

Research Article

Self-Adaptive and Relaxed Self-Adaptive Projection Methods for Solving the Multiple-Set Split Feasibility Problem

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Given nonempty closed convex subsets $C_i \subseteq R^m$, $i = 1, 2, \dots, t$ and nonempty closed convex subsets $Q_j \subseteq R^n$, $j = 1, 2, \dots, r$, in the n - and m -dimensional Euclidean spaces, respectively. The multiple-set split feasibility problem (MSSFP) proposed by Censor is to find a vector $x \in \bigcap_{i=1}^t C_i$ such that $Ax \in \bigcap_{j=1}^r Q_j$, where A is a given $M \times N$ real matrix. It serves as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operator's range. MSSFP has a variety of specific applications in real world, such as medical care, image reconstruction, and signal processing. In this paper, for the MSSFP, we first propose a new self-adaptive projection method by adopting Armijo-like searches, which does not require estimating the Lipschitz constant and calculating the largest eigenvalue of the matrix $A^T A$; besides, it makes a sufficient decrease of the objective function at each iteration. Then we introduce a relaxed self-adaptive projection method by using projections onto half-spaces instead of those onto convex sets. Obviously, the latter are easy to implement. Global convergence for both methods is proved under a suitable condition.

1. Introduction

The multiple-sets split feasibility problem (MSSFP) requires to find a point closest to a family of closed convex sets in one space such that its image under a linear transformation will be closest to another family of closed convex sets in the image space. It is formulated as follows:

$$\text{Find a } x \in C := \bigcap_{i=1}^t C_i \text{ such that } Ax \in Q := \bigcap_{j=1}^r Q_j, \quad (1.1)$$

where nonempty closed convex sets $C_i \subseteq R^n$, $i = 1, 2, \dots, t$, in the n -dimensional Euclidean space R^n , and nonempty closed convex sets $Q_j \subseteq R^m$, $j = 1, 2, \dots, r$, in the m -dimensional Euclidean space R^m . A is an $m \times n$ real matrix. Specially, the problem with only a single set C in R^n and a single set Q in R^m was introduced by Censor and Elfving [1] and was called the split feasibility problem (SFP).

Such MSSFPs (1.1), proposed in [2], arise in signal processing, image reconstruction and so on. Various algorithms have been invented to solve MSSFP (1.1). See [2–5] and references therein.

In [2], Censor and Elfving were handling the MSSFP (1.1), for both the consistent and the inconsistent cases, where they aim at minimizing the proximity function

$$P(x) = \left(\frac{1}{2}\right) \sum_{i=1}^t \alpha_i \|P_{C_i}(x) - x\|^2 + \left(\frac{1}{2}\right) \sum_{j=1}^r \beta_j \|P_{Q_j}(Ax) - Ax\|^2. \quad (1.2)$$

For convenience reasons, they consider an additional closed convex set $\Omega \subseteq R^n$. Their algorithm for the MSSFP (1.1) involves orthogonal projection onto $\Omega \subseteq R^n$, $C_i \subseteq R^n$, $i = 1, 2, \dots, t$, and $Q_j \subseteq R^m$, $j = 1, 2, \dots, r$, which were assumed to be easily calculated and has the following iterative step:

$$x^{k+1} = P_{\Omega} \left(x^k + \gamma \left(\sum_{i=1}^t \alpha_i (P_{C_i}(x^k) - x^k) + \sum_{j=1}^r \beta_j A^T (P_{Q_j}(Ax^k) - Ax^k) \right) \right), \quad (1.3)$$

where $\alpha_i > 0$, $i = 1, 2, \dots, t$. $\beta_j > 0$, $j = 1, 2, \dots, r$. $\gamma \in (0, 2/L)$, $L_1 = \sum_{i=1}^t \alpha_i + \lambda \sum_{j=1}^r \beta_j$ is the Lipschitz constant of $\nabla P(x)$, which is the gradient of the proximity function $P(x)$ defined by (1.2), and λ is the spectral radius of the matrix $A^T A$. For any starting vector $x^0 \in R^n$, the algorithm converges to a solution of the MSSFP (1.1), whenever MSSFP (1.1) has a solution. In the inconsistent case, it find a point “closest” to all sets.

This algorithm uses a fixed stepsize related to the Lipschitz constant L , which sometimes computing it may be hard. On the other hand, even if we know the Lipschitz constant L , the method with fixed stepsize may lead to slow speed of convergence.

In 2005, Qu and Xiu [6] modified the CQ algorithm [7] and relaxed CQ algorithm [8] by adopting Armijo-like searches to solve the SFP, where the second algorithm used orthogonal projections onto half-spaces instead of projections onto the original convex sets, just as Yang’s relaxed CQ algorithm [8]. This may reduce a lot of work for computing projections, since projections onto half-spaces can be directly calculated.

Motivated by Qu and Xiu’s idea, Zhao and Yang in [4] introduce a self-adaptive projection method by adopting Armijo-like searches to solve the MSSFP (1.1) and propose a relaxed self-adaptive projection method by using orthogonal projections onto half-spaces instead of these projections onto the original convex sets, which is more practical. But the same as Algorithm 1.3, Zhao and Yang’s algorithm involves an addition projection P_{Ω} . Though the MSSFP (1.1) includes the SFP as a special case, the Zhao and Yang’s algorithm does not reduce to Qu and Xiu’s modifications of the CQ algorithm [6].

In this paper, We first proposed a self-adaptive method by adopting Armijo-like searches to solve the MSSFP (1.1) without an addition projection P_{Ω} , then a relaxed self-adaptive projection method was introduced which only involves orthogonal projections onto

half-spaces, so that the algorithm is implementable. We need not estimate the Lipschitz constant and make a sufficient decrease of the objection function at each iteration; besides, these projection algorithms can reduce to the modifications of the CQ algorithm [6] when the MSSFP (1.1) is reduced to the SFP. We also show convergence the algorithms under mild conditions.

2. Preliminaries

In this section, we give some definitions and basic results that will be used in this paper.

Definition 2.1. An operator F from a set $X \subset R^n$ into R^n is called

(a) monotone on X , if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in X; \quad (2.1)$$

(b) cocoercive on X with constant $\alpha > 0$, if

$$\langle F(x) - F(y), x - y \rangle \geq \alpha \|F(x) - F(y)\|^2, \quad \forall x, y \in X; \quad (2.2)$$

(c) Lipschitz continuous on X with constant $\lambda > 0$, if

$$\|F(x) - F(y)\| \leq \lambda \|x - y\|, \quad \forall x, y \in X. \quad (2.3)$$

In particular, if $\lambda = 1$, F is said to be nonexpansive. It is easily seen from the definitions that cocoercive mappings are monotone.

Definition 2.2. Functions $f(x)$, differentiable on a nonempty convex set S , is pseudoconvex if for every $x_1, x_2 \in S$, the condition $f(x_1) < f(x_2)$ implies that

$$\nabla f(x_2)^T (x_1 - x_2) \leq 0. \quad (2.4)$$

It is known that differentiable convex functions are pseudoconvex (see [9]).

For a given nonempty closed convex set Ω in R^n , the orthogonal projection from R^n onto Ω is defined by

$$P_\Omega(x) = \operatorname{argmin}\{\|x - y\| \mid y \in \Omega\}, \quad x \in R^n. \quad (2.5)$$

Lemma 2.3 (see [10]). *Let Ω be a nonempty closed convex subset in R^n , then, for any $x, y \in R^n$ and $z \in \Omega$.*

- (1) $\langle P_\Omega(x) - x, z - P_\Omega(x) \rangle \geq 0$;
- (2) $\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \langle P_\Omega(x) - P_\Omega(y), x - y \rangle$;
- (3) $\|P_\Omega(x) - z\|^2 \leq \|x - z\|^2 - \|P_\Omega(x) - x\|^2$.

From Lemma 2.3 (2), one sees that the orthogonal projection mapping $P_\Omega(x)$ is cocoercive with modulus 1, monotone, and nonexpansive.

Let F be a mapping from R^n into R^n . For any $x \in R^n$ and $\alpha > 0$, define $x(\alpha) = P_\Omega(x - \alpha F(x))$, $e(x, \alpha) = x - x(\alpha)$.

Lemma 2.4 (see [6]). Let F be a mapping from R^n into R^n . For any $x \in R^n$ and $\alpha > 0$, one has $\min\{1, \alpha\}\|e(x, 1)\| \leq \|e(x, \alpha)\| \leq \max\{1, \alpha\}\|e(x, 1)\|$.

Lemma 2.5 (see [5, 9]). Suppose $h : R^n \rightarrow R$ is a convex function, then it is subdifferentiable everywhere and its subdifferentials are uniformly bounded on any bounded subset of R^n .

3. Self-Adaptive Projection Iterative Scheme and Convergence Results

It is easily seen that if the solution set of MSSFP (1.1) is nonempty, then the MSSFP (1.1) is equivalent to the minimization problem of q over all $x \in C := \bigcap_{i=1}^r C_i$. $q(x)$ is defined by

$$q(x) = \left(\frac{1}{2}\right) \sum_{j=1}^r \beta_j \left\| P_{Q_j}(Ax) - Ax \right\|^2, \quad (3.1)$$

where $\beta_j > 0$. Note that the gradient of $q(x)$ is

$$\nabla q(x) = \sum_{j=1}^r \beta_j A^T (I - P_{Q_j}) Ax. \quad (3.2)$$

Consider the following constrained minimization problem:

$$\min\{q(x), x \in C\}. \quad (3.3)$$

We say that a point $x^* \in C$ is a stationary point of the problem (3.3) if it satisfies the condition

$$\langle \nabla q(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (3.4)$$

This optimization problem is proposed by Xu [3] for solving the MSSFP (1.1); the ∇q defined by (3.2) is L -Lipschitzian with $L = \|A\|^2 \sum_{j=1}^r \beta_j$ and ∇q is $(1/L)$ -ism.

Algorithm 3.1. Given constant $\beta > 0$, $\sigma \in (0, 1)$. Let x^0 be arbitrary. For $k = 0, 1, \dots$, calculate

$$x^{k+1} = P_{C_{[k+1]}}(x^k - \tau_k \nabla q(x) x^k), \quad (3.5)$$

where $C_{[n]} = C_{n \bmod N}$ and mod function takes values in $1, 2, \dots, N$, $\tau_k = \beta \gamma^{l_k}$ and l_k is the smallest nonnegative integer l such that

$$q(P_{C_{[k+1]}}(x^k - \beta \gamma^l \nabla q(x) x^k)) \leq q(x^k) - \sigma \langle \nabla q(x^k), x^k - P_{C_{[k+1]}}(x^k - \beta \gamma^l \nabla q(x) x^k) \rangle. \quad (3.6)$$

Algorithm 3.1 need not estimate the largest eigenvalue of the matrix $A^T A$, and the stepsize τ_k is chosen so that the objective function $q(x)$ has a sufficient decrease. It is in fact a special case of the standard gradient projection method with the Armijo-like search rule for solving the constrained optimization problem:

$$\min\{g(x); x \in \Omega\}, \quad (3.7)$$

where $\Omega \subseteq R^n$ is a nonempty closed convex set, and the function $g(x)$ is continuously differentiable on Ω , then the following convergence result ensures the convergence of Algorithm 3.1.

Lemma 3.2 (see [6]). *Let $g \in C_{\Omega}^1$ be pseudoconvex and x^k be an infinite sequence generated by the gradient projection method with the Armijo-like searches. Then, the following conclusions hold:*

- (1) $\lim_{k \rightarrow \infty} g(x^k) = \inf\{g(x) : x \in \Omega\}$;
- (2) $\emptyset \neq \Omega^*$, which denotes the set of the optimal solutions to (3.7), if and only if there exists at least one limit point of $\{x^k\}$. In this case, $\{x^k\}$ converges to a solution of (3.7).

Since the function $q(x)$ is convex and continuously differentiable on C , therefore it is pseudoconvex. Then, by Lemma 3.2, one immediately obtains the following convergence result.

Theorem 3.3. *Let $\{x^k\}$ be a sequence generated by Algorithm 3.1, then the following conclusions hold:*

- (1) $\{x^k\}$ is bounded if and only if the solution set of (3.3) is nonempty. In such a case, $\{x^k\}$ must converge to a solution of (3.3).
- (2) $\{x^k\}$ is bounded and $\lim_{k \rightarrow \infty} q(x^k) = 0$ if and only if the MSSFP (1.1) is solvable. In such a case, $\{x^k\}$ must converge to a solution of MSSFP (1.1).

However, in Algorithm 3.1, it costs a large amount of work to compute the orthogonal projections P_{C_i} and P_{Q_j} ; therefore, the same as Censor's method, these projections are assumed to be easily calculated. But, in some cases it is difficult or costs too much work to exactly compute the orthogonal projection, then the efficiency of these methods will be deeply affected. In what follows, one assume that the projections are not easily calculated. One present a relaxed self-adaptive projection method. Carefully speaking, the convex sets C_i and Q_j satisfy the following assumptions.

- (1) The sets $C_i, i = 1, 2, \dots, t$, are given by

$$C_i = \{x \in R^n \mid c_i(x) \leq 0\}, \quad (3.8)$$

where $c_i : R^n \rightarrow R, i = 1, 2, \dots, t$, are convex functions.

The sets $Q_j, j = 1, 2, \dots, r$, are given by

$$Q_j = \{y \in R^m \mid q_j(y) \leq 0\}, \quad (3.9)$$

where $q_j : R^m \rightarrow R, j = 1, 2, \dots, r$, are convex functions.

(2) For any $x \in R^n$, at least one subgradient $\xi_i \in \partial c_i(x)$ can be calculated, where $\partial c_i(x)$ is a generalized gradient (subdifferential) of $c_i(x)$ at x and it is defined as follows:

$$\partial c_i(x) = \{ \xi_i \in R^n \mid c_i(z) \geq c_i(x) + \langle \xi_i, z - x \rangle \quad \forall z \in R^n \}. \quad (3.10)$$

For any $y \in R^m$, at least one subgradient $\eta_j \in \partial q_j(y)$ can be calculated, where $\partial q_j(y)$ is a generalized gradient (subdifferential) of $q_j(y)$ at y and it is defined as the following:

$$\partial q_j(y) = \{ \eta_j \in R^m \mid q_j(u) \geq q_j(y) + \langle \eta_j, u - y \rangle \quad \forall u \in R^m \}. \quad (3.11)$$

In the k th iteration, let

$$C_i^k = \{ x \in R^n \mid c_i(x^k) + \langle \xi_i^k, x - x^k \rangle \leq 0 \}, \quad (3.12)$$

where ξ_i^k is an element in $\partial c_i(x^k)$, $i = 1, 2, \dots, t$.

Consider

$$Q_j^k = \{ y \in R^m \mid q_j(Ax^k) + \langle \eta_j^k, y - Ax^k \rangle \leq 0 \}, \quad (3.13)$$

where η_j^k is an element in $\partial q_j(Ax^k)$, $j = 1, 2, \dots, r$.

By the definition of the subgradient, it is clear that $C_i \subseteq C_i^k$, $Q_j \subseteq Q_j^k$ and the orthogonal projections onto C_i^k and Q_j^k can be calculated [4, 6, 8]. Define

$$q_k(x) = \left(\frac{1}{2} \right) \sum_{j=1}^r \beta_j \left\| P_{Q_j^k}(Ax) - Ax \right\|^2, \quad (3.14)$$

where $\beta_j > 0$. Then

$$\nabla q_k(x) = \sum_{j=1}^r \beta_j A^T (I - P_{Q_j^k}) Ax. \quad (3.15)$$

For the Lipschitz constant and the cocoercive modulus of ∇q defined by (3.2) are not related to the nonempty closed convex sets C_i and Q_j [3], one can obtain that the $\nabla q_k(x)$ is L -Lipschitzian with $L = \|A\|^2 \sum_{j=1}^r \beta_j$ and $(1/L)$ -ism. So, $\nabla q_k(x)$ is monotone.

Algorithm 3.4. Given constant $\gamma > 0$, $\alpha \in (0, 1)$, $\mu \in (0, 1)$. Let x^0 be arbitrary. For $k = 0, 1, 2, \dots$, compute

$$\bar{x}^k = P_{C_{[k+1]}^k} \left(x^k - \rho_k \nabla q_k(x^k) \right), \quad (3.16)$$

where $\rho_k = \gamma\alpha^{l_k}$ and l_k is the smallest non-negative interger l such that

$$\left\| \nabla q_k(x^k) - \nabla q_k(\bar{x}^k) \right\| \leq \mu \frac{\|x^k - \bar{x}^k\|}{\rho_k}. \quad (3.17)$$

Set

$$x^{k+1} = P_{C_{[k+1]}^k} \left(x^k - \rho_k \nabla q_k(\bar{x}^k) \right). \quad (3.18)$$

Lemma 3.5. *The Armijo-like search rule (3.17) is well defined, and $\mu\alpha/L \leq \rho_k \leq \gamma$.*

Proof. Obviously, from (3.17), $\rho_k \leq \gamma$ for $k = 0, 1, \dots$. We know that ρ_k/α must violate inequality (3.17). That is,

$$\left\| \nabla q_k(x^k) - \nabla q_k \left(P_{C_{[k+1]}^k} \left(x^k - \frac{\rho_k}{\alpha} \nabla q_k(x^k) \right) \right) \right\| \geq \mu \frac{\|x^k - P_{C_{[k+1]}^k} (x^k - (\rho_k/\alpha) \nabla q_k(x^k))\|}{\rho_k/\alpha}. \quad (3.19)$$

Since ∇q_k is Lipschitz continuous with constant L , which together with (3.19), we have

$$\rho_k > \frac{\mu\alpha}{L}, \quad (3.20)$$

which completes the proof. \square

Theorem 3.6. *Let $\{x^k\}$ be a sequence generated by Algorithm 3.4. If the solution set of the MSSFP (1.1) is nonempty, the $\{x^k\}$ converges to a solution of the MSSFP (1.1).*

Proof. Let x^* be a solution of the MSSFP (1.1), then $x^* = P_C(x^*) = P_{C_i}(x^*)$, $i = 1, 2, \dots, t$. and $Ax^* = P_Q(Ax^*) = P_{Q_j}(x^*)$, $j = 1, 2, \dots, r$.

Since $C_i \subseteq C_i^k$, $Q_j \subseteq Q_j^k$ for all i and j , we have $x^* \in C_i^k$, $Ax^* \in Q_j^k$, and, $q_k(x^*) = 0$; thus, we have $\nabla q_k(x^*) = 0$ for all $k = 0, 1, \dots$

Using the monotonicity of ∇q_k , we have for all $k = 0, 1, \dots$

$$\langle \nabla q_k(\bar{x}^k) - \nabla q_k(x^*), \bar{x}^k - x^* \rangle \geq 0. \quad (3.21)$$

This implies

$$\langle \nabla q_k(\bar{x}^k), \bar{x}^k - x^* \rangle \geq \langle \nabla q_k(x^*), \bar{x}^k - x^* \rangle = 0. \quad (3.22)$$

Therefore, we have

$$\langle \nabla q_k(\bar{x}^k), x^{k+1} - x^* \rangle \geq \langle \nabla q_k(\bar{x}^k), x^{k+1} - \bar{x}^k \rangle. \quad (3.23)$$

Thus, using part (3) of Lemma 2.3 and (3.23), we obtain

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|P_{C_{[k+1]}^k}(x^k - \rho_k \nabla q_k(\bar{x}^k)) - x^*\|^2 \\
&\leq \|x^k - \rho_k \nabla q_k(\bar{x}^k) - x^*\|^2 - \|x^{k+1} - x^k + \rho_k \nabla q_k(\bar{x}^k)\|^2 \\
&= \|x^k - x^*\|^2 - 2\rho_k \langle \nabla q_k(\bar{x}^k), x^k - x^* \rangle - \|x^{k+1} - x^k\|^2 \\
&\quad - 2\rho_k \langle \nabla q_k(\bar{x}^k), x^{k+1} - x^k \rangle \\
&\leq \|x^k - x^*\|^2 - 2\rho_k \langle \nabla q_k(\bar{x}^k), x^{k+1} - \bar{x}^k \rangle - \|x^{k+1} - \bar{x}^k + \bar{x}^k - x^k\|^2 \\
&= \|x^k - x^*\|^2 - 2\rho_k \langle \nabla q_k(\bar{x}^k), x^{k+1} - \bar{x}^k \rangle - \|x^{k+1} - \bar{x}^k\|^2 \\
&\quad - \|\bar{x}^k - x^k\|^2 - 2\langle \bar{x}^k - x^k, x^{k+1} - \bar{x}^k \rangle \\
&= \|x^k - x^*\|^2 - \|\bar{x}^k - x^k\|^2 - \|x^{k+1} - \bar{x}^k\|^2 + 2\langle x^k - \bar{x}^k - \rho_k \nabla q_k(\bar{x}^k), x^{k+1} - \bar{x}^k \rangle.
\end{aligned} \tag{3.24}$$

Since $\bar{x}^k = P_{C_{[k+1]}^k}(x^k - \rho_k \nabla q_k(x^k))$, $x^{k+1} \in C_{[k+1]}^k$. By Lemma 2.3(1), we have $\langle \bar{x}^k - x^k + \rho_k \nabla q_k(x^k), x^{k+1} - \bar{x}^k \rangle \geq 0$; also, by search rule (3.17), it follows that

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \|\bar{x}^k - x^k\|^2 - \|x^{k+1} - \bar{x}^k\|^2 \\
&\quad + 2\langle x^k - \bar{x}^k - \rho_k \nabla q_k(\bar{x}^k), x^{k+1} - \bar{x}^k \rangle \\
&\quad + 2\langle \bar{x}^k - x^k + \rho_k \nabla q_k(x^k), x^{k+1} - \bar{x}^k \rangle \\
&= \|x^k - x^*\|^2 - \|\bar{x}^k - x^k\|^2 - \|x^{k+1} - \bar{x}^k\|^2 \\
&\quad + 2\rho_k \langle \nabla q_k(x^k) - \nabla q_k(\bar{x}^k), x^{k+1} - \bar{x}^k \rangle \\
&\leq \|x^k - x^*\|^2 - \|\bar{x}^k - x^k\|^2 - \|x^{k+1} - \bar{x}^k\|^2 \\
&\quad + \rho_k^2 \|\nabla q_k(x^k) - \nabla q_k(\bar{x}^k)\|^2 + \|x^{k+1} - \bar{x}^k\|^2 \\
&\leq \|x^k - x^*\|^2 - \|\bar{x}^k - x^k\|^2 + \mu^2 \|x^k - \bar{x}^k\|^2 \\
&= \|x^k - x^*\|^2 - (1 - \mu^2) \|\bar{x}^k - x^k\|^2,
\end{aligned} \tag{3.25}$$

which implies that the sequence $\{\|x^k - x^*\|\}$ is monotonically decreasing and hence $\{x^k\}$ is bounded. Consequently, we get from (3.25)

$$\lim_{k \rightarrow \infty} \|x^k - \bar{x}^k\| = 0. \tag{3.26}$$

On the other hand

$$\begin{aligned} \|x^{k+1} - x^k\| &\leq \|x^{k+1} - \bar{x}^k\| + \|\bar{x}^k - x^k\| \\ &\leq \|x^k - \rho_k \nabla q_k(\bar{x}^k) - x^k + \rho_k \nabla q_k(x^k)\| + \|\bar{x}^k - x^k\| \\ &= \rho_k \|\nabla q_k(\bar{x}^k) - \nabla q_k(x^k)\| + \|\bar{x}^k - x^k\| \\ &\leq (\mu + 1) \|\bar{x}^k - x^k\|, \end{aligned} \tag{3.27}$$

which results in

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0. \tag{3.28}$$

Let \tilde{x} be an accumulation point of $\{x^k\}$ and $x^{k_n} \rightarrow \tilde{x}$, where $\{x^{k_n}\}_{n=1}^\infty$ is a subsequence of $\{x^k\}$. We will show that \tilde{x} is a solution of the MSSFP (1.1). Thus, we need to show that $\tilde{x} \in C = \bigcap_{i=1}^t C_i$ and $A\tilde{x} \in Q = \bigcap_{j=1}^r Q_j$.

Since $x^{k_n+1} \in C_{[k_n+1]}^{k_n}$ for all $n = 1, 2, \dots$, then by the definition of $C_{[k_n+1]}^{k_n}$, we have

$$c_{[k_i+1]}(x^{k_n}) + \langle \xi_{[k_n+1]}^{k_n+1}, x^{k_n+1} - x^{k_n} \rangle \leq 0. \tag{3.29}$$

Passing onto the limit in this inequality and taking into account (3.29) and Lemma 3.2, we obtain that $c_{[k_n+1]}(\tilde{x}) \leq 0$ with $k_n \rightarrow \infty$.

Because $C_{[k_n+1]}$ is repeated regularly of C_1, C_2, \dots, C_t , so $\tilde{x} \in C_i$ for every $1 \leq i \leq t$; thus, $\tilde{x} \in C = \bigcap_{i=1}^t C_i$. Next, we need to show $A\tilde{x} \in Q = \bigcap_{j=1}^r Q_j$.

Let $e_k(x, \rho) = x - P_{C_{[k+1]}^k}(x - \rho \nabla q_k(x))$, $k = 0, 1, 2, \dots$, then $e_{k_n}(x^{k_n}, \rho_{k_n}) = x^{k_n} - \bar{x}^{k_n}$ and we get from Lemmas 2.4 and 3.2, and (3.26) that

$$\lim_{k_n \rightarrow \infty} \|e_{k_n}(x^{k_n}, 1)\| \leq \lim_{k_n \rightarrow \infty} \frac{\|x^{k_n} - \bar{x}^{k_n}\|}{\min\{1, \rho_{k_n}\}} \leq \lim_{k_n \rightarrow \infty} \frac{\|x^{k_n} - \bar{x}^{k_n}\|}{\min\{1, \hat{\rho}\}} = 0, \tag{3.30}$$

where $\hat{\rho} = \mu\alpha/L$. Since x^* is a solution point of MSSFP (1.1), so $x^* \in C_i$, $i = 1, 2, \dots, t$. By using Lemma 2.3, we have

$$\begin{aligned} 0 &\leq \langle x^{k_n} - \nabla q_{k_n}(x^{k_n}) - P_{C_{[k_n+1]}^{k_n}}(x^{k_n} - \nabla q_{k_n}(x^{k_n})), P_{C_{[k_n+1]}^{k_n}}(x^{k_n} - \nabla q_{k_n}(x^{k_n})) - x^* \rangle \\ &= \langle e_{k_n}(x^{k_n}, 1) - \nabla q_{k_n}(x^{k_n}), x^{k_n} - x^* - e_{k_n}(x^{k_n}, 1) \rangle. \end{aligned} \tag{3.31}$$

From Lemma 2.3, we see that orthogonal projection mappings are cocoercive with modulus 1, taking into account the fact that the mapping F is cocoercive with modulus 1 if and only if $I - F$ is cocoercive with modulus 1 [6], we obtain from (3.31) and $\nabla q_{k_n}(x^*) = 0$ that

$$\begin{aligned}
\langle x^{k_n} - x^*, e_{k_n}(x^{k_n}, 1) \rangle &\geq \|e_{k_n}(x^{k_n}, 1)\|^2 - \langle \nabla q_{k_n}(x^{k_n}), e_{k_n}(x^{k_n}, 1) \rangle \\
&\quad + \langle \nabla q_{k_n}(x^{k_n}) - \nabla q_{k_n}(x^*), x^{k_n} - x^* \rangle \\
&= \|e_{k_n}(x^{k_n}, 1)\|^2 - \langle \nabla q_{k_n}(x^{k_n}), e_{k_n}(x^{k_n}, 1) \rangle \\
&\quad + \sum_{j=1}^r \beta_j \langle A^T(I - P_{Q_j^{k_n}})(Ax^{k_n}) - A^T(I - P_{Q_j^{k_n}})(Ax^*), x^{k_n} - x^* \rangle \\
&= \|e_{k_n}(x^{k_n}, 1)\|^2 - \langle \nabla q_{k_n}(x^{k_n}), e_{k_n}(x^{k_n}, 1) \rangle \\
&\quad + \sum_{j=1}^r \beta_j \langle (I - P_{Q_j^{k_n}})(Ax^{k_n}) - (I - P_{Q_j^{k_n}})(Ax^*), Ax^{k_n} - Ax^* \rangle \\
&\geq \|e_{k_n}(x^{k_n}, 1)\|^2 - \langle \nabla q_{k_n}(x^{k_n}), e_{k_n}(x^{k_n}, 1) \rangle \\
&\quad + \sum_{j=1}^r \beta_j \|Ax^{k_n} - P_{Q_j^{k_n}}(Ax^{k_n})\|^2.
\end{aligned} \tag{3.32}$$

Since $\|\nabla q_{k_n}(x^{k_n})\| = \|\nabla q_{k_n}(x^{k_n}) - \nabla q_{k_n}(x^*)\| \leq L\|x^{k_n} - x^*\|$ and $\{x^{k_n}\}$ is bounded, the sequence $\nabla q_{k_n}(x^{k_n})$ is also bounded. Therefore, from (3.30) and (3.32), we get for all $j = 1, 2, \dots$, that

$$\lim_{k_n \rightarrow \infty} \|Ax^{k_n} - P_{Q_j^{k_n}}(Ax^{k_n})\| = 0. \tag{3.33}$$

Moreover, since $P_{Q_j^{k_n}}(Ax^{k_n}) \in Q_j^{k_n}$, we have

$$q_j(Ax^{k_n}) + \langle \eta_j^{k_n}, P_{Q_j^{k_n}}(Ax^{k_n}) - Ax^{k_n} \rangle \leq 0, \quad j = 1, 2, \dots, r. \tag{3.34}$$

Again passing onto the limits and combining those inequalities with Lemma 2.5, we conclude by (3.33) that

$$q_j(A\tilde{x}) \leq 0, \quad j = 1, 2, \dots, r. \tag{3.35}$$

Thus, $\tilde{x} \in C = \bigcap_{i=1}^t C_i$ and $A\tilde{x} \in Q = \bigcap_{j=1}^r Q_j$.

Therefore, \tilde{x} is a solution of the MSSFP (1.1). So we may use \tilde{x} in place of x^* in (3.23) and get that the sequence $\{\|x^k - \tilde{x}\|\}$ is convergent. Furthermore, noting that a subsequence of $\{x^k\}$, that is, $\{x^{k_n}\}$, converges to \tilde{x} , we obtain that $x^k \rightarrow \tilde{x}$, as $k \rightarrow \infty$.

This completes the proof. \square

4. Concluding Remarks

This paper introduced two self-adaptive projection methods with the Armijo-like searches for solving the multiple-set split feasibility problem MSSFP (1.1). They need not compute the additional projection P_Ω and avoid the difficult task of estimating the Lipschitz constant. It makes a sufficient decrease of the objective function at each iteration; thus the efficiency is enhanced greatly. Moreover in the second algorithm, we use the relaxed projection technology to calculate orthogonal projections onto convex sets, which may reduce a large amount of computing work and make the method more practical. The corresponding convergence theories have been established.

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