

Research Article

Numerical Algorithms for Computing Eigenvalues of Discontinuous Dirac System Using Sinc-Gaussian Method

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The eigenvalues of a discontinuous regular Dirac systems with transmission conditions at the point of discontinuity are computed using the sinc-Gaussian method. The error analysis of this method for solving discontinuous regular Dirac system is discussed. It shows that the error decays exponentially in terms of the number of involved samples. Therefore, the accuracy of the new method is higher than the classical sinc-method. Numerical results indicating the high accuracy and effectiveness of these algorithms are presented. Comparisons with the classical sinc-method are given.

1. Introduction

The mathematical modeling of many practical problems in mechanics and other areas of mathematical physics requires solutions of boundary value problems (see, e.g., [1–7]). Boundary value problems with discontinuity conditions arise in different branches of mathematics, mechanics, radio, electronics, geophysics, and other fields of natural science and technology. For example, discontinuous conditions inside an interval are connected with discontinuous or nonsmooth properties of media (see [8–11] and references there in).

Sampling theory is one of the most important mathematical tools used in communication engineering since it enables engineers to reconstruct signals from some of their sampled data. A fundamental result in information theory is the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem [12–14]. It states that any $f \in PW_{\sigma}^2$, where PW_{σ}^2 is the space of all entire functions of exponential type $\sigma > 0$ which lie in $L^2(\mathbb{R})$ when restricted to \mathbb{R} , can be reconstructed from its sampled values $\{f(n\pi/\sigma) : n \in \mathbb{Z}\}$ by the formula

$$f(\lambda) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{\sigma}\right) \text{sinc}(\sigma\lambda - n\pi), \quad \lambda \in \mathbb{C}, \quad (1.1)$$

where

$$\text{sinc}(\sigma\lambda - n\pi) := \begin{cases} \frac{\sin(\sigma\lambda - n\pi)}{(\sigma\lambda - n\pi)}, & \lambda \neq \frac{n\pi}{\sigma}, \\ 1, & \lambda = \frac{n\pi}{\sigma}. \end{cases} \quad (1.2)$$

Series (1.1) converges absolutely and uniformly on compact subsets of \mathbb{C} , and uniform on \mathbb{R} , cf. [15]. Expansion (1.1) is used in several approximation problems which are known as sinc methods, see, for example, [16–19]. In particular the sinc-method is used to approximate eigenvalues of boundary value problems, see, for examples, [20–24]. The sinc-method has a slow rate of decay at infinity, which is as slow as $O(|\lambda|^{-1})$. There are several attempts to improve the rate of decay. One of the interesting ways is to multiply the sinc-function in (1.1) by a kernel function, see, for example, [25–27]. Let $h \in (0, \pi/\sigma]$ and $\gamma \in (0, \pi - h\sigma)$. Assume that $\Phi \in PW_\gamma^2$ such that $\Phi(0) = 1$, then for $f \in PW_\sigma^2$ we have the expansion [28],

$$f(\lambda) = \sum_{n=-\infty}^{\infty} f(nh) \text{sinc}(h^{-1}\pi\lambda - n\pi) \Phi(h^{-1}\lambda - n). \quad (1.3)$$

The speed of convergence of the series in (1.3) is determined by the decay of $|\Phi(\lambda)|$. But the decay of an entire function of exponential type cannot be as fast as $e^{-c|x|}$ as $|x| \rightarrow \infty$, for some positive c , [28]. In [29], Qian has introduced the following regularized sampling formula. For $h \in (0, \pi/\sigma]$, $N \in \mathbb{N}$ and $r > 0$,

$$(G_{h,N}f)(x) = \sum_{n \in \mathbb{Z}_N(x)} f(nh) S_n(h^{-1}\pi x) G\left(\frac{x - nh}{\sqrt{2}rh}\right), \quad x \in \mathbb{R}, \quad (1.4)$$

where $G(t) := \exp(-t^2)$, which is called the Gaussian function, $S_n(h^{-1}\pi x) := \text{sinc}(h^{-1}\pi x - n\pi)$, $\mathbb{Z}_N(x) := \{n \in \mathbb{Z} : |[h^{-1}x] - n| \leq N\}$ and $[x]$ denotes the integer part of $x \in \mathbb{R}$, see also [30, 31]. Qian also derived the following error bound. If $f \in PW_\sigma^2$, $h \in (0, \pi/\sigma]$ and $a := \min\{r(\pi - h\sigma), (N - 2)/r\} \geq 1$, then

$$|f(x) - (G_{h,N}f)(x)| \leq \frac{2\sqrt{\sigma\pi} \|f\|_2}{\pi^2 a^2} \left(\sqrt{2\pi}a + e^{3/2r^2} \right) e^{-a^2/2}, \quad x \in \mathbb{R}. \quad (1.5)$$

In [28], Schmeisser and Stenger extended the operator (1.4) to the complex domain \mathbb{C} . For $\sigma > 0$, $h \in (0, \pi/\sigma]$ and $\omega := (\pi - h\sigma)/2$, they defined the operator [28],

$$(\mathcal{G}_{h,N}f)(\lambda) := \sum_{n \in \mathbb{Z}_N(\lambda)} f(nh) S_n\left(\frac{\pi\lambda}{h}\right) G\left(\frac{\sqrt{\omega}(\lambda - nh)}{\sqrt{N}h}\right), \quad (1.6)$$

where $\mathbb{Z}_N(\lambda) := \{n \in \mathbb{Z} : |[h^{-1}\Re\lambda + 1/2] - n| \leq N\}$ and $N \in \mathbb{N}$. Note that the summation limits in (1.6) depend on the real part of λ . Schmeisser and Stenger, [28], proved that if f is an entire

function of exponential type $\sigma > 0$, then for $h \in (0, \pi/\sigma)$, $\omega := (\pi - h\sigma)/2$, $N \in \mathbb{N}$, $|\Im \lambda| < N$, we have

$$|f(\lambda) - (\mathcal{G}_{h,N}f)(\lambda)| \leq 2 \left| \sin(h^{-1}\pi\lambda) \right| \|f\|_{\infty} \frac{e^{-\omega N}}{\sqrt{\pi\omega N}} \beta_N(h^{-1}\Im \lambda), \quad \lambda \in \mathbb{C}, \quad (1.7)$$

where

$$\beta_N(t) := \cosh(2\omega t) + \frac{2e^{\omega t^2/N}}{\sqrt{\pi\omega N} \left[1 - (t/N)^2\right]} + \frac{1}{2} \left[\frac{e^{2\omega t}}{e^{2\pi(N-t)} - 1} + \frac{e^{-2\omega t}}{e^{2\pi(N+t)} - 1} \right]. \quad (1.8)$$

The amplitude error arises when the exact values $f(nh)$ of (1.6) are replaced by the approximations $\tilde{f}(nh)$. We assume that $\tilde{f}(nh)$ are close to $f(nh)$, that is, there is $\varepsilon > 0$, sufficiently small such that

$$\sup_{n \in \mathbb{Z}_n(\lambda)} |f(nh) - \tilde{f}(nh)| < \varepsilon. \quad (1.9)$$

Let $h \in (0, \pi/\sigma)$, $\omega := (\pi - h\sigma)/2$ and $N \in \mathbb{N}$ be fixed numbers. The authors in [9] proved that if (1.9) is held, then for $|\Im \lambda| < N$, we have

$$\left| (\mathcal{G}_{h,N}f)(\lambda) - (\mathcal{G}_{h,N}\tilde{f})(\lambda) \right| \leq A_{\varepsilon,N}(\Im \lambda), \quad (1.10)$$

where

$$A_{\varepsilon,N}(\Im \lambda) = 2\varepsilon e^{-\omega/4N} \left(1 + \sqrt{\frac{N}{\omega\pi}} \right) \exp\left((\omega + \pi)h^{-1}|\Im \lambda|\right). \quad (1.11)$$

We are concerned with the computation of eigenvalues the Dirac system

$$\begin{aligned} u_2'(x, \lambda) - p_1(x)u_1(x, \lambda) &= \lambda u_1(x, \lambda), \\ u_1'(x, \lambda) + p_2(x)u_2(x, \lambda) &= -\lambda u_2(x, \lambda), \\ x &\in [-1, 0) \cup (0, 1], \end{aligned} \quad (1.12)$$

with boundary conditions

$$U_1(\mathbf{u}) := \sin \alpha u_1(-1, \lambda) + \cos \alpha u_2(-1, \lambda) = 0, \quad (1.13)$$

$$U_2(\mathbf{u}) := \sin \beta u_1(1, \lambda) + \cos \beta u_2(1, \lambda) = 0, \quad (1.14)$$

and transmission conditions

$$U_3(\mathbf{u}) := u_1(0^-, \lambda) - \delta u_1(0^+, \lambda) = 0, \quad (1.15)$$

$$U_4(\mathbf{u}) := u_2(0^-, \lambda) - \delta u_2(0^+, \lambda) = 0, \quad (1.16)$$

where $\lambda \in \mathbb{C}$; the real valued function $p_1(\cdot)$ and $p_2(\cdot)$ are continuous in $[-1, 0)$ and $(0, 1]$, and have finite limits $p_1(0^\pm) := \lim_{x \rightarrow 0^\pm} p_1(x)$, $p_2(0^\pm) := \lim_{x \rightarrow 0^\pm} p_2(x)$; $\delta \in \mathbb{R}$, $\alpha, \beta \in [0, \pi)$ and $\delta \neq 0$.

In this paper we will use the sinc-Gaussian sampling formula (1.6) to compute eigenvalues of the Dirac system (1.12)–(1.16). As expected, the new method reduced the error bounds remarkably, see examples at the end of this paper. Special attention is given to the comparison of the numerical results obtained by the new method with those found by classical sinc-method. We would like to mention that works in direction of computing eigenvalues with the sinc-Gaussian, are few, see for example, [9]. Also papers in computing of eigenvalues with discontinuous are few, see [10, 32]. However, the computing of eigenvalues by sinc-Gaussian technique which has discontinuity conditions, do not exist as far as we know.

The paper is organized as follows: Section 2 contains some preliminary results and the approximated values of the eigenvalues of the Dirac system with discontinuous. The method with error estimates are contained in Section 3. The last section involves some illustrative examples for showing the high accuracy of the proposed technique.

2. The Approximated Eigenvalues of Dirac System

In this section we derive approximate values of the eigenvalues of problem (1.12)–(1.16). Recall that problem (1.12)–(1.16) has denumerable set of real and simple eigenvalues, compare with [33, 34]. Let

$$\phi(\cdot, \lambda) = \begin{pmatrix} \phi_1(\cdot, \lambda) \\ \phi_2(\cdot, \lambda) \end{pmatrix}, \quad \phi_i(x, \lambda) = \begin{cases} \phi_{i1}(x, \lambda), & x \in [-1, 0), \\ \phi_{i2}(x, \lambda), & x \in (0, 1], \end{cases} \quad i = 1, 2, \quad (2.1)$$

be the solution of (1.12) satisfying the following initial conditions:

$$\begin{pmatrix} \phi_{11}(-1, \lambda) & \phi_{12}(0^+, \lambda) \\ \phi_{21}(-1, \lambda) & \phi_{22}(0^+, \lambda) \end{pmatrix} = \begin{pmatrix} \cos \alpha & \delta^{-1} \phi_{11}(0^-, \lambda) \\ -\sin \alpha & \delta^{-1} \phi_{21}(0^-, \lambda) \end{pmatrix}. \quad (2.2)$$

Since $\phi(\cdot, \lambda)$ satisfies (1.13), then the eigenvalues of the problem (1.12)–(1.16) are the zeros of the function

$$\Delta(\lambda) = -\delta^2 (\sin \beta \phi_{12}(1, \lambda) + \cos \beta \phi_{22}(1, \lambda)). \quad (2.3)$$

Notice that both $\phi(\cdot, \lambda)$ and $\Delta(\lambda)$ are entire functions of λ and $\phi(\cdot, \lambda)$ satisfies the system of integral equations, see [32, 35],

$$\phi_{11}(x, \lambda) = \cos(\lambda(x+1) - \alpha) - \mathcal{T}_{-1,1}\phi_{11}(x, \lambda) - \tilde{\mathcal{T}}_{-1,2}\phi_{21}(x, \lambda), \quad (2.4)$$

$$\phi_{21}(x, \lambda) = \sin(\lambda(x+1) - \alpha) + \tilde{\mathcal{T}}_{-1,1}\phi_{11}(x, \lambda) - \mathcal{T}_{-1,2}\phi_{21}(x, \lambda), \quad (2.5)$$

$$\begin{aligned} \phi_{12}(x, \lambda) &= \frac{1}{\delta}\phi_{11}(0^-, \lambda)\cos(\lambda x) - \frac{1}{\delta}\phi_{21}(0^-, \lambda)\sin(\lambda x) \\ &\quad - \mathcal{T}_{0,1}\phi_{12}(x, \lambda) - \tilde{\mathcal{T}}_{0,2}\phi_{22}(x, \lambda), \end{aligned} \quad (2.6)$$

$$\begin{aligned} \phi_{22}(x, \lambda) &= \frac{1}{\delta}\phi_{11}(0^-, \lambda)\sin(\lambda x) + \frac{1}{\delta}\phi_{21}(0^-, \lambda)\cos(\lambda x) \\ &\quad + \tilde{\mathcal{T}}_{0,1}\phi_{12}(x, \lambda) - \mathcal{T}_{0,2}\phi_{22}(x, \lambda), \end{aligned} \quad (2.7)$$

where $\mathcal{T}_{-1,i}$, $\tilde{\mathcal{T}}_{-1,i}$, $\mathcal{T}_{0,i}$ and $\tilde{\mathcal{T}}_{0,i}$, $i = 1, 2$, are the Volterra integral operators defined by

$$\begin{aligned} \mathcal{T}_{-1,1}\varphi(x, \lambda) &:= \int_{-1}^x \sin \lambda(x-t)p_1(t)\varphi(t, \lambda)dt, \\ \mathcal{T}_{-1,2}\varphi(x, \lambda) &:= \int_{-1}^x \sin \lambda(x-t)p_2(t)\varphi(t, \lambda)dt, \\ \tilde{\mathcal{T}}_{-1,1}\varphi(x, \lambda) &:= \int_{-1}^x \cos \lambda(x-t)p_1(t)\varphi(t, \lambda)dt, \\ \tilde{\mathcal{T}}_{-1,2}\varphi(x, \lambda) &:= \int_{-1}^x \cos \lambda(x-t)p_2(t)\varphi(t, \lambda)dt, \\ \mathcal{T}_{0,1}\varphi(x, \lambda) &:= \int_0^x \sin \lambda(x-t)p_1(t)\varphi(t, \lambda)dt, \\ \mathcal{T}_{0,2}\varphi(x, \lambda) &:= \int_0^x \sin \lambda(x-t)p_2(t)\varphi(t, \lambda)dt, \\ \tilde{\mathcal{T}}_{0,1}\varphi(x, \lambda) &:= \int_0^x \cos \lambda(x-t)p_1(t)\varphi(t, \lambda)dt, \\ \tilde{\mathcal{T}}_{0,2}\varphi(x, \lambda) &:= \int_0^x \cos \lambda(x-t)p_2(t)\varphi(t, \lambda)dt. \end{aligned} \quad (2.8)$$

For convenience, we define the constants

$$\begin{aligned} c_1 &:= \int_{-1}^0 [|p_1(t)| + |p_2(t)|]dt, & c_2 &:= c_1 \exp(c_1), \\ c_3 &:= \int_0^1 [|p_1(t)| + |p_2(t)|]dt, & c_4 &:= c_2 + \frac{2}{|\delta|}(1 + c_2). \end{aligned} \quad (2.9)$$

Define $f_{-1,i}(\cdot, \lambda)$ and $f_{0,i}(\cdot, \lambda)$, $i = 1, 2$, to be

$$f_{-1,1}(x, \lambda) := \tau_{-1,1}\phi_{11}(x, \lambda) + \tilde{\tau}_{-1,2}\phi_{21}(x, \lambda), \quad (2.10)$$

$$f_{-1,2}(x, \lambda) := \tilde{\tau}_{-1,1}\phi_{11}(x, \lambda) - \tau_{-1,2}\phi_{21}(x, \lambda),$$

$$f_{0,1}(x, \lambda) := \tau_{0,1}\phi_{12}(x, \lambda) + \tilde{\tau}_{0,2}\phi_{22}(x, \lambda), \quad (2.11)$$

$$f_{0,2}(x, \lambda) := \tilde{\tau}_{0,1}\phi_{12}(x, \lambda) - \tau_{0,2}\phi_{22}(x, \lambda).$$

Lemma 2.1. *The functions $f_{-1,1}(x, \lambda)$ and $f_{-1,2}(x, \lambda)$ are entire in λ for any fixed $x \in [-1, 0)$ and satisfy the growth condition*

$$|f_{-1,1}(x, \lambda)|, \quad |f_{-1,2}(x, \lambda)| \leq 2c_2 e^{|\Im \lambda|(x+1)}, \quad \lambda \in \mathbb{C}. \quad (2.12)$$

Proof. Since $f_{-1,1}(x, \lambda) = \tau_{-1,1}\phi_{11}(x, \lambda) + \tilde{\tau}_{-1,2}\phi_{21}(x, \lambda)$, then from (2.4) and (2.5) we obtain

$$\begin{aligned} f_{-1,1}(x, \lambda) &= \tau_{-1,1} \cos(\lambda(x+1) - \alpha) + \tilde{\tau}_{-1,2} \sin(\lambda(x+1) - \alpha) - \tau_{-1,1} f_{-1,1}(x, \lambda) \\ &\quad + \tilde{\tau}_{-1,2} f_{-1,2}(x, \lambda). \end{aligned} \quad (2.13)$$

Using the inequalities $|\sin z| \leq e^{|\Im z|}$ and $|\cos z| \leq e^{|\Im z|}$ for $z \in \mathbb{C}$, leads for $\lambda \in \mathbb{C}$ to

$$\begin{aligned} |f_{-1,1}(x, \lambda)| &\leq |\tau_{-1,1} \cos(\lambda(x+1) - \alpha)| + |\tilde{\tau}_{-1,2} \sin(\lambda(x+1) - \alpha)| \\ &\quad + |\tau_{-1,1} f_{-1,1}(x, \lambda)| + |\tilde{\tau}_{-1,2} f_{-1,2}(x, \lambda)| \\ &\leq e^{|\Im \lambda|(x+1)} \int_{-1}^x [|p_1(t)| |f_{-1,1}(t, \lambda)| + |p_2(t)| |f_{-1,2}(t, \lambda)|] e^{-|\Im \lambda|(t+1)} dt \\ &\quad + 2e^{|\Im \lambda|(x+1)} \int_{-1}^x [|p_1(t)| + |p_2(t)|] dt \leq 2c_1 e^{|\Im \lambda|(x+1)} \\ &\quad + e^{|\Im \lambda|(x+1)} \int_{-1}^x [|p_1(t)| |f_{-1,1}(t, \lambda)| + |p_2(t)| |f_{-1,2}(t, \lambda)|] e^{-|\Im \lambda|(t+1)} dt. \end{aligned} \quad (2.14)$$

The above inequality can be reduced to

$$e^{-|\Im \lambda|(x+1)} |f_{-1,1}(x, \lambda)| \leq 2c_1 + \int_{-1}^x [|p_1(t)| |f_{-1,1}(t, \lambda)| + |p_2(t)| |f_{-1,2}(t, \lambda)|] e^{-|\Im \lambda|(t+1)} dt. \quad (2.15)$$

Similarly, we can prove that

$$e^{-|\Im \lambda|(x+1)} |f_{-1,2}(x, \lambda)| \leq 2c_1 + \int_{-1}^x [|p_1(t)| |f_{-1,1}(t, \lambda)| + |p_2(t)| |f_{-1,2}(t, \lambda)|] e^{-|\Im \lambda|(t+1)} dt. \quad (2.16)$$

Then from (2.12) and (2.15) and Lemma 3.1 of [34, pp. 204], we obtain (2.12). \square

In a similar manner, we will prove the following lemma for $f_{0,1}(\cdot, \lambda)$ and $f_{0,2}(\cdot, \lambda)$.

Lemma 2.2. *The functions $f_{0,1}(x, \lambda)$ and $f_{0,2}(x, \lambda)$ are entire in λ for any fixed $x \in (0, 1]$ and satisfy the growth condition*

$$|f_{0,1}(x, \lambda)|, |f_{0,2}(x, \lambda)| \leq 2c_3c_4e^{|\Im \lambda|(x+1)}, \quad \lambda \in \mathbb{C}. \quad (2.17)$$

Proof. Since $f_{0,1}(x, \lambda) = \tau_{0,1}\phi_{11}(x, \lambda) + \tilde{\tau}_{0,2}\phi_{21}(x, \lambda)$, then from (2.6) and (2.7) we obtain

$$\begin{aligned} f_{0,1}(x, \lambda) &= \frac{1}{\delta}\phi_{11}(0^-, \lambda)\tau_{0,1}\cos(\lambda x) - \frac{1}{\delta}\phi_{21}(0^-, \lambda)\tau_{0,1}\sin(\lambda x) - \tau_{0,1}f_{-1,2}(x, \lambda) \\ &\quad + \frac{1}{\delta}\phi_{11}(0^-, \lambda)\tilde{\tau}_{0,2}\sin(\lambda x) + \frac{1}{\delta}\phi_{21}(0^-, \lambda)\tilde{\tau}_{0,2}\cos(\lambda x) + \tilde{\tau}_{0,2}f_{-1,2}(x, \lambda). \end{aligned} \quad (2.18)$$

Then from (2.4) and (2.5) and Lemma 2.1, we get

$$\begin{aligned} f_{0,1}(x, \lambda) &\leq \frac{1}{|\delta|}|\phi_{11}(0^-, \lambda)||\tau_{0,1}\cos(\lambda x)| + \frac{1}{|\delta|}|\phi_{21}(0^-, \lambda)||\tau_{0,1}\sin(\lambda x)| + |\tau_{0,1}f_{-1,2}(x, \lambda)| \\ &\quad + \frac{1}{|\delta|}|\phi_{11}(0^-, \lambda)||\tilde{\tau}_{0,2}\sin(\lambda x)| + \frac{1}{|\delta|}|\phi_{21}(0^-, \lambda)||\tilde{\tau}_{0,2}\cos(\lambda x)| + |\tilde{\tau}_{0,2}f_{-1,2}(x, \lambda)| \\ &\leq 2\left(c_2 + \frac{2}{|\delta|}(1 + c_2)\right)c_3e^{|\Im \lambda|(x+1)} = 2c_3c_4e^{|\Im \lambda|(x+1)}. \end{aligned} \quad (2.19)$$

Similarly, we can prove that

$$f_{0,2}(x, \lambda) \leq 2c_3c_4e^{|\Im \lambda|(x+1)}. \quad (2.20)$$

□

3. The Method and Error Analysis

In this section we derive the method of computing the eigenvalues of problem (1.12)–(1.16) numerically. We aim to approximate $\Delta(\lambda)$ and hence its zeros, that is, the eigenvalues. The idea is to split $\Delta(\lambda)$ into two parts, one is known and the other is unknown, but is an entire function of exponential type. Then we approximate the unknown part using (1.6) to get the approximate $\Delta(\lambda)$ and then compute the approximate zeros. Now, let us split $\Delta(\lambda)$ into

$$\Delta(\lambda) := \mathcal{K}(\lambda) + \mathcal{U}(\lambda), \quad (3.1)$$

where $\mathcal{U}(\lambda)$ is the unknown part involving integral operators

$$\begin{aligned} \mathcal{U}(\lambda) &:= \delta^2 \sin \beta f_{0,1}(1, \lambda) - \delta^2 \cos \beta f_{0,2}(1, \lambda) - \delta \sin(\lambda - \beta) f_{-1,1}(0^-, \lambda) \\ &\quad - \delta \cos(\lambda + \beta) f_{-1,2}(0^-, \lambda), \end{aligned} \quad (3.2)$$

and $\mathcal{K}(\lambda)$ is the known part

$$\mathcal{K}(\lambda) := -\delta \sin(2\lambda - \alpha + \beta). \quad (3.3)$$

Then, from Lemmas 2.1 and 2.2, we have the following result.

Lemma 3.1. *The function $\mathcal{U}(\lambda)$ is entire in λ and the following estimate holds:*

$$|\mathcal{U}(\mu)| \leq M e^{2|\Im \lambda|}, \quad (3.4)$$

where

$$M := 4|\delta|(c_2 + |\delta|c_3c_4). \quad (3.5)$$

Proof. From (3.2), we have

$$\begin{aligned} |\mathcal{U}(\lambda)| &\leq |\delta|^2 |\sin \beta| |f_{0,1}(1, \lambda)| + |\delta|^2 |\cos \beta| |f_{0,2}(1, \lambda)| + |\delta| |\sin(\lambda - \beta)| |f_{-1,1}(0^-, \lambda)| \\ &\quad + |\delta| |\cos(\lambda + \beta)| |f_{-1,2}(0^-, \lambda)|. \end{aligned} \quad (3.6)$$

Using the inequalities $|\sin \lambda| \leq e^{|\Im \lambda|}$ and $|\cos \lambda| \leq e^{|\Im \lambda|}$ for $\lambda \in \mathbb{C}$, and Lemmas 2.1 and 2.2 imply (3.4). \square

Then $\mathcal{U}(\lambda)$ is an entire function of exponential type 2. In the following we let $\lambda \in \mathbb{R}$ since all eigenvalues are real. Now we approximate the function $\mathcal{U}(\lambda)$ using the operator (1.6) where $h \in (0, \pi/2]$ and $\omega := (\pi - 2h)/2$ and then we obtain

$$|\mathcal{U}(\lambda) - (\mathcal{G}_{h,N}\mathcal{U})(\lambda)| \leq T_{h,N}(\lambda), \quad (3.7)$$

where

$$T_{h,N}(\lambda) := 2 \left| \sin(h^{-1}\pi\lambda) \right| \|\mathcal{U}\|_{\infty} \frac{e^{-\omega N}}{\sqrt{\pi\omega N}} \beta_N(0), \quad \lambda \in \mathbb{R}. \quad (3.8)$$

The samples $\mathcal{U}(nh) = \Delta(nh) - \mathcal{K}(nh)$, $n \in \mathbb{Z}_N(\lambda)$ cannot be computed explicitly in the general case. We approximate these samples numerically by solving the initial-value problems defined by (1.12) and (2.2) to obtain the approximate values $\tilde{\mathcal{U}}(nh)$, $n \in \mathbb{Z}_N(\lambda)$, that is, $\tilde{\mathcal{U}}(nh) = \tilde{\Delta}(nh) - \mathcal{K}(nh)$. Accordingly we have the explicit expansion

$$(\mathcal{G}_{h,N}\tilde{\mathcal{U}})(\lambda) := \sum_{n \in \mathbb{Z}_N(\lambda)} \tilde{\mathcal{U}}(nh) S_n\left(\frac{\pi\lambda}{h}\right) G\left(\frac{\sqrt{\omega}(\lambda - nh)}{\sqrt{N}h}\right). \quad (3.9)$$

Therefore we get, compare with (1.10),

$$\left| (\mathcal{G}_{h,N}\mathcal{U})(\lambda) - (\mathcal{G}_{h,N}\tilde{\mathcal{U}})(\lambda) \right| \leq A_{\varepsilon,N}(0), \quad \lambda \in \mathbb{R}. \quad (3.10)$$

Now let $\tilde{\Delta}_N(\lambda) := \mathcal{K}(\lambda) + (\mathcal{G}_{h,N}\tilde{\mathcal{U}})(\lambda)$. From (3.7) and (3.10) we obtain

$$\left| \Delta(\lambda) - \tilde{\Delta}_N(\lambda) \right| \leq T_{h,N}(\lambda) + A_{\varepsilon,N}(0), \quad \lambda \in \mathbb{R}. \quad (3.11)$$

Let λ^* be an eigenvalue and λ_N be its desired approximation, that is, $\Delta(\lambda^*) = 0$ and $\tilde{\Delta}_N(\lambda_N) = 0$. From (3.11) we have $|\tilde{\Delta}_N(\lambda^*)| \leq T_{h,N}(\lambda^*) + A_{\varepsilon,N}(0)$. Now we define an enclosure interval $I_{\varepsilon,N}$ for λ^* . Define the curves

$$a_{\pm}(\lambda) = \tilde{\Delta}_N(\lambda) \pm (T_{h,N}(\lambda) + A_{\varepsilon,N}(0)). \quad (3.12)$$

The curves $a_+(\lambda)$, $a_-(\lambda)$ trap the curve of $\Delta(\lambda)$ for suitably large N . Hence the closure interval is determined by solving $a_{\pm}(\lambda) = 0$, gives an interval $I_{\varepsilon,N} := [a_-, a_+]$. Next we estimate the error $|\lambda^* - \lambda_N|$ for the eigenvalue λ^* .

Theorem 3.2. *Let λ^* be an eigenvalue of (1.12)–(1.16) and λ_N be its approximation. Then, for $\lambda \in \mathbb{R}$, one has the following estimate:*

$$|\lambda^* - \lambda_N| < \frac{T_{h,N}(\lambda_N) + A_{\varepsilon,N}(0)}{\inf_{\zeta \in I_{\varepsilon,N}} |\Delta'(\zeta)|}, \quad (3.13)$$

where the interval $I_{\varepsilon,N}$ is defined above.

Proof. Replacing λ by λ_N in (3.11) we obtain

$$|\Delta(\lambda_N) - \Delta(\lambda^*)| < T_{h,N}(\lambda_N) + A_{\varepsilon,N}(0), \quad (3.14)$$

where we have used $\tilde{\Delta}_N(\lambda_N) = \Delta(\lambda^*) = 0$. Using the mean value theorem yields that for some $\zeta \in J_{\varepsilon,N} := [\min(\lambda^*, \lambda_N), \max(\lambda^*, \lambda_N)]$,

$$|(\lambda^* - \lambda_N)\Delta'(\zeta)| \leq T_{h,N}(\lambda_N) + A_{\varepsilon,N}(0), \quad \zeta \in J_{\varepsilon,N} \subset I_{\varepsilon,N}. \quad (3.15)$$

Since λ^* is simple and N is sufficiently large, then $\inf_{\zeta \in I_{\varepsilon,N}} |\Delta'(\zeta)| > 0$ and we get (3.13). \square

4. Numerical Examples and Comparisons

This section includes two examples illustrating the sinc-Gaussian method. In the following examples, we consider $\lambda_{k,N}$ being the k th root of $\mathcal{K}(\lambda) + (\mathcal{G}_{h,N}\tilde{\mathcal{U}}) = 0$. Also, it is observed that the approximation $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$. We indicate in these two examples the effect of the amplitude error in the proposed method by determining enclosure intervals for different values of ε . All examples are computed in [32] by using the classical sinc method. We see that the sinc-Gaussian method gives remarkably better results.

Table 1: The absolute errors of sinc-Gaussian and the classical sinc-method with $N = 20$, $h = 0.2$, $\omega = 1.3708$.

λ_k	Sinc	Exact λ_k	Sinc-Gaussian $\lambda_{k,N}$	E	E_G
λ_{-2}	-3.5582594249860	-3.55825932025645	-3.5582593202564	1.04×10^{-7}	2.75×10^{-14}
λ_{-1}	-1.98746291522753	-1.98746299346156	-1.98746299346155	7.82×10^{-8}	3.77×10^{-15}
λ_0	-0.41666654116886	-0.416666666666666	-0.416666666666666	1.25×10^{-7}	1.83×10^{-15}
λ_1	1.15412963731329	1.15412966012822	1.15412966012823	2.28×10^{-8}	9.77×10^{-15}
λ_2	2.72492581024868	2.72492598692312	2.72492598692315	1.76×10^{-7}	3.28×10^{-14}

Example 4.1. Consider the system

$$\begin{aligned} u_2'(x) - p(x)u_1(x) &= \lambda u_1(x), \\ u_1'(x) + p(x)u_2(x) &= -\lambda u_2(x), \quad x \in [-1, 0) \cup (0, 1], \end{aligned} \quad (4.1)$$

subject to

$$\begin{aligned} u_1(-1) + u_2(-1) &= 0, & u_1(1) + u_2(1) &= 0, \\ u_1(0^-) - 2u_1(0^+) &= 0, & u_2(0^-) - 2u_2(0^+) &= 0. \end{aligned} \quad (4.2)$$

Here

$$p_1(x) = p_2(x) = p(x) = \begin{cases} x^2, & x \in [-1, 0) \\ x, & x \in (0, 1], \end{cases} \quad (4.3)$$

$\alpha = \beta = \pi/4$ and $\delta = 2$.

Direct calculations give

$$\begin{aligned} \mathcal{K}(\lambda) &= -2 \sin[2\lambda], \\ \Delta(\lambda) &= -2 \sin\left[\frac{5}{6} + 2\lambda\right], \end{aligned} \quad (4.4)$$

therefore the eigenvalues are $\lambda_k = (6k\pi - 5)/12$, $k \in \mathbb{Z}$.

Let E and E_G denote the absolute errors associated with the results of classical sinc method and sinc-Gaussian method, respectively. In Table 1, we give comparison between the absolute error of sinc-Gaussian and the classical sinc-method.

In Table 2, we observe that the approximation $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ for different values of ε .

Example 4.2. In this example we consider the system

$$\begin{aligned} u_2'(x) - p(x)u_1(x) &= \lambda u_1(x), \\ u_1'(x) + p(x)u_2(x) &= -\lambda u_2(x), \quad x \in [-1, 0) \cup (0, 1], \end{aligned} \quad (4.5)$$

Table 2: The approximation $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ for different values of ε .

λ_k	Exact λ_k	$[a_-, a_+], \varepsilon = 10^{-2}$	$[a_-, a_+], \varepsilon = 10^{-5}$	$\lambda_{k,N}$
λ_{-2}	-3.55825932025645	$[-3.59648, -3.52003]$	$[-3.55829, -3.55822]$	-3.5582593202564
λ_{-1}	-1.98746299346156	$[-2.02569, -1.94923]$	$[-1.98750, -1.98746]$	-1.98746299346155
λ_0	-0.416666666666666	$[-0.454895, -0.37843]$	$[-0.416704, -0.416628]$	-0.416666666666666
λ_1	1.15412966012822	$[1.115901, 1.19235]$	$[1.15409, 1.15416]$	1.15412966012823
λ_2	2.72492598692312	$[2.68669, 2.76315]$	$[2.72488, 2.72496]$	2.72492598692315

$\|\mathcal{H}\|_\infty = 67.9396, \beta_N(0) = 1.2155.$

Table 3: The absolute errors of sinc-Gaussian and the classical sinc-method with $N = 20, h = 0.3, \omega = 1.2708$.

λ_k	Sinc	Exact λ_k	Sinc-Gaussian $\lambda_{k,N}$	E	E_G
λ_{-2}	-3.29646018123751	-3.29645993245731	-3.29645993245735	2.48×10^{-7}	4.21×10^{-14}
λ_{-1}	-1.7256638185270	-1.7256636056624	-1.7256636056621	2.12×10^{-7}	2.68×10^{-13}
λ_0	-0.154867300702813	-0.154867278867517	-0.154867278867127	2.18×10^{-8}	3.89×10^{-13}
λ_1	1.41592898901685	1.41592904792737	1.41592904792752	5.89×10^{-8}	1.44×10^{-13}
λ_2	2.98672509121710	2.9867253747222	2.98672537472225	2.83×10^{-7}	1.86×10^{-14}

subject to

$$\begin{aligned} \sqrt{3}u_1(-1) + u_2(-1) &= 0, & u_1(1) + \sqrt{3}u_2(1) &= 0, \\ u_1(0^-) - 3u_1(0^+) &= 0, & u_2(0^-) - 3u_2(0^+) &= 0, \end{aligned} \quad (4.6)$$

where

$$p_1(x) = p_2(x) = p(x) = \begin{cases} x, & x \in [-1, 0) \\ x^2 + 1, & x \in (0, 1], \end{cases} \quad (4.7)$$

$\alpha = \pi/3, \beta = \pi/6$ and $\delta = 3$.

Direct calculations give

$$\begin{aligned} \mathcal{K}(\lambda) &= 3 \sin \left[\frac{\pi}{6} - 2\lambda \right], \\ \Delta(\lambda) &= \frac{3}{2} \left(\cos \left[\frac{5}{6} + 2\lambda \right] - \sqrt{3} \sin \left[\frac{5}{6} + 2\lambda \right] \right), \end{aligned} \quad (4.8)$$

therefore the eigenvalues are $\lambda_k = ((6k + 1)\pi - 5)/12, k \in \mathbb{Z}$.

Table 3 gives comparison between the absolute error of sinc-Gaussian and the classical sinc-method. Moreover, in Table 4 the approximation $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ for different values of ε .

Table 4: The approximation $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ for different values of ε .

λ_k	Exact λ_k	$[a_-, a_+], \varepsilon = 10^{-2}$	$[a_-, a_+], \varepsilon = 10^{-5}$	$\lambda_{k,N}$
λ_{-2}	-3.29645993245731	$[-3.322822, -3.270097]$	$[-3.296486, -3.296433]$	-3.29645993245735
λ_{-1}	-1.7256636056624	$[-1.752025, -1.699301]$	$[-1.725689, -1.725637]$	-1.7256636056621
λ_0	-0.154867278867517	$[-0.181229, -0.128505]$	$[-0.154893, -0.1548409]$	-0.154867278867127
λ_1	1.41592904792737	$[1.389566, 1.442291]$	$[1.415902, 1.415955]$	1.41592904792752
λ_2	2.9867253747222	$[2.960363, 3.013087]$	$[2.986699, 3.013087]$	2.98672537472225

$$\|\mathcal{U}\|_\infty = 531.544, \beta_N(0) = 1.22382.$$

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