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Research Article

Asymptotic Behavior for a Nondissipative and Nonlinear System of the Kirchhoff Viscoelastic Type

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A wave equation of the Kirchhoff type with several nonlinearities is stabilized by a viscoelastic damping. We consider the case of nonconstant (and unbounded) coefficients. This is a nondissipative case, and as a consequence the nonlinear terms cannot be estimated in the usual manner by the initial energy. We suggest a way to get around this difficulty. It is proved that if the solution enters a certain region, which we determine, then it will be attracted exponentially by the equilibrium.

1. Introduction

We will consider the following wave equation with a viscoelastic damping term:

$$
\begin{gathered}
u_{tt} + \sum_{i=1}^{m} b_i(t)|u|^{p_i} u = \left(1 + \sum_{j=1}^{k} a_j(t)\|\nabla u\|_2^{2q_j}\right)\Delta u \\
- \int_0^t h(t-s)\Delta u(s)\,ds, \quad \text{in } \Omega \times \mathbf{R}_+, \qquad (1.1) \\
u = 0, \quad \text{on } \Gamma \times \mathbf{R}_+, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{gathered}
$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ and $p_i, q_j > 0$, $i = 1,\ldots,m$, $j = 1,\ldots,k$. The functions $u_0(x)$ and $u_1(x)$ are given initial data, and the (nonnegative) functions $a_j(t)$, $b_i(t)$, and $h(t)$ are at least absolutely continuous and will be specified later on. This problem arises in viscoelasticity where it has been shown by experiments that when subject to sudden changes, the viscoelastic response not only does depend on the current state of stress but also on all past states of stress. This gives rise to the integral term called the memory term. One may find a rich literature in this regard (with or without the Kirchhoff terms) treating mainly the stabilization of such systems for different classes of functions h. We refer the reader to [1–25] and the references therein. For problems of the Kirchhoff type, one can consult [26–35] and in particular [36–46] where the equations are supplemented by a nonlinear source. Several questions, such as well-posedness and asymptotic behavior, have been discussed in these references, to cite but a few.

As is clear from the equation in (1.1), we consider here several nonlinearities and the relaxation function is not necessarily decreasing or even nonincreasing. These issues are important but do not constitute the main contribution in the present paper. In case that $a_j(t)$ and $b_i(t)$ are not nonincreasing, then we are in a nondissipative situation. This is the case also when the relaxation function oscillates (in case $a_j(t)$, $b_i(t)$ are nonincreasing). Our argument here is simple and flexible. It relies on a Gronwall-type inequality involving several nonlinearities. We prove that there exists a sufficiently large $T > 0$ and a constant U after which (the modified energy of) global solutions are bounded below by U or decay to zero exponentially. We were not able to find conditions directly on the initial data because the Gronwall inequality is applicable only after some large values of time.

For simplicity we shall consider the simpler case $p_1 = p$, $p_i = 0$, $b_1 = b$, $b_i = 0$, $i = 2, \ldots, m$ and $q_1 = q$, $q_j = 0$, $a_1 = a$, $a_j = 0$, $j = 2, \ldots, k$.

The local existence and uniqueness may be found in [36, 37].

Theorem 1.1. *Assume that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $h(t)$ *is a nonnegative summable kernel. If* $0 < p < 2/(n-2)$ *when* $n \geq 3$ *and* $p > 0$ *when* $n = 1, 2$, *then there exists a unique solution* u *to problem* (1.1) *such that*

$$u \in C\Big([0,T]; H_0^1(\Omega)\Big) \cap C^1\Big([0,T]; L^2(\Omega)\Big) \tag{1.2}$$

for T small enough.

The plan of the paper is as follows. In the next section we prepare some materials needed to prove our result. Section 3 is devoted to the statement and proof of our theorem.

2. Preliminaries

In this section we define the different functionals we will work with. We prove an equivalence result between two functionals. Further, some useful lemmas are presented. We define the (classical) energy by

$$E(t) = \frac{1}{2}\Big(\|u_t\|_2^2 + \|\nabla u\|_2^2\Big) + \frac{a(t)}{2(q+1)}\|\nabla u\|_2^{2(q+1)} + \frac{b(t)}{p+2}\|u\|_{p+2}^{p+2}, \quad t \geq 0, \tag{2.1}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. Then by (1.1) it is easy to see that for $t \geq 0$

$$E'(t) = \int_\Omega \nabla u_t \cdot \int_0^t h(t-s)\nabla u(s)ds\,dx + \frac{a'(t)}{2(q+1)}\|\nabla u\|_2^{2(q+1)} + \frac{b'(t)}{p+2}\|u\|_{p+2}^{p+2}. \tag{2.2}$$

The first term in the right-hand side of (2.2) may be written as the derivative of some expression; namely,

$$\begin{aligned}\int_\Omega \nabla u_t \cdot \int_0^t h(t-s)\nabla u(s)ds\,dx = \frac{1}{2}\int_\Omega (h'\Box\nabla u)dx - \frac{1}{2}h(t)\|\nabla u\|_2^2 \\ - \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\Box\nabla u)dx - \left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2\right\},\end{aligned} \tag{2.3}$$

where

$$(h\Box v)(t) := \int_0^t h(t-s)|v(t)-v(s)|^2 ds. \tag{2.4}$$

Therefore, if we modify $E(t)$ to

$$\begin{aligned}\mathcal{E}(t) := \frac{1}{2}\left\{\|u_t\|_2^2 + \left(1 - \int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \int_\Omega (h\Box\nabla u)dx\right\} \\ + \frac{a(t)}{2(q+1)}\|\nabla u\|_2^{2(q+1)} + \frac{b(t)}{p+2}\|u\|_{p+2}^{p+2},\end{aligned} \tag{2.5}$$

we obtain for $t \geq 0$

$$\mathcal{E}'(t) = \frac{1}{2}\int_\Omega \left((h'\Box\nabla u) - h(t)|\nabla u|^2\right)dx + \frac{a'(t)}{2(q+1)}\|\nabla u\|_2^{2(q+1)} + \frac{b'(t)}{p+2}\|u\|_{p+2}^{p+2}. \tag{2.6}$$

Assuming that

$$1 - \int_0^{+\infty} h(s)ds =: 1 - \kappa > 0 \tag{2.7}$$

makes $\mathcal{E}(t)$ a nonnegative functional. The following functionals

$$\begin{aligned}\Phi_1(t) &:= \int_\Omega u_t u\,dx, \\ \Phi_2(t) &:= -\int_\Omega u_t \int_0^t h(t-s)(u(t)-u(s))ds\,dx\end{aligned} \tag{2.8}$$

are standard and will be used here. The next ones have been introduced by the present author in [24]

$$\Phi_3(t) := \int_0^t H_\gamma(t-s)\|\nabla u(s)\|_2^2 ds, \qquad \Phi_4(t) := \int_0^t \Psi_\gamma(t-s)\|\nabla u(s)\|_2^2 ds, \tag{2.9}$$

where

$$H_\gamma(t) := \gamma(t)^{-1}\int_t^\infty h(s)\gamma(s)ds, \quad \Psi_\gamma(t) := \gamma(t)^{-1}\int_t^\infty \xi(s)\gamma(s)ds, \quad t \geq 0, \tag{2.10}$$

and $\gamma(t)$ and $\xi(t)$ are two nonnegative functions which will be precised later (see (H2), (H3)). The functional

$$L(t) := \mathcal{E}(t) + \sum_{i=1}^{4}\lambda_i \Phi_i(t) \tag{2.11}$$

for some $\lambda_i > 0$, $i = 1, 2, 3, 4$, to be determined is equivalent to $\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t)$.

Proposition 2.1. *There exist $\rho_i > 0$, $i = 1, 2$ such that*

$$\rho_1[\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t)] \leq L(t) \leq \rho_2[\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t)] \tag{2.12}$$

for all $t \geq 0$ and small λ_i, $i = 1, 2$.

Proof. By the inequalities

$$\begin{aligned} \Phi_1(t) &= \int_\Omega u_t u\, dx \leq \frac{1}{2}\|u_t\|_2^2 + \frac{C_p}{2}\|\nabla u\|_2^2, \\ \Phi_2(t) &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{C_p\kappa}{2}\int_\Omega (h\Box\nabla u)dx, \end{aligned} \tag{2.13}$$

where C_p is the Poincaré constant, we have

$$\begin{aligned} L(t) \leq{}& \frac{1}{2}(1+\lambda_1+\lambda_2)\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)ds + \lambda_1 C_p\right)\|\nabla u\|_2^2 + \frac{b(t)}{p+2}\|u\|_{p+2}^{p+2} \\ &+ \frac{1}{2}(1+\lambda_2 C_p\kappa)\int_\Omega (h\Box\nabla u)dx + \frac{a(t)}{2(q+1)}\|\nabla u\|_2^{2(q+1)} + \lambda_3\Phi_3(t) + \lambda_4\Phi_4(t), \quad t \geq 0. \end{aligned} \tag{2.14}$$

On the other hand,

$$2L(t) \geq (1-\lambda_1-\lambda_2)\|u_t\|_2^2 + (1-\lambda_2 C_p\kappa)\int_\Omega (h\Box\nabla u)dx + \frac{2b(t)}{p+2}\|u\|_{p+2}^{p+2} + \frac{a(t)}{q+1}\|\nabla u\|_2^{2(q+1)} + \left[1-\kappa-\lambda_1 C_p\right]\|\nabla u\|_2^2 + 2\lambda_3\Phi_3(t) + 2\lambda_4\Phi_4(t). \tag{2.15}$$

Therefore, $\rho_1[\mathcal{E}(t)+\Phi_3(t)+\Phi_4(t)] \leq L(t) \leq \rho_2[\mathcal{E}(t)+\Phi_3(t)+\Phi_4(t)]$ for some constant $\rho_i > 0$, $i=1,2$ and small λ_i, $i=1,2$ such that $\lambda_1 < \min\{1,(1-\kappa)/C_p\}$ and $\lambda_2 < \min\{1/C_p\kappa, 1-\lambda_1\}$.

The identity to follow is easy to justify and is helpful to prove our result. □

Lemma 2.2. *One has for $h \in C(0,\infty)$ and $v \in C((0,\infty);L^2(\Omega))$*

$$\int_\Omega v(t)\int_0^t h(t-s)v(s)ds\,dx = \frac{1}{2}\left(\int_0^t h(s)ds\right)\|v(t)\|_2^2 + \frac{1}{2}\int_0^t h(t-s)\|v(s)\|_2^2 ds - \frac{1}{2}\int_\Omega (h\Box v)dx, \quad t\geq 0. \tag{2.16}$$

The next lemma is crucial in estimating (partially) our nonlinear terms. It can be found in [47].

Let $I \subset \mathbf{R}$, and let $g_1, g_2 : I \to \mathbf{R}\setminus\{\mathbf{0}\}$. We write $g_1 \propto g_2$ if g_2/g_1 is nondecreasing in I.

Lemma 2.3. *Let $a(t)$ be a positive continuous function in $J := [\alpha,\beta)$, $k_j(t)$, $j=1,\dots,n$ are nonnegative continuous functions, $g_j(u)$, $j=1,\dots,n$ are nondecreasing continuous functions in $\mathbf{R}_+$, with $g_j(u)>0$ for $u>0$, and $u(t)$ is a nonnegative continuous functions in J. If $g_1 \propto g_2 \propto \cdots \propto g_n$ in $(0,\infty)$, then the inequality*

$$u(t) \leq a(t) + \sum_{j=1}^{n}\int_\alpha^t k_j(s)g_j(u(s))ds, \quad t\in J, \tag{2.17}$$

implies that

$$u(t) \leq c_n(t), \quad \alpha \leq t < \beta_0, \tag{2.18}$$

where $c_0(t) := \sup_{0\leq s\leq t} a(s)$,

$$c_j(t) := G_j^{-1}\left[G_j(c_{j-1}(t)) + \int_\alpha^t k_j(s)ds\right], \quad j=1,\dots,n,$$
$$G_j(u) := \int_{u_j}^u \frac{dx}{g_j(x)}, \quad u>0\ (u_j>0,\ j=1,\dots,n), \tag{2.19}$$

and β_0 is chosen so that the functions $c_j(t)$, $j=1,\dots,n$ are defined for $\alpha \leq t < \beta_0$.

Lemma 2.4. *Assume that* $2 \leq q < +\infty$ *if* $n = 1, 2$ *or* $2 \leq q < 2n/(n-2)$ *if* $n \geq 3$. *Then there exists a positive constant* $C_e = C_e(\Omega, q)$ *such that*

$$\|u\|_q \leq C_e \|\nabla u\|_2 \tag{2.20}$$

for $u \in H_0^1(\Omega)$.

3. Asymptotic Behavior

In this section we state and prove our result. To this end we need some notation. For every measurable set $\mathcal{A} \subset \mathbf{R}^+$, we define the probability measure $\widehat{h}$ by

$$\widehat{h}(\mathcal{A}) := \frac{1}{\kappa} \int_{\mathcal{A}} h(s) ds. \tag{3.1}$$

The nondecreasingness set and the non-decreasingness rate of h are defined by

$$\mathcal{Q}_h := \left\{ s \in \mathbf{R}^+ : h(s) > 0,\, h'(s) \geq 0 \right\}, \tag{3.2}$$

$$\mathcal{R}_h := \widehat{h}(\mathcal{Q}_h), \tag{3.3}$$

respectively.

The following assumptions on the kernel $h(t)$ will be adopted.

(H1) $h(t) \geq 0$ for all $t \geq 0$ and $0 < \kappa = \int_0^{+\infty} h(s) ds < 1$.

(H2) h is absolutely continuous and of bounded variation on $(0, \infty)$ and $h'(t) \leq \xi(t)$ for some nonnegative summable function $\xi(t)$ $(= \max\{0, h'(t)\}$ where $h'(t)$ exists) and almost all $t > 0$.

(H3) There exists a nondecreasing function $\gamma(t) > 0$ such that $\gamma'(t)/\gamma(t) = \eta(t)$ is a nonincreasing function: $\int_0^{+\infty} h(s)\gamma(s) ds < +\infty$ and $\int_0^{+\infty} \xi(s)\gamma(s) ds < +\infty$.

Note that a wide class of functions satisfies the assumption (H3). In particular, exponentially and polynomially (or power type) decaying functions are in this class.

Let $t_* > 0$ be a number such that $\int_0^{t_*} h(s) ds = h_* > 0$. We denote by $\mathcal{B}_t$ the set $\mathcal{B}_t := \mathcal{B} \cap [0, t]$.

Lemma 3.1. *One has for* $t \geq t_*$ *and* $\delta_i > 0$, $i = 1, \ldots, 5$

$$\begin{aligned}
\Phi_2'(t) \leq & (1 - h_*) \left[\delta_1 + \frac{3}{2} \int_{\mathcal{Q}_t} h(t-s) ds \right] \|\nabla u\|_2^2 + (\delta_3 - h_*) \|u_t\|_2^2 \\
& + \left[(1 - h_*) \frac{\kappa}{4\delta_1} + \left(1 + \frac{1}{\delta_2} \right) \kappa \right] \int_\Omega \int_{\mathcal{A}_t} h(t-s) |\nabla u(t) - \nabla u(s)|^2 ds\, dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(1-h_*)\int_{Q_t} h(t-s)\|\nabla u(s)\|_2^2 ds - \frac{C_p}{4\delta_3} BV[h]\int_\Omega (h'\Box\nabla u)dx \\
&+(1+\delta_2)\left(\int_{Q_t} h(t-s)ds\right)\int_\Omega\int_{Q_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2 ds\,dx \\
&+\frac{2^{p+1}\delta_4 C_e b^2(t)}{(1-\kappa)^{p+1}}\mathcal{E}^{p+1}(t)+\frac{C_p\kappa}{4\delta_4}\int_\Omega (h\Box\nabla u)dx+\delta_5 a(t)\|\nabla u\|_2^{2(q+1)}+\frac{2^{q-1}\kappa a(t)}{\delta_5(1-\kappa)^q}\mathcal{E}^{q+1}(t) \\
&+\frac{C_p}{4\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)\int_\Omega\int_{Q_t}\xi(t-s)|\nabla u(t)-\nabla u(s)|^2 ds\,dx,
\end{aligned}
\tag{3.4}
$$

where $BV[h]$ is the total variation of h.

Proof. This lemma is proved by a direct differentiation of $\Phi_2(t)$ along solutions of (1.1) and estimation of the different terms in the obtained expression of the derivative. Indeed, we have

$$
\begin{aligned}
\Phi_2'(t) = &-\int_\Omega u_{tt}\int_0^t h(t-s)(u(t)-u(s))ds\,dx \\
&-\int_\Omega u_t\left[\int_0^t h'(t-s)(u(t)-u(s))ds+u_t\int_0^t h(s)ds\right]dx
\end{aligned}
\tag{3.5}
$$

or

$$
\begin{aligned}
\Phi_2'(t) = &-\int_\Omega\left[\left(1-\int_0^t h(s)ds\right)\Delta u - b(t)|u|^p u + a(t)\|\nabla u\|_2^{2q}\Delta u\right. \\
&\left.+\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right]\int_0^t h(t-s)(u(t)-u(s))ds\,dx \\
&-\left(\int_0^t h(s)ds\right)\|u_t\|_2^2-\int_\Omega u_t\int_0^t h'(t-s)(u(t)-u(s))ds\,dx,\qquad t\geq 0.
\end{aligned}
\tag{3.6}
$$

Therefore,

$$
\begin{aligned}
\Phi_2'(t) = &\left(1-\int_0^t h(s)ds + a(t)\|\nabla u\|_2^{2q}\right) \\
&\times\int_\Omega \nabla u\cdot\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))ds\,dx - \left(\int_0^t h(s)ds\right)\|u_t\|_2^2
\end{aligned}
$$

$$+ b(t)\int_\Omega |u|^p u \int_0^t h(t-s)(u(t)-u(s))ds\,dx - \int_\Omega u_t \int_0^t h'(t-s)(u(t)-u(s))ds\,dx$$
$$+ \int_\Omega \left| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))ds \right|^2 dx, \qquad t \geq 0. \tag{3.7}$$

For all measurable sets $\mathcal{A}$ and Q such that $\mathcal{A} = \mathbf{R}^+ \setminus Q$, it is clear that

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))ds\,dx \\
&\quad = \int_\Omega \nabla u \cdot \int_{\mathcal{A}\cap[0,t]} h(t-s)(\nabla u(t) - \nabla u(s))ds\,dx \\
&\qquad + \int_\Omega \nabla u \cdot \int_{Q\cap[0,t]} h(t-s)(\nabla u(t) - \nabla u(s))ds\,dx \\
&\quad = \int_\Omega \nabla u \cdot \int_{\mathcal{A}\cap[0,t]} h(t-s)(\nabla u(t) - \nabla u(s))ds\,dx \\
&\qquad + \left(\int_{Q\cap[0,t]} h(t-s)ds\right)\|\nabla u\|_2^2 - \int_\Omega \nabla u \cdot \int_{Q\cap[0,t]} h(t-s)\nabla u(s)ds\,dx, \qquad t \geq 0.
\end{aligned} \tag{3.8}$$

For $\delta_1 > 0$, the first term in the right-hand side of (3.8) satisfies

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \int_{\mathcal{A}_t} h(t-s)(\nabla u(t) - \nabla u(s))ds\,dx \\
&\quad \leq \delta_1 \|\nabla u\|_2^2 + \frac{\kappa}{4\delta_1}\int_\Omega \int_{\mathcal{A}_t} h(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx, \quad t \geq 0,
\end{aligned} \tag{3.9}$$

and the third one fulfills

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \int_{Q_t} h(t-s)\nabla u(s)ds\,dx \\
&\quad \leq \frac{1}{2}\left(\int_{Q_t} h(t-s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\int_{Q_t} h(t-s)\|\nabla u(s)\|_2^2 ds, \quad t \geq 0.
\end{aligned} \tag{3.10}$$

Back to (3.8) we may write

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\
&\quad \leq \delta_1 \|\nabla u\|_2^2 + \frac{\kappa}{4\delta_1} \int_\Omega \int_{\mathcal{A}_t} h(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx \\
&\qquad + \frac{3}{2}\|\nabla u\|_2^2 \int_{Q_t} h(t-s)ds + \frac{1}{2}\int_{Q_t} h(t-s)\|\nabla u(s)\|_2^2 ds, \quad t \geq 0.
\end{aligned} \tag{3.11}$$

The last term in the right-hand side of (3.7) will be estimated as follows:

$$\begin{aligned}
&\int_\Omega \left| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))ds \right|^2 dx \\
&\quad \leq \left(1 + \frac{1}{\delta_2}\right)\kappa \int_\Omega \int_{\mathcal{A}_t} h(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx \\
&\qquad + (1+\delta_2)\left(\int_{Q_t} h(t-s)ds\right) \int_\Omega \int_{Q_t} h(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx, \quad \delta_2 > 0.
\end{aligned} \tag{3.12}$$

For the fourth term in (3.7), it holds that

$$\begin{aligned}
&-\int_\Omega u_t \int_0^t h'(t-s)(u(t) - u(s))ds\,dx \\
&\quad \leq \delta_3 \|u_t\|_2^2 - \frac{C_p}{4\delta_3} BV[h] \int_\Omega (h' \square \nabla u)dx \\
&\qquad + \frac{C_p}{4\delta_3}\left(\int_{Q_t} \xi(t-s)ds\right) \int_\Omega \int_{Q_t} \xi(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx, \quad \delta_3 > 0,\ t \geq 0.
\end{aligned} \tag{3.13}$$

Moreover, from Lemma 2.4, for $p > 0$ if $n = 1,2$ and $0 < p < 2/(n-2)$ if $n \geq 3$, we find

$$\begin{aligned}
&b(t) \int_\Omega |u|^p u \int_0^t h(t-s)(u(t) - u(s))ds\,dx \\
&\quad \leq \delta_4 b^2(t) \|u\|_{2(p+1)}^{2(p+1)} + \frac{C_p}{4\delta_4}\left(\int_0^t h(s)ds\right) \int_\Omega (h \square \nabla u)dx \\
&\quad \leq \delta_4 C_e b^2(t) \|\nabla u\|_2^{2(p+1)} + \frac{C_p \kappa}{4\delta_4} \int_\Omega (h \square \nabla u)dx \\
&\quad \leq \frac{2^{p+1}\delta_4 C_e b^2(t)}{(1-\kappa)^{p+1}} \mathcal{E}^{p+1}(t) + \frac{C_p \kappa}{4\delta_4} \int_\Omega (h \square \nabla u)dx, \quad \delta_4 > 0,\ t \geq 0.
\end{aligned} \tag{3.14}$$

The definition of $\mathcal{E}(t)$ in (2.5) allows us to write

$$\begin{aligned} & a(t)\|\nabla u\|_2^{2q}\int_\Omega \nabla u\cdot\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))ds\,dx \\ & \le a(t)\|\nabla u\|_2^{2q}\left\{\delta_5\|\nabla u\|_2^2+\frac{\kappa}{4\delta_5}\int_\Omega (h\Box\nabla u)dx\right\} \\ & \le \delta_5 a(t)\|\nabla u\|_2^{2(q+1)}+\frac{2^{q-1}\kappa a(t)}{\delta_5(1-\kappa)^q}\mathcal{E}^{q+1}(t),\quad \delta_5>0,\ t\ge 0. \end{aligned} \tag{3.15}$$

Gathering all the relations (3.11)–(3.15) together with (3.7), we obtain for $t\ge t_*$

$$\begin{aligned} \Phi_2'(t) \le{}& (1-h_*)\left[\delta_1+\frac{3}{2}\int_{Q_t} h(t-s)ds\right]\|\nabla u\|_2^2+(\delta_3-h_*)\|u_t\|_2^2 \\ &+\left[(1-h_*)\frac{\kappa}{4\delta_1}+\left(1+\frac{1}{\delta_2}\right)\kappa\right]\int_\Omega\int_{\mathcal{A}_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2ds\,dx \\ &+\frac{1}{2}(1-h_*)\int_{Q_t} h(t-s)\|\nabla u(s)\|_2^2ds-\frac{C_p}{4\delta_3}BV[h]\int_\Omega (h'\Box\nabla u)dx \\ &+(1+\delta_2)\left(\int_{Q_t} h(t-s)ds\right)\int_\Omega\int_{Q_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2ds\,dx \\ &+\frac{2^{p+1}\delta_4 C_e b^2(t)}{(1-\kappa)^{p+1}}\mathcal{E}^{p+1}(t)+\frac{C_p\kappa}{4\delta_4}\int_\Omega (h\Box\nabla u)dx+\delta_5 a(t)\|\nabla u\|_2^{2(q+1)}+\frac{2^{q-1}\kappa a(t)}{\delta_5(1-\kappa)^q}\mathcal{E}^{q+1}(t) \\ &+\frac{C_p}{4\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)\int_\Omega\int_{Q_t}\xi(t-s)|\nabla u(t)-\nabla u(s)|^2ds\,dx. \end{aligned} \tag{3.16}$$

□

In the following theorem we will assume that $p<q$ just to fix ideas. The result is also valid for $p>q$. It suffices to interchange $p\leftrightarrow q$ and $A(t)\leftrightarrow B(t)$ in the proof following it. The case $p=q$ is easier.

We will make use of the following hypotheses for some positive constants A, B, U, and V to be determined.

(A) $a(t)$ is a continuously differentiable function such that $a'(t)<Aa(t)$, $t\ge 0$.

(B) $b(t)$ is a continuously differentiable function such that $b'(t)<Bb(t)$, $t\ge 0$.

(C) $p>0$ if $n=1,2$ and $0<p<2/(n-2)$ if $n\ge 3$.

(D) $[\int_0^\infty a(s)e^{-qs}ds]^{1/q}[\int_0^\infty b^2(s)e^{-ps}ds]^{1/p}<U$.

(E) $[\int_0^\infty a(s)e^{-q\int_0^s\eta(\tau)d\tau}ds]^{1/q}[\int_0^\infty b^2(s)e^{-p\int_0^s\eta(\tau)d\tau}ds]^{1/p}<V$.

Theorem 3.2. *Assume that the hypotheses (H1)–(H3), (A)–(C) hold and* $\mathcal{R}_h < 1/4$. *If* $\lim_{t\to\infty}\eta(t) = \overline{\eta} \neq 0$, *then, for global solutions and small* $\int_Q \xi(s)ds$, *there exist* $T_1 > 0$ *and* $U > 0$ *such that* $L(t) > U$, $t \geq T_1$ *or*

$$E(t) \leq M_1 e^{-\nu_1 t}, \qquad t \geq 0 \tag{3.17}$$

for some positive constants M_1 *and* ν_1 *as long as (D) holds. If* $\overline{\eta} = 0$, *then there exist* $T_2 > 0$ *and* $V > 0$ *such that* $L(t) > V$, $t \geq T_2$ *or*

$$E(t) \leq M_2 e^{-\nu_2 \int_0^t \eta(s)ds}, \qquad t \geq 0 \tag{3.18}$$

for some positive constants M_2 *and* ν_2 *as long as (E) holds.*

Proof. A differentiation of $\Phi_1(t)$ with respect to t along trajectories of (1.1) gives

$$\begin{aligned} \Phi_1'(t) := \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega \nabla u \cdot \int_0^t h(t-s)\nabla u(s)ds\,dx \\ - a(t)\|\nabla u\|_2^{2(q+1)} - b(t)\|u\|_{p+2}^{p+2}, \end{aligned} \tag{3.19}$$

and Lemma 2.2 implies

$$\begin{aligned} \Phi_1'(t) \leq \|u_t\|_2^2 - \left(1 - \frac{\kappa}{2}\right)\|\nabla u\|_2^2 + \frac{1}{2}\int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds \\ - \frac{1}{2}\int_\Omega (h\Box\nabla u)dx - a(t)\|\nabla u\|_2^{2(q+1)} - b(t)\|u\|_{p+2}^{p+2}, \qquad t \geq 0. \end{aligned} \tag{3.20}$$

Next, a differentiation of $\Phi_3(t)$ and $\Phi_4(t)$ yields

$$\begin{aligned} \Phi_3'(t) &= H_\gamma(0)\|\nabla u\|_2^2 + \int_0^t H_\gamma'(t-s)\|\nabla u(s)\|_2^2 ds \\ &= H_\gamma(0)\|\nabla u\|_2^2 - \int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)} H_\gamma(t-s)\|\nabla u(s)\|_2^2 ds - \int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds \\ &\leq H_\gamma(0)\|\nabla u\|_2^2 - \eta(t)\Phi_3(t) - \int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds, \qquad t \geq 0, \\ \Phi_4'(t) &= \Psi_\gamma(0)\|\nabla u\|_2^2 + \int_0^t \Psi_\gamma'(t-s)\|\nabla u(s)\|_2^2 ds \\ &= \Psi_\gamma(0)\|\nabla u\|_2^2 - \int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)} \Psi_\gamma(t-s)\|\nabla u(s)\|_2^2 ds - \int_0^t \xi(t-s)\|\nabla u(s)\|_2^2 ds \\ &\leq \Psi_\gamma(0)\|\nabla u\|_2^2 - \eta(t)\Phi_4(t) - \int_0^t \xi(t-s)\|\nabla u(s)\|_2^2 ds, \quad t \geq 0. \end{aligned} \tag{3.21}$$

Taking into account Lemma 3.1 and the relations (2.6), (3.20)-(3.21), we see that

$$\begin{aligned}
L'(t) \leq{} & \left(\frac{1}{2}-\frac{C_p}{4\delta_3}\lambda_2 BV[h]\right)\int_\Omega (h'\Box\nabla u)dx+[\lambda_1+(\delta_3-h_*)\lambda_2]\|u_t\|_2^2 \\
&+\left\{\lambda_2(1-h_*)\left[\delta_1+\frac{3}{2}\int_{Q_t}h(t-s)ds\right]+\lambda_3 H_\gamma(0)+\lambda_4\Psi_\gamma(0)-\lambda_1\left(1-\frac{\kappa}{2}\right)\right\} \\
&\times\|\nabla u\|_2^2+\left(\frac{\lambda_1}{2}+\frac{\lambda_2(1-h_*)}{2}-\lambda_3\right)\int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds \\
&+\left[\frac{a'(t)}{2(q+1)}+\delta_5\lambda_2 a(t)-\lambda_1 a(t)\right]\|\nabla u\|_2^{2(q+1)}+\left[(1+\delta_2)\lambda_2\int_{Q_t}h(t-s)ds-\frac{\lambda_1}{2}\right] \\
&\times\int_\Omega (h\Box\nabla u)dx+\lambda_2\kappa\left[1+\frac{1-h_*}{4\delta_1}+\frac{1}{\delta_2}\right]\int_\Omega\int_{\mathcal{A}_t}h(t-s)|\nabla u(t)-\nabla u(s)|^2 ds\,dx \\
&-\lambda_3\eta(t)\Phi_3(t)-\lambda_4\eta(t)\Phi_4(t)+\left[\frac{b'(t)}{p+2}-\lambda_1 b(t)\right]\|u\|_{p+2}^{p+2}+\frac{\lambda_2 C_p\kappa}{4\delta_4}\int_\Omega (h\Box\nabla u)dx \\
&+\frac{2^{q-1}\kappa a(t)}{\delta_5(1-\kappa)^q}\lambda_2\mathcal{E}^{q+1}(t)+\frac{2^{p+1}\delta_4 C_e b^2(t)}{(1-\kappa)^{p+1}}\lambda_2\mathcal{E}^{p+1}(t)-\lambda_4\int_0^t \xi(t-s)\|\nabla u(s)\|_2^2 ds \\
&+\frac{\lambda_2 C_p}{4\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)\int_{Q_t}\xi(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds, \quad t\geq t_*.
\end{aligned} \tag{3.22}$$

Next, as in [17], we introduce the sets

$$\mathcal{A}_n := \left\{s\in\mathbf{R}^+ : nh'(s)+h(s)\leq 0\right\}, \quad n\in\mathbf{N}, \tag{3.23}$$

and observe that

$$\bigcup_n \mathcal{A}_n = \mathbf{R}^+ \setminus \{Q_h\cup\mathcal{N}_h\}, \tag{3.24}$$

where $\mathcal{N}_h$ is the null set where h' is not defined and Q_h is as in (3.2). Furthermore, if we denote $Q_n := \mathbf{R}^+\setminus\mathcal{A}_n$, then $\lim_{n\to\infty}\widehat{h}(Q_n)=\widehat{h}(Q_h)$ because $Q_{n+1}\subset Q_n$ for all n and $\bigcap_n Q_n = Q_h\cup\mathcal{N}_h$. Moreover, we designate by $\widetilde{A}_{nt}$ the sets

$$\widetilde{A}_{nt} := \left\{s\in\mathbf{R}^+ : 0\leq s\leq t,\ nh'(t-s)+h(t-s)\leq 0\right\}, \quad n\in\mathbf{N}. \tag{3.25}$$

In (3.22), we take $\mathcal{A}_t := \widetilde{A}_{nt}$ and $Q_t := \widetilde{Q}_{nt}$. Choosing $\lambda_1 = (h_*-\varepsilon)\lambda_2$, it is clear that

$$(1+\delta_2)\lambda_2\kappa\widehat{h}(Q_n)-\frac{\lambda_1}{2}\leq 0 \tag{3.26}$$

for small ε and δ_2, large n and t_*, if $\widehat{h}(Q) < 1/4$. We deduce that

$$(1+\delta_2)\lambda_2 \int_{\tilde{Q}_{nt}} h(t-s)ds - \frac{\lambda_1}{2} < 0. \tag{3.27}$$

Furthermore, if $\widehat{h}(Q) < 1/4$, then

$$\frac{3(1-h_*)}{2} \int_{\tilde{Q}_{nt}} h(t-s)ds < \delta h_* \left(1 - \frac{\kappa}{2}\right) \tag{3.28}$$

with

$$\delta = \frac{3(1-h_*)\kappa}{4(2-\kappa)h_*} + \beta \tag{3.29}$$

and a small $\beta > 0$. Pick

$$\lambda_3 = \frac{1}{2}[\lambda_1 + \lambda_2(1-h_*)] \tag{3.30}$$

and $H_\gamma(0)$ such that

$$\lambda_3 H_\gamma(0) < \lambda_2 \frac{(1-\delta)h_*(2-\kappa)}{2}. \tag{3.31}$$

Note that this is possible if t_* is so large that $h_* > 7\kappa/(8-\kappa)$ even though

$$H_\gamma(0) = \gamma(0)^{-1} \int_0^\infty h(s)\gamma(s)ds \geq \int_0^\infty h(s)ds = \kappa. \tag{3.32}$$

Taking the relations (3.22)–(3.30) into account and selecting $\lambda_2 < \delta_3/C_p BV[h]$ so that

$$\frac{1}{2} - \frac{C_p\lambda_2}{4\delta_3} BV[h] \geq \frac{1}{4}, \tag{3.33}$$

and small enough so that

$$\lambda_2\kappa\left(1 + \frac{1-h_*}{4\delta_1} + \frac{1}{\delta_2} + \frac{C_p}{4\delta_4}\right) < \frac{1}{4n}, \tag{3.34}$$

we find for $\delta_3 = \varepsilon/2$, large δ_4, small $\Psi_\gamma(0)$, and $t \geq t_*$

$$
\begin{aligned}
L'(t) \leq & -C_1\left\{\|u_t\|_2^2 + \|\nabla u\|_2^2 + \int_\Omega (h\Box\nabla u)dx\right\} + \frac{2^{p+1}\delta_4 C_e b^2(t)}{(1-\kappa)^{p+1}}\lambda_2 \mathcal{E}^{p+1}(t) \\
& + \left[\frac{a'(t)}{2(q+1)} + \delta_5\lambda_2 a(t) - \lambda_1 a(t)\right]\|\nabla u\|_2^{2(q+1)} + \frac{2^{q-1}\kappa a(t)}{\delta_5(1-\kappa)^q}\lambda_2\mathcal{E}^{q+1}(t) - \lambda_3\eta(t)\Phi_3(t) \\
& - \lambda_4\eta(t)\Phi_4(t) + \frac{1}{2}\left[1 + \frac{\lambda_2 C_p}{2\delta_3}\int_{Q_t}\xi(t-s)ds\right]\int_\Omega\int_{Q_{nt}} h'(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx \\
& + \left[\frac{b'(t)}{p+2} - \lambda_1 b(t)\right]\|u\|_{p+2}^{p+2} - \lambda_4\int_0^t \xi(t-s)\|\nabla u(s)\|_2^2 ds
\end{aligned}
\tag{3.35}
$$

for some positive constant C_1. Take $\lambda_4 > 1 + (\lambda_2 C_p/2\delta_3)\int_Q \xi(s)ds$, δ_5, $\int_Q \xi(s)ds$ small, and

$$
\frac{a'(t)}{2(q+1)} < (\lambda_1 - \alpha)a(t) \tag{3.36}
$$

(i.e., $A = 2(q+1)(\lambda_1 - \alpha)$ for some $0 < \alpha < \lambda_1$) and

$$
\frac{b'(t)}{p+2} < (\lambda_1 - \beta)b(t) \tag{3.37}
$$

(i.e., $B = (p+2)(\lambda_1 - \beta)$ for some $0 < \beta < \lambda_1$) to derive that

$$
\begin{aligned}
L'(t) \leq & -C_2\mathcal{E}(t) + \frac{2^{p+1}\delta_4 C_e b^2(t)}{(1-\kappa)^{p+1}}\lambda_2\mathcal{E}^{p+1}(t) + \frac{2^{q-1}\kappa a(t)}{\delta_5(1-\kappa)^q}\lambda_2\mathcal{E}^{q+1}(t) \\
& - \lambda_3\eta(t)\Phi_3(t) - \lambda_4\eta(t)\Phi_4(t)
\end{aligned}
\tag{3.38}
$$

for some positive constant C_2.

If $\lim_{t\to\infty}\eta(t) \neq 0$, then there exist a $\hat{t} \geq t_*$ and $C_3 > 0$ such that $\eta(t) \geq C_3$ for $t \geq \hat{t}$. Thus, in virtue of Proposition 2.1, for $C_3 > 0$, we have

$$
L'(t) \leq -C_3 L(t) + B(t)L^{p+1}(t) + A(t)L^{q+1}(t), \tag{3.39}
$$

where

$$
\begin{aligned}
B(t) &:= \frac{2^{p+1}\delta_4 C_e\lambda_2}{(1-\kappa)^{p+1}\rho_1^{p+1}} b^2(t), \\
A(t) &:= \frac{2^{q-1}\kappa\lambda_2}{\delta_5(1-\kappa)^q\rho_1^{q+1}} a(t).
\end{aligned}
\tag{3.40}
$$

If there exists a $T \geq \hat{t}$ such that $L(T) < [p\int_0^\infty \widetilde{B}(s)ds]^{-1/p}$, where $\widetilde{B}(t) := B(t)e^{-pC_3(t-T)}$, then from (3.39)

$$\left[e^{C_3(t-T)}L(t)\right]' \leq e^{C_3(t-T)}B(t)L^{p+1}(t) + e^{C_3(t-T)}A(t)L^{q+1}(t), \tag{3.41}$$

and it follows that for $t \geq T$

$$\widetilde{L}(t) \leq L(T) + \int_T^t e^{-pC_3(s-T)}B(s)\widetilde{L}^{p+1}(s)ds + \int_T^t e^{-qC_3(s-T)}A(s)\widetilde{L}^{q+1}(s)ds \tag{3.42}$$

with $\widetilde{L}(t) := e^{C_3(t-T)}L(t)$. Now we apply Lemma 2.3 to get

$$\widetilde{L}(t) \leq \left[N^{-q} - q\int_0^t \widetilde{A}(s)ds\right]^{-1/q}, \quad t \geq T, \tag{3.43}$$

where $\widetilde{A}(t) := e^{-qC_3(t-T)}A(t)$ and $N := [L(T)^{-p} - p\int_0^\infty \widetilde{B}(s)ds]^{-1/p}$. If, in addition, $q\int_0^\infty \widetilde{A}(s)ds < L(T)^{-q}$, then $\widetilde{L}(t)$ is uniformly bounded by a positive constant C_4. Thus

$$L(t) \leq C_4 e^{-C_3 t}, \quad t \geq T, \tag{3.44}$$

and by continuity (3.44) holds for all $t \geq 0$.

If $\lim_{t\to\infty}\eta(t) = 0$, then for any $C > 0$ there exists a $\bar{t}(C) \geq t_*$ such that $\eta(t) \leq C$ for $t \geq \bar{t}(C)$. Therefore,

$$L'(t) \leq -C_5\eta(t)L(t) + B(t)L^{p+1}(t) + A(t)L^{q+1}(t), \quad t \geq \bar{t} = \bar{t}(C_2) \tag{3.45}$$

for some $C_5 > 0$. The previous argument carries out with $e^{C_3(t-T)}$ replaced by $e^{C_5\int_T^t \eta(d)ds}$.

In case that $q < p$, we reverse the roles of p and q in the argument above. The case $p = q$ is clear. □

Remark 3.3. The case where the derivative of the kernel does not approach zero on $\mathcal{A}$ (as is the case, for instance, when $h' \leq -Ch$ on $\mathcal{A}$) is interesting. Indeed, the right-hand side in condition (3.34) will be replaced by $C/4$ with a possibly large constant C.

Remark 3.4. The argument clearly works for all kinds of kernels previously treated where derivatives cannot be positive or even take the value zero. In these cases there will be no need for the smallness conditions on the kernels. This work shows that derivatives may be positive (i.e., kernels may be increasing) on some "small" subintervals and open the door for (optimal) estimations and improvements of these sets.

Remark 3.5. The assumptions $a'(t) < 2(q+1)(\lambda_1 - \alpha)a(t)$ and $b'(t) < (p+2)(\lambda_1 - \beta)b(t)$ may be relaxed to $a'(t) < 2(q+1)\lambda_1 a(t)$ and $b'(t) < (p+2)\lambda_1 b(t)$, respectively. In this case $\alpha = \alpha(t)$ and $\beta = \beta(t)$ would depend on t.

Remark 3.6. The assertion in Theorem 3.2 is an "alternative" statement. As a next step it would be nice to discuss the (sufficient conditions of) occurrence of each case in addition to the global existence.

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