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Research Article

Landen Inequalities for Zero-Balanced Hypergeometric Functions

Slavko Simić[1] and Matti Vuorinen[2]

[1] *Mathematical Institute, SANU, Kneza Mihaila 36, 11000 Belgrade, Serbia*
[2] *Department of Mathematics, University of Turku, 20014 Turku, Finland*

Correspondence should be addressed to Slavko Simić, ssimic@turing.mi.sanu.ac.rs

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For zero-balanced Gaussian hypergeometric functions $F(a,b;a+b;x)$, $a,b>0$, we determine maximal regions of ab plane where well-known Landen identities for the complete elliptic integral of the first kind turn on respective inequalities valid for each $x\in(0,1)$. Thereby an exhausting answer is given to the open problem from the work by Anderson et al., 1990.

1. Introduction

Among special functions, the hypergeometric function has perhaps the widest range of applications. For instance, several well-known classes of mathematical physics are particular or limiting cases of it. For real numbers a, b, and c with $c\neq 0,-1,-2,\ldots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) := {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}, \tag{1.1}$$

for $x\in(-1,1)$, where

$$(a,n) := a(a+1)(a+2)\cdots(a+n-1), \tag{1.2}$$

for $n=1,2,\ldots$, and $(a,0)=1$ for $a\neq 0$. For many rational triples (a,b,c) the function (1.1) can be expressed in terms of elementary functions and long lists of such particular cases are given in [1].

It is clear that small changes of the parameters a, b, c will have small influence on the value of $F(a,b;c;x)$. In this paper we will study to what extent some well-known properties of the complete elliptic integral of the first kind,

$$\mathcal{K}(x) \equiv \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) = \int_0^{\pi/2} \left(1 - x^2 \sin^2 t\right)^{-1/2} dt, \quad x \in (0,1), \tag{1.3}$$

can be extended to $F(a,b;a+b;x)$ for (a,b) close to $(1/2, 1/2)$. Recall that $F(a,b;c;r)$ is called *zero-balanced* if $c = a + b$. In the zero-balanced case, there is a logarithmic singularity at $r = 1$ and Gauss proved the asymptotic formula

$$F(a,b;a+b;r) \sim -\frac{1}{B(a,b)} \log(1-r), \tag{1.4}$$

as r tends to 1, where

$$B(z,w) \equiv \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \operatorname{Re} z > 0, \ \operatorname{Re} w > 0 \tag{1.5}$$

is the classical beta function. Note that $\Gamma(1/2) = \sqrt{\pi}$ and $B(1/2, 1/2) = \pi$, see [2, Chapter 6].

Ramanujan found a much sharper asymptotic formula

$$B(a,b)F(a,b;a+b;r) + \log(1-r) = R(a,b) + O\big((1-r)\log(1-r)\big), \tag{1.6}$$

as r tends to 1 (see also [3].) Here and in the sequel,

$$\begin{gathered} R(a,b) \equiv -\Psi(a) - \Psi(b) - 2\gamma, \qquad R\left(\frac{1}{2}, \frac{1}{2}\right) = \log 16, \\ \Psi(z) \equiv \frac{d}{dz}\big(\log \Gamma(z)\big) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \operatorname{Re} z > 0, \end{gathered} \tag{1.7}$$

and γ is the Euler-Mascheroni constant. Ramanujan's formula (1.6) is a particular case of another well-known formula given in [2, 15.3.10].

We shall use in the sequel the following assertion which is a mixture of Biernacki-Krzyz and related results on the ratio of formal power series [4, 5].

Lemma 1.1. *Suppose that the power series $f(x) = \sum_{n\ge 0} \widehat{f}_n x^n$ and $g(x) = \sum_{n\ge 0} \widehat{g}_n x^n$ have the radius of convergence $r > 0$ and $\widehat{g}_n > 0$ for all $n \in \{0, 1, 2, \ldots\}$. Denote also*

$$h(x) = \frac{f(x)}{g(x)} = \sum_{n\ge 0} \widehat{h}_n x^n. \tag{1.8}$$

(1) *If the sequence $\{\widehat{f}_n / \widehat{g}_n\}_{n\ge 0}$ is monotone increasing then $h(x)$ is also monotone increasing on $(0, r)$.*

(2) *If the sequence* $\{\widehat{f}_n/\widehat{g}_n\}_{n\geq 0}$ *is monotone decreasing then* $h(x)$ *is also monotone decreasing on* $(0,r)$.

(3) *If the sequence* $\{\widehat{f}_n/\widehat{g}_n\}$ *is monotone increasing (decreasing) for* $0 < n \leq n_0$ *and monotone decreasing (increasing) for* $n > n_0$*, then there exists* $x_0 \in (0,r)$ *such that* $h(x)$ *is increasing (decreasing) on* $(0,x_0)$ *and decreasing (increasing) on* (x_0,r).

Some of the most important properties of the elliptic integral $\mathcal{K}(r)$ are the Landen identities [6, p. 507]:

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \qquad \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathcal{K}'(r), \tag{1.9}$$

where $\mathcal{K}'(r) = \mathcal{K}(\sqrt{1-r^2})$, $r \in (0,1)$. In [4, Page 79], the following problem was raised.

Open Problem 1. Find an analog of Landen's transformation formulas in (1.9) for $F(a,b;a+b;r)$. In particular, if $k(r) = F(a,b;a+b;r^2)$ and $a,b \in (0,1)$, is it true that

$$k\left(\frac{2\sqrt{r}}{1+r}\right) \leq Ck(r) \tag{1.10}$$

for some constant C and all $r \in (0,1)$?

Since $2\sqrt{r}/(1+r) > r$ for $r \in (0,1)$, C must be greater than 1.

Some other forms of Landen inequalities can be found in [7, 8].

In [4, pp. 20-21] and [9, Theorem 1.4] Gauss' asymptotic formula (1.4) was refined by finding the lower and upper bounds for

$$W(r) = B(a,b)F(a,b;a+b;r) + \left(\frac{1}{x}\right)\log(1-r), \tag{1.11}$$

when $a,b \in (0,1)$ or $a,b \in (1,\infty)$. Our second result gives a full solution to Open Problem 1.

We wish to point out that in [10, Theorem 1.2(1)] it was claimed that for $a,b \in (0,1)$, $c = a+b \leq 1$, the function

$$s(r) = (1+\sqrt{r})F(a,b;c;r) - F\left(a,b;c;\frac{4\sqrt{r}}{(1+\sqrt{r})^2}\right) \tag{1.12}$$

is increasing in $r \in (0,1)$. As pointed out by Baricz [11] the proof contains a gap and the correct proof will be given here.

We also found another area in *ab* plane where the function $s(r)$ is monotone decreasing in $r \in (0,1)$.

2. Main Results

Our first result shows that Landen inequalities hold not only in the neighborhood of the point $a = b = 1/2$ but also in some unbounded parts of *ab* plane.

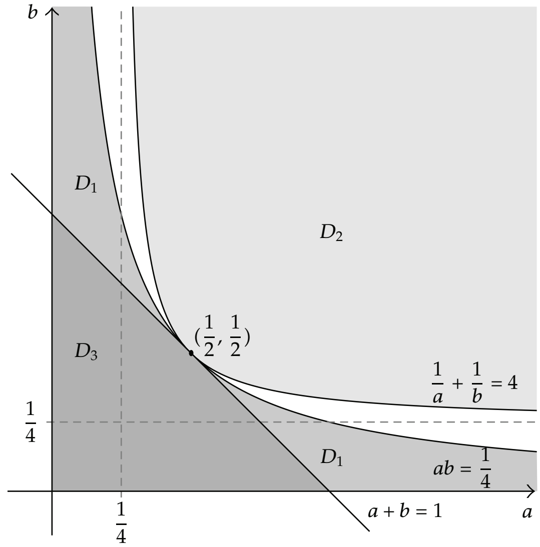

Figure 1: The domains D_j, $j = 1, 2, 3$ visualized.

Theorem 2.1. *For all $a, b > 0$ with $ab \leq 1/4$ one has that the inequality*

$$F\left(a, b; a+b; \frac{4r}{(1+r)^2}\right) \leq (1+r) F\left(a, b; a+b; r^2\right) \tag{2.1}$$

holds for each $r \in (0,1)$. Also, for $a, b > 0$, $1/a + 1/b \leq 4$, the reversed inequality

$$F\left(a, b; a+b; \frac{4r}{(1+r)^2}\right) \geq (1+r) F\left(a, b; a+b; r^2\right), \tag{2.2}$$

takes place for each $r \in (0,1)$.

In the remaining region $a, b > 0 \bigwedge ab > 1/4 \bigwedge 1/a + 1/b > 4$ neither of the above inequalities holds for each $r \in (0,1)$.

The disjoint regions in ab plane $D_1 = \{(a,b) \mid a, b > 0, ab \leq 1/4\}$ and $D_2 = \{(a,b) \mid a, b > 0, 1/a + 1/b \leq 4\}$, where Landen inequalities hold, are shown on the Figure 1.

The only common point of the graphs in Figure 1 is $(1/2, 1/2)$ where equality sign holds.

Two-sided bounds for the ratio of target functions are also possible.

Theorem 2.2. *For each $r \in (0,1)$ and $(a,b) \in D_1$, one has*

$$1 \leq \frac{(1+r) F\left(a, b; a+b; r^2\right)}{F\left(a, b; a+b; 4r/(1+r)^2\right)} \leq \frac{B(a,b)}{\pi}. \tag{2.3}$$

For $(a,b) \in D_2$ *the inequalities are reversed,*

$$\frac{B(a,b)}{\pi} \leq \frac{(1+r)F(a,b;a+b;r^2)}{F\left(a,b;a+b;4r/(1+r)^2\right)} \leq 1. \tag{2.4}$$

The bounds in both pairs of inequalities are sharp and equality is reached for $a = b = 1/2$*.*

Some numerical estimations of the constant C in Open Problem 1 follows.

Corollary 2.3. *Let* $k(\cdot)$ *be defined as in Open Problem 1. Then, for each* $r \in (0,1)$ *and* $(a,b) \in D_1$*, one has*

$$\frac{\pi}{B(a,b)}k(r) < k\left(\frac{2\sqrt{r}}{1+r}\right) < 2k(r). \tag{2.5}$$

In the region D_2 *one has*

$$k(r) < k\left(\frac{2\sqrt{r}}{1+r}\right) < \frac{2\pi}{B(a,b)}k(r). \tag{2.6}$$

Two-sided bounds for the difference exist in a smaller region $D_3 \subset D_1$ (see Figure 1), where $D_3 = \{(a,b) \mid a,b > 0, a+b \leq 1\}$ and in D_2.

Theorem 2.4. *Let* $B = B(a,b)$ *be the classical Beta function and* $R = R(a,b)$ *be defined by* (1.7)*.*
For $a,b > 0$*,* $a+b \leq 1$*, one has*

$$0 \leq (1+\sqrt{r})F(a,b;a+b;r) - F\left(a,b;a+b;\frac{4\sqrt{r}}{(1+\sqrt{r})^2}\right) \leq \frac{R - \log 16}{B}. \tag{2.7}$$

If $a,b > 0$*,* $1/a + 1/b \leq 4$*, then*

$$0 \leq F\left(a,b;a+b;\frac{4\sqrt{r}}{(1+\sqrt{r})^2}\right) - (1+\sqrt{r})F(a,b;a+b;r) \leq \frac{\log 16 - R}{B}. \tag{2.8}$$

The second Landen identity has the following counterpart for hypergeometric functions. The resulting inequalities might be called Landen inequalities for zero-balanced hypergeometric functions.

Theorem 2.5. *Let* $F(x) = F(a,b;a+b;x)$*.*
For $(a,b) \in D_1$ *and each* $x \in (0,1)$*, one has*

$$\frac{1}{2} < \frac{F\left(((1-x)/(1+x))^2\right)}{(1+x)F(1-x^2)} < \frac{B(a,b)}{2\pi}. \tag{2.9}$$

If $(a,b) \in D_3$, *then*

$$(1+x)F\left(1-x^2\right) \le 2F\left(\left(\frac{1-x}{1+x}\right)^2\right) \le (1+x)\left[F\left(1-x^2\right)+\frac{R-\log 16}{B}\right]. \tag{2.10}$$

For $(a,b) \in D_2$, *one has*

$$\begin{gathered}\frac{B(a,b)}{2\pi} < \frac{F\left(((1-x)/(1+x))^2\right)}{(1+x)F(1-x^2)} < \frac{1}{2}, \\ 0 \le (1+x)F\left(1-x^2\right) - 2F\left(\left(\frac{1-x}{1+x}\right)^2\right) \le \frac{(1+x)(\log 16 - R)}{B}.\end{gathered} \tag{2.11}$$

3. Proofs

Throughout this section we denote

$$F(x) = F(a,b;a+b;x), \qquad G(x) = F(a,b;a+b+1;x), \tag{3.1}$$

where $a, b, (a,b) \neq (1/2, 1/2)$ are fixed positive parameters and

$$F_0(x) = F\left(\frac{1}{2},\frac{1}{2};1;x\right), \qquad G_0(x) = F\left(\frac{1}{2},\frac{1}{2};2;x\right), \tag{3.2}$$

with the regions D_1, D_2, D_3 defined as above.

The basic results, which makes possible all proofs in the sequel, are contained in the following.

Lemma 3.1. (1) *The function* $f(r) = F(r)/F_0(r)$ *is monotone decreasing in* $r \in (0,1)$ *on* D_1 *and monotone increasing on* D_2.

(2) *The function* $g(r) = G(r)/G_0(r)$ *is monotone decreasing on* D_3 *and monotone increasing on* D_2.

It should be noted that a general result of this kind was given in [12, Theorem 2.31].

Proof. We shall use Lemma 1.1 in the proof.

Since $\widehat{F}_n = (a)_n(b)_n/(a+b)_n(1)_n$, $\widehat{F_{0n}} = ((1/2)_n/(1)_n)^2$, applying the lemma one can see that the monotonicity of $\{\widehat{F}_n/\widehat{F_{0n}}\}$ depends on the sign of

$$T_n = T(a,b;n) = n\left(ab - \frac{1}{4}\right) + ab - \frac{a+b}{4} = C_1 n + C_2. \tag{3.3}$$

Since $(a, b) \neq (1/2, 1/2)$ and

$$C_2 = \frac{\sqrt{ab}}{\sqrt{ab} + 1/2} C_1 - \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{4}, \tag{3.4}$$

it follows that

(1) if $C_1 \leq 0$, that is, $(a, b) \in D_1$, then $C_2 < 0$; hence $T_n < 0$ for $n = 0, 1, 2, \ldots$ and $f(r)$ is monotone decreasing in $r \in (0, 1)$;

(2) if $C_2 \geq 0$, that is, $(a, b) \in D_2$ then $C_1 > 0$, that is, $T_n > 0$, $n = 0, 1, 2, \ldots$ and $f(r)$ is monotone increasing in r.

In the second case we have $\widehat{G}_n = (a)_n (b)_n / (a+b+1)_n (1)_n$, $\widehat{G_{0n}} = ((1/2)_n / (1)_n)^2 / (n+1)$ and, proceeding analogously, we get

$$T_n' = n\left(ab + a + b - \frac{5}{4}\right) + 2ab - \frac{a+b}{4} - \frac{1}{4} = C_3 n + C_4. \tag{3.5}$$

(3) If $(a, b) \in D_3$, that is, $a, b > 0$, $a + b \leq 1$, let $a + b = k > 0$. Then $ab \leq k^2/4$ and

$$C_3 \leq \frac{k^2}{4} + k - \frac{5}{4} = \frac{(k-1)(k+5)}{4}; \qquad C_4 \leq \frac{k^2}{2} - \frac{k}{4} - \frac{1}{4} = \frac{(k-1)(2k+1)}{4}. \tag{3.6}$$

Since $0 < k \leq 1$, it follows that both C_3, C_4 are nonpositive. Therefore $T_n' < 0$, $n = 0, 1, 2, \ldots$ because both constants cannot be zero simultaneously. By Lemma 1.1, we conclude that the function $g(r)$ is monotone decreasing in $r \in (0, 1)$.

(4) If $(a, b) \in D_2$, that is, $a, b > 0$, $1/a + 1/b \leq 4$, then $4ab \geq a + b \geq 2\sqrt{ab}$, hence $ab \geq 1/4$. Also $a + b \geq 2\sqrt{ab} \geq 2 \cdot (1/2) = 1$. Therefore $C_3 \geq 0$ and $C_4 = (ab - 1/4) + (4ab - a - b)/4 \geq 0$. As above, we conclude that $T_n' > 0$, $n = 0, 1, 2, \ldots$ and $g(r)$ is monotone increasing in this case. □

Proof of Theorem 2.1. By the above lemma, for each $0 < x < y < 1$ we have $f(x) > f(y)$ on D_1 and $f(x) < f(y)$ on D_2.

Putting $x = x(r) = r^2$, $y = y(r) = 4r/(1+r)^2$, we get on D_1,

$$\frac{F(r^2)}{F_0(r^2)} > \frac{F(y)}{F_0(y)}, \tag{3.7}$$

that is, by Landen's identity,

$$F(y) < \frac{F_0(y)}{F_0(r^2)} F\left(r^2\right) = (1+r) F\left(r^2\right). \tag{3.8}$$

The second inequality is obtained analogously.

It is easily seen by (3.3) that in the remaining region the sequence $\{\widehat{F}_n/\widehat{F_{0n}}\}$ decreases and then increases. By Lemma 1.1, part 3, this means that the function $f(r)$, for some $r_0 \in (0,1)$, decreases in $(0,r_0)$ and increases in $(r_0,1)$. Therefore, putting $0 < x(r) < y(r) < r_0$ and $r_0 < x(r) < y(r) < 1$, one concludes that neither of the given inequalities holds for each $r \in (0,1)$. □

Proof of Theorem 2.2. Since $f(r)$ is monotone decreasing on D_1, applying Gauss formula, we obtain

$$1 = \lim_{r\to 0^+}\frac{F(r)}{F_0(r)} > \frac{F(r)}{F_0(r)} > \lim_{r\to 1^-}\frac{F(r)}{F_0(r)} = \frac{B(1/2,1/2)}{B(a,b)} = \frac{\pi}{B(a,b)}. \tag{3.9}$$

Therefore,

$$\frac{F(y(r))}{F(x(r))} < \frac{B(a,b)}{\pi}\frac{F_0(y(r))}{F_0(x(r))} = (1+r)\frac{B(a,b)}{\pi}, \tag{3.10}$$

by the Landen identity.

The inequality valid on D_2 can be proved similarly. □

Proof of Theorem 2.4. Both assertions of this theorem are a consequence of the following.

Lemma 3.2. *The function*

$$s(r) = (1+\sqrt{r})F(a,b;a+b;r) - F\left(a,b;a+b;\frac{4\sqrt{r}}{(1+\sqrt{r})^2}\right) \tag{3.11}$$

is monotone increasing in $r \in (0,1)$ on D_3 and monotone decreasing on D_2.

Proof. Let $z = 4\sqrt{r}/(1+\sqrt{r})^2$. Then

$$1 - z = \frac{(1-\sqrt{r})^2}{(1+\sqrt{r})^2}; \qquad \frac{dz}{dr} = \frac{2(1-\sqrt{r})}{\sqrt{r}(1+\sqrt{r})^3}. \tag{3.12}$$

Hence

$$\begin{aligned}
s_1(r) := 2\sqrt{r}(1-\sqrt{r})s'(r) &= (1-\sqrt{r})F(a,b;a+b;r) + 2\sqrt{r}(1-r)F'(a,b;a+b;r) \\
&\quad - \frac{4}{1+\sqrt{r}}(1-z)F'(a,b;a+b;z) \\
&= (1-\sqrt{r})F(a,b;a+b;r) + 2\frac{ab}{a+b}\sqrt{r}F(a,b;a+b+1;r) \\
&\quad - \frac{4ab}{(a+b)(1+\sqrt{r})}F(a,b;a+b+1;z) \\
&= (1-\sqrt{r})F(r) + 2\frac{ab}{a+b}\sqrt{r}G(r) - \frac{4ab}{(a+b)(1+\sqrt{r})}G(z).
\end{aligned} \tag{3.13}$$

We used here the well-known formula

$$(1-x)F'(a,b;a+b;x) = \frac{ab}{a+b}F(a,b;a+b+1;x). \tag{3.14}$$

On the other hand, differentiating the first Landen identity we get

$$\frac{1}{1+\sqrt{r}}G_0(z) = \left(1-\sqrt{r}\right)F_0(r) + \frac{1}{2}\sqrt{r}G_0(r). \tag{3.15}$$

Since $g(r)$ is monotone decreasing on D_3 and $0 < r < z < 1$, we get $g(r) > g(z)$, that is,

$$G(z) < \frac{G_0(z)}{G_0(r)}G(r). \tag{3.16}$$

This, together with (3.15), yields

$$\begin{aligned} s_1(r) &> (1-\sqrt{r})F(r) + 2\frac{ab}{a+b}\sqrt{r}G(r) - \frac{4ab}{(a+b)\left(1+\sqrt{r}\right)}\frac{G_0(z)}{G_0(r)}G(r) \\ &= (1-\sqrt{r})F(r) + 2\frac{ab}{a+b}\sqrt{r}G(r) - \frac{4ab}{(a+b)}\left(\left(1-\sqrt{r}\right)\frac{F_0(r)}{G_0(r)} + \frac{1}{2}\sqrt{r}\right)G(r) \\ &= (1-\sqrt{r})\left(F(r) - \frac{4ab}{(a+b)}\frac{F_0(r)}{G_0(r)}G(r)\right). \end{aligned} \tag{3.17}$$

By (3.14) again, we get

$$\frac{4ab}{(a+b)}\frac{G(r)}{G_0(r)} = \frac{F'(r)}{F_0'(r)}. \tag{3.18}$$

Hence,

$$2\sqrt{r}s'(r) > F(r) - \frac{F'(r)}{F_0'(r)}F_0(r) = \frac{F^2(r)}{F_0'(r)}\left(\frac{F_0(r)}{F(r)}\right)'. \tag{3.19}$$

The last expression is positive on D_3 because $D_3 \subset D_1$ and, by Lemma 3.1, the function $f(r) = F(r)/F_0(r)$ is monotone decreasing on D_1.

Therefore we proved that the function $s(r)$ is monotone increasing in $r \in (0,1)$ on D_3.

Remark 3.3. Due to the remark in Section 1, this proof gives an affirmative answer to the 12-years-old hypothesis risen in [10].

Since $g(r)$ is increasing on D_2, we get

$$G(z) > \frac{G_0(z)}{G_0(r)}G(r). \tag{3.20}$$

Hence, proceeding as before, it follows that

$$2\sqrt{r}s'(r) < \frac{F^2(r)}{F_0'(r)}\left(\frac{F_0(r)}{F(r)}\right)' < 0, \tag{3.21}$$

since $f(r) = F(r)/F_0(r)$ is monotone increasing on D_2.

Therefore $s(r)$ is monotone decreasing in $r \in (0,1)$ on D_2 and the proof of Lemma 3.2 is done. □

By Lemma 3.2 we obtain $\lim_{r\to 0^+} s(r) < s(r) < \lim_{r\to 1^-} s(r)$ on D_3 and $\lim_{r\to 1^-} s(r) < s(r) < \lim_{r\to 0^+} s(r)$ on D_2.

Evidently, $\lim_{r\to 0^+} s(r) = 0$.

Applying Ramanujan formula (1.6), we get

$$\begin{aligned}\lim_{r\to 1^-} s(r) &= \frac{\lim_{r\to 1^-}\left(R - 2\log(1-r) + \log(1-z) + o(1)\right)}{B} \\ &= \frac{\lim_{r\to 1^-}\left(R - 2\log\left(1-\sqrt{r}\right)\left(1+\sqrt{r}\right) + 2\log\left(\left(1-\sqrt{r}\right)/\left(1+\sqrt{r}\right)\right) + o(1)\right)}{B} \\ &= \frac{(R - \log 16)}{B}.\end{aligned} \tag{3.22}$$

The assertion of Theorem 2.4 follows. □

Proof of Theorem 2.5. Changing variable $(1-r)/(1+r) = x \in (0,1)$, we obtain

$$r = \frac{1-x}{1+x}; \qquad 1 + r = \frac{2}{1+x}; \qquad \frac{4r}{(1+r)^2} = 1 - x^2. \tag{3.23}$$

Putting this in Theorems 2.2 and 2.4, we obtain the assertions of Theorem 2.5. □

Remark 3.4. As the referee notes, the results from Theorems 2.1 and 2.2 can be generalized for $F(a,b,c;r)$. This is left to the readers.

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