaotic,

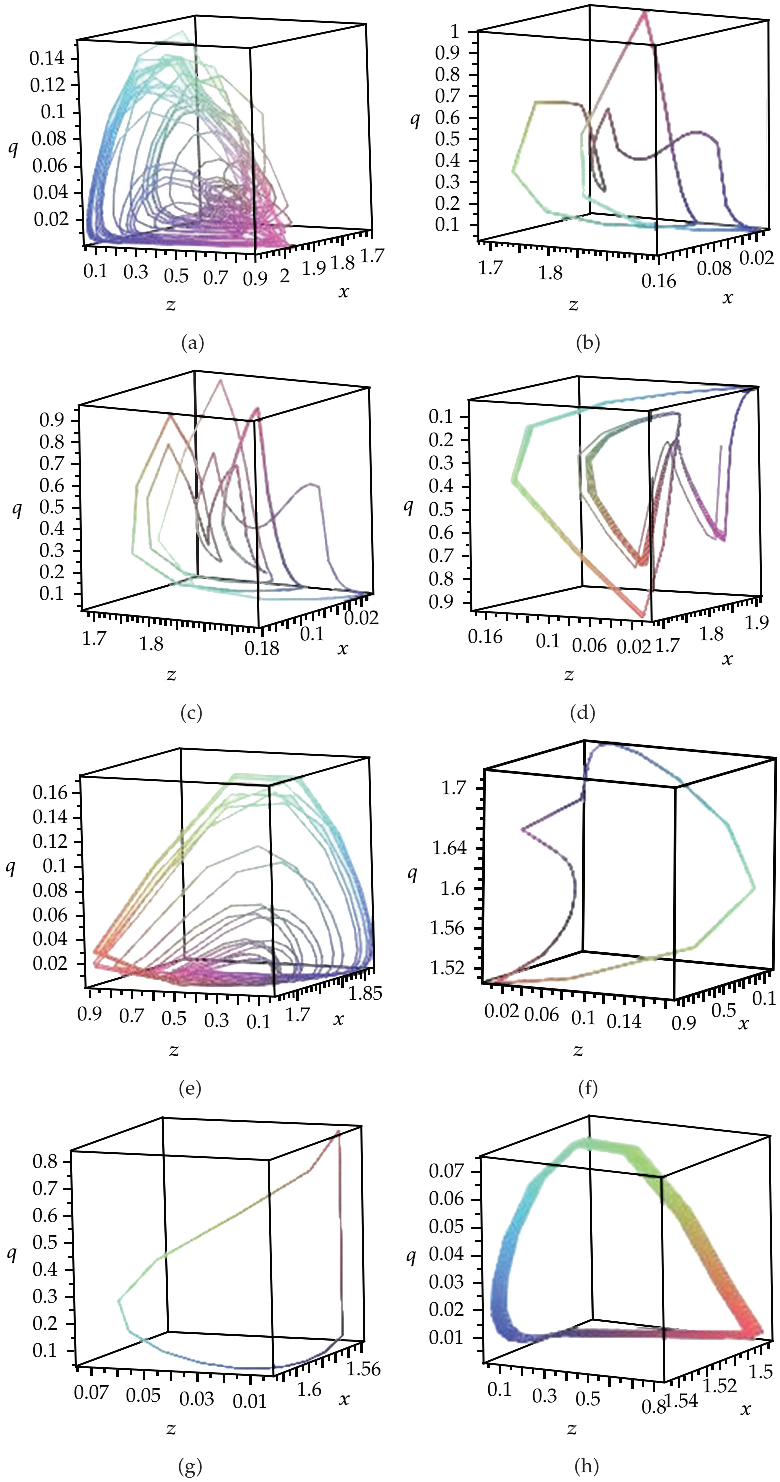

Figure 6: Periodic and chaotic behavior corresponding to Figure 3, as shown in phase diagrams: (a) $p = 0.13$, (b) $p = 0.25$, (c) $p = 0.3$, (d) $p = 0.35$, (e) $p = 0.45$, (f) $p = 0.7$, (g) $p = 0.9$, and (h) $p = 1.0009$.

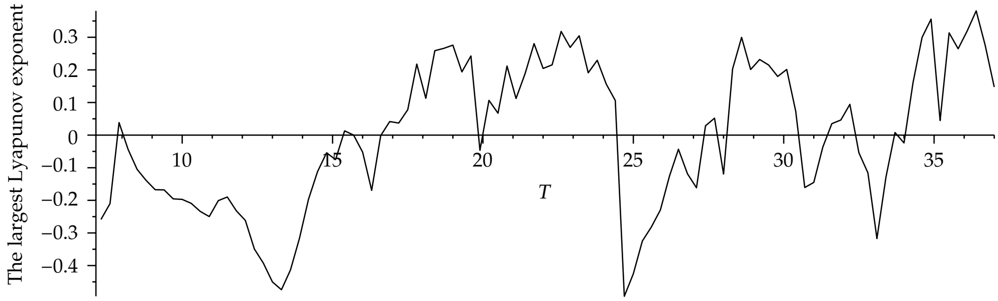

Figure 7: The largest Lyapunov exponents (LLE) corresponding to Figure 2.

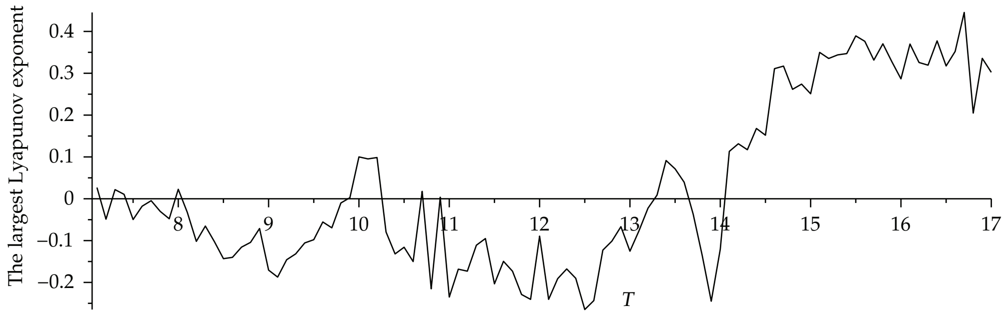

Figure 8: The largest Lyapunov exponents (LLE) corresponding to Figure 3.

and at what time the system is stable. The largest Lyapunov exponents corresponding to Figures 2, 3, and 4 can be calculated and are shown in Figures 7, 8, and 9, which shows the accuracy and effectiveness of numerical simulation. Moreover, using the simulation of the largest Lyapunov exponents, the existence of chaotic behavior in system (1.3) can be further confirmed.

3.3. *Strange Attractors and Power Spectra*

To understand the qualitative nature of strange attractors, power spectra are used [33]. From Section 3.2, it is known that the largest Lyapunov exponent for strange attractor (a) is 0.0413, and for strange attractor (b) is 0.124. Therefore, they are both chaotic attractors, and the exponent of (b) is larger than that of (a), which means that the chaotic dynamics of (b) are more extreme than those of (a). The power spectrum of strange attractor (a) is composed of strong broadband components and sharp peaks, as are shown in Figure 10(c). On the contrary, in the spectrum of strong chaotic attractor (b), it is difficult to distinguish any sharp peaks, as can be seen in Figure 10(d). These power spectra can be interpreted as meaning that (a) comes from a strong limit cycle, but that (b) experiences some weak limit cycles. Hence, it is obvious that the impulsive control strategy has a strong effect on the dynamical behaviors of system (1.3) with t the period of the impulsive control T varying but that (b) experiences some weak limit cycles.

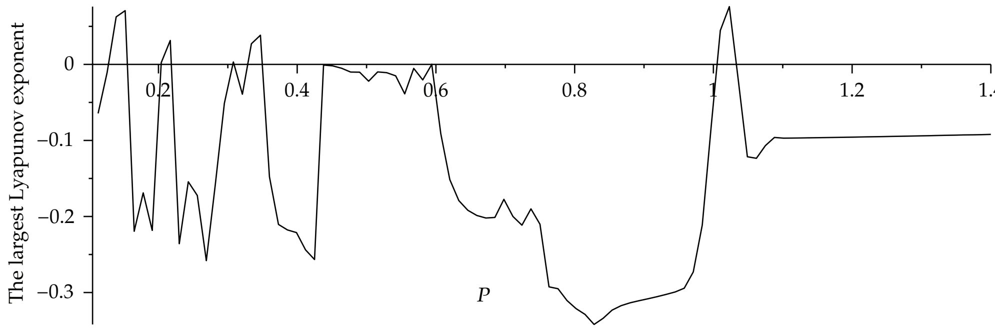

Figure 9: The largest Lyapunov exponents (LLE) corresponding to Figure 4.

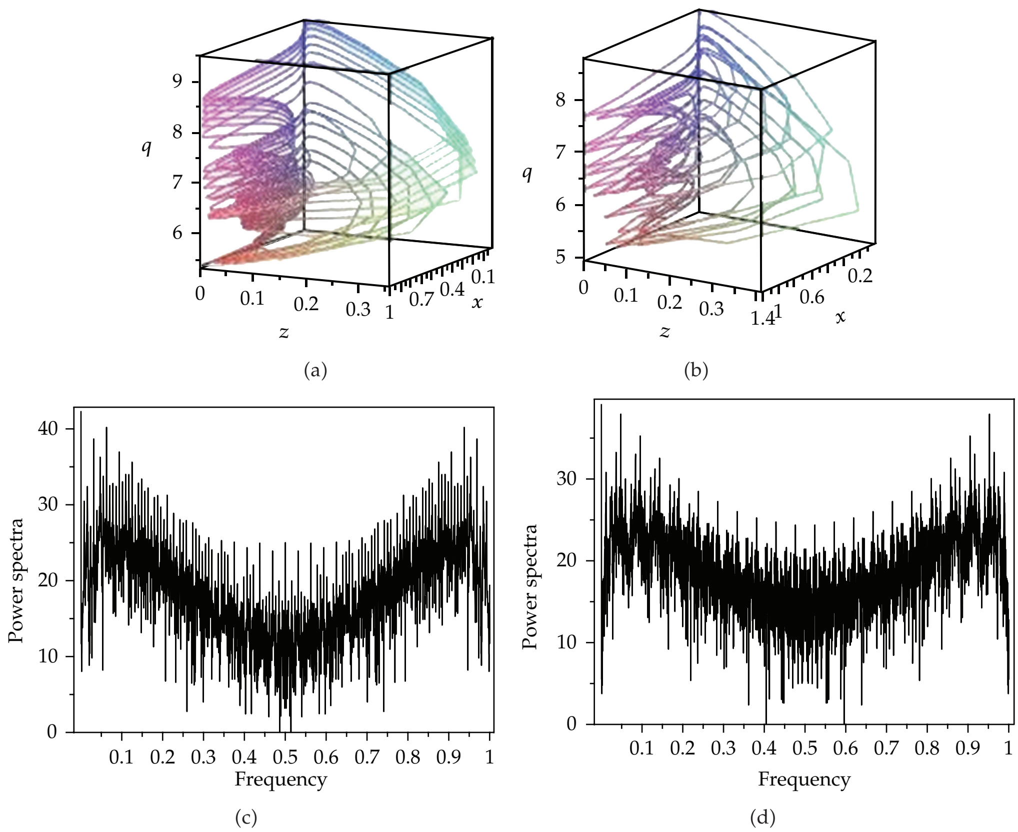

Figure 10: Strange attractors and power spectra: (a) strange attractor when $T = 32$, (b) strange attractor when $T = 21$, (c) power spectrum of attractor (a), and (d) power spectrum of attractor (b).

4. Conclusions and Remarks

In the paper, the dynamic behaviors of a predator-prey (pest) model with disease in prey and involving an impulsive control strategy are presented analytically and numerically. The critical conditions are obtained to ensure the local asymptotical stability and global attractivity of semitrivial periodic solution as well as population permanence. Numerical analysis indicates that the impulsive control strategy has a strong effect on the dynamical complexity and population persistent using bifurcation diagrams and power spectra diagrams. In addition, the largest Lyapunov exponents are computed. This computation further confirms the existence of chaotic behavior and the accuracy of numerical simulation. These results revealed that the introduction of disease and the use of an impulsive control strategy can change the dynamic behaviors of the system. The same results also have been observed in continuous-time models of predator-prey or three-species food-chain models [34–37] and other systems [38]. In a word, it should be stressed that the impulsive control strategy is an effective method to control complex dynamics of predator-prey (pest) model.

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References

[1] R. M. Anderson and R. M. May, "The invasion, persistence and spread of infectious diseases within animal and plant communities," *Philosophical transactions of the Royal Society of London. Series B*, vol. 314, no. 1167, pp. 533–570, 1986.
[2] K. P. Hadeler and H. I. Freedman, "Predator-prey populations with parasitic infection," *Journal of Mathematical Biology*, vol. 27, no. 6, pp. 609–631, 1989.
[3] G.-P. Hu and X.-L. Li, "Stability and Hopf bifurcation for a delayed predator-prey model with disease in the prey," *Chaos, Solitons and Fractals*, vol. 45, no. 3, pp. 229–237, 2012.
[4] W. Bob, A.K. George, V Voornb, and D. Krishna, "Stabilization and complex dynamics in a predator-prey model with predator suffering from an infectious disease," *Ecological Complexity*, vol. 8, no. 1, pp. 113–122, 2011.
[5] X. Niu, T. Zhang, and Z. Teng, "The asymptotic behavior of a nonautonomous eco-epidemic model with disease in the prey," *Applied Mathematical Modelling*, vol. 35, no. 1, pp. 457–470, 2011.
[6] E. Venturino, "Epidemics in predator-prey models: disease in the predators," *IMA Journal of Mathematics Applied in Medicine and Biology*, vol. 19, no. 3, pp. 185–205, 2002.
[7] Y. Xiao and L. Chen, "Modeling and analysis of a predator-prey model with disease in the prey," *Mathematical Biosciences*, vol. 171, no. 1, pp. 59–82, 2001.
[8] Y. Pei, S. Li, and C. Li, "Effect of delay on a predator-prey model with parasitic infection," *Nonlinear Dynamics*, vol. 63, no. 3, pp. 311–321, 2011.
[9] S. Chatterjee, M. Bandyopadhyay, and J. Chattopadhyay, "Proper predation makes the system disease free - Conclusion drawn from an eco-epidemiological model," *Journal of Biological Systems*, vol. 14, no. 4, pp. 599–616, 2006.
[10] K. P. Das, S. Chatterjee, and J. Chattopadhyay, "Disease in prey population and body size of intermediate predator reduce the prevalence of chaos-conclusion drawn from Hastings-Powell model," *Ecological Complexity*, vol. 6, no. 3, pp. 363–374, 2009.
[11] A. Hastings and T. Powell, "Chaos in a three-species food chain," *Ecology*, vol. 72, no. 3, pp. 896–903, 1991.
[12] X. Liu and L. Chen, "Complex dynamics of Holling type II Lotka-Volterra predator-prey system with impulsive perturbations on the predator," *Chaos, Solitons and Fractals*, vol. 16, no. 2, pp. 311–320, 2003.
[13] R. Shi, X. Jiang, and L. Chen, "A predator-prey model with disease in the prey and two impulses for integrated pest management," *Applied Mathematical Modelling*, vol. 33, no. 5, pp. 2248–2256, 2009.
[14] H. Liu, H. Xu, J. Yu, and G. Zhu, "Stability on coupling SIR epidemic model with vaccination," *Journal of Applied Mathematics*, no. 4, pp. 301–319, 2005.

[15] G. Ballinger and X. Liu, "Permanence of population growth models with impulsive effects," *Mathematical and Computer Modelling*, vol. 26, no. 12, pp. 59–72, 1997.
[16] L. Dong, L. Chen, and L. Sun, "Optimal harvesting policies for periodic Gompertz systems," *Nonlinear Analysis*, vol. 8, no. 2, pp. 572–578, 2007.
[17] X. Song and Y. Li, "Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect," *Nonlinear Analysis*, vol. 9, no. 1, pp. 64–79, 2008.
[18] X. Liu and L. Chen, "Complex dynamics of Holling type II Lotka-Volterra predator-prey system with impulsive perturbations on the predator," *Chaos, Solitons and Fractals*, vol. 16, no. 2, pp. 311–320, 2003.
[19] A. Lakmeche and O. Arino, "Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 7, no. 2, pp. 265–287, 2000.
[20] H. Yu, S. Zhong, R. P. Agarwal, and S. K. Sen, "Three-species food web model with impulsive control strategy and chaos," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 2, pp. 1002–1013, 2011.
[21] H. Yu and M. Zhao, "Seasonally perturbed prey-predator ecological system with the beddington-deangelis functional response," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 150359, 12 pages, 2012.
[22] M. Zhao, X. Wang, H. Yu, and J. Zhu, "Dynamics of an ecological model with impulsive control strategy and distributed time delay," *Mathematics and Computers in Simulation*, vol. 82, no. 8, pp. 1432–1444, 2012.
[23] Y. Wang and M. Zhao, "Dynamic analysis of an impulsively controlled predator-prey model with holling type iv functional response," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 141272, 18 pages, 2012.
[24] J. Yang and M. Zhao, "Complex behavior in a fish algae consumption model with impulsive control strategy," *Discrete Dynamics in Nature and Society. An International Multidisciplinary Research and Review Journal*, vol. 2011, Article ID 163541, 17 pages, 2011.
[25] J. Yang and M. Zhao, "A mathematical model for the dynamics of a fish algae consumption model with impulsive control strategy," *Journal of Applied Mathematics*, vol. 2012, Article ID 452789, 17 pages, 2012.
[26] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific Publishing, Singapore, 1989.
[27] V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 1993.
[28] K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, vol. 14 of *Classics in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1996.
[29] H. Yu, S. Zhong, and R. P. Agarwal, "Mathematics analysis and chaos in an ecological model with an impulsive control strategy," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 2, pp. 776–786, 2011.
[30] Z. W. Gao and D. W. C. Ho, "On state-space realization of Bezout factorizations in singular systems," *Dynamics of Continuous, Discrete & Impulsive Systems*, vol. 13, no. 3-4, pp. 387–410, 2006.
[31] H. Yu, M. Zhao, S. Lv, and L. Zhu, "Dynamic complexities in a parasitoid-host-parasitoid ecological model," *Chaos, Solitons and Fractals*, vol. 39, no. 1, pp. 39–48, 2009.
[32] F. Grond, H. H. Diebner, S. Sahle, A. Mathias, S. Fischer, and O. E. Rossler, "A robust, locally interpretable algorithm for Lyapunov exponents," *Chaos, Solitons and Fractals*, vol. 16, no. 5, pp. 841–852, 2003.
[33] C. Masoller, A. C. S. Schifino, and L. Romanelli, "Characterization of strange attractors of lorenz model of general circulation of the atmosphere," *Chaos, Solitons and Fractals*, vol. 6, no. C, pp. 357–366, 1995.
[34] S. Lv and M. Zhao, "The dynamic complexity of a three species food chain model," *Chaos, Solitons and Fractals*, vol. 37, no. 5, pp. 1469–1480, 2008.
[35] M. Zhao and S. Lv, "Chaos in a three-species food chain model with a Beddington-DeAngelis functional response," *Chaos, Solitons and Fractals*, vol. 40, no. 5, pp. 2305–2316, 2009.
[36] S. Lv and M. Zhao, "The dynamic complexity of a host-parasitoid model with a lower bound for the host," *Chaos, Solitons and Fractals*, vol. 36, no. 4, pp. 911–919, 2008.
[37] M. Zhao, H. Yu, and J. Zhu, "Effects of a population floor on the persistence of chaos in a mutual interference host-parasitoid model," *Chaos, Solitons and Fractals*, vol. 42, no. 2, pp. 1245–1250, 2009.
[38] Z. Gao and S. X. Ding, "Actuator fault robust estimation and fault-tolerant control for a class of nonlinear descriptor systems," *Automatica*, vol. 43, no. 5, pp. 912–920, 2007.