Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 925092, 18 pages
doi:10.1155/2012/925092

*Research Article*

# Zero Triple Product Determined Matrix Algebras

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Received 9 August 2011; Accepted 20 December 2011

Academic Editor: Xianhua Tang



Let $A$ be an algebra over a commutative unital ring $\mathcal{C}$. We say that $A$ is zero triple product determined if for every $\mathcal{C}$-module $X$ and every trilinear map $\{\cdot,\cdot,\cdot\}$, the following holds: if $\{x,y,z\} = 0$ whenever $xyz = 0$, then there exists a $\mathcal{C}$-linear operator $T : A^3 \longrightarrow X$ such that $\{x,y,z\} = T(xyz)$ for all $x,y,z \in A$. If the ordinary triple product in the aforementioned definition is replaced by Jordan triple product, then $A$ is called zero Jordan triple product determined. This paper mainly shows that matrix algebra $M_n(B)$, $n \geq 3$, where $B$ is any commutative unital algebra even different from the above mentioned commutative unital algebra $\mathcal{C}$, is always zero triple product determined, and $M_n(F)$, $n \geq 3$, where $F$ is any field with $\mathrm{ch}F \neq 2$, is also zero Jordan triple product determined.

## 1. Introduction

Over the last couple of years, several papers characterizing bilinear maps on algebras through their action on elements whose certain product is zero have been written; see [1–6]. The philosophy in these papers is that certain classical problems concerning linear maps that preserve zero product, Jordan product, commutativity, and so forth can be sometimes effectively solved by considering bilinear maps that preserve certain zero product properties. For example, in [1], in order to determine whether a linear map preserving zero product (resp., zero Jordan product, zero Lie product) is "closed" to a homomorphism, the authors introduced the definitions of zero product (resp., zero Jordan product, zero Lie product) determined algebras. The core idea of these definitions is to answer the aforementioned questions by determining the bilinear maps preserving zero product (resp., zero Jordan product, zero Lie product). Furthermore, as the main task, they gave the positive answer that the matrix algebra $M_n(B)$ of $n \times n$ matrices over a unital algebra $B$ is zero product determined, and under some further restrictions on $B$, $M_n(B)$ is still zero Jordan (resp., Lie) product determined.

Meanwhile, in preserver problems, there have appeared many kinds of preserving product forms, such as the forms in papers [7–10]. Particularly, in [10], there appears a definition of Jordan triple product. This definition has applications not only in Banach algebra (see [11]) but also in generalized inverse of matrices (see [12]).

Inspired by the aforementioned, this gives rise to a question: whether can we generalize the products in paper [1] to more generalized forms, such as triple product and Jordan triple product? And immediately, it follows another interesting preserver problem: whether a linear map preserving zero triple product (resp., zero Jordan triple product) is still "closed" to a homomorphism? In order to answer the aforementioned questions, we introduce the following definitions. Let $\mathcal{C}$ be a (fixed) commutative unital ring and $A$ an algebra over $\mathcal{C}$. By $A^3$, we denote the $\mathcal{C}$-linear span of all elements of the form $xyz$, where $x, y, z \in A$. X is a $\mathcal{C}$-module and we denote by $\{\cdot,\cdot,\cdot\} : A \times A \times A \to X$ a $\mathcal{C}$-trilinear map. Consider the following conditions:

(a) for all $x, y, z \in A$ such that $xyz = 0$, we have $\{x, y, z\} = 0$;

(b) there exists a $\mathcal{C}$-linear map $T : A^3 \to X$ such that $\{x, y, z\} = T(xyz)$ for all $x, y, z \in A$.

Trivially, (b) implies (a). We call that $A$ is a zero triple product determined algebra if for every $\mathcal{C}$-module $X$ and every $\mathcal{C}$-trilinear map $\{\cdot,\cdot,\cdot\}$, (a) implies (b). And we say that $A$ is a zero Jordan triple product determined algebra if we replace the above triple product by Jordan triple product ($x \circ y \circ z = xyz + zyx$).

From the previous definitions, it is interesting to examine whether the matrix algebra $M_n(B)$ of $n \times n$ matrices over a unital algebra $B$ is still zero triple product (resp., zero Jordan triple product) determined. If the unital algebra $B$ has no further restrictions, this problem will be difficult. Therefore, the purpose of this paper is to characterize the zero triple product (resp., zero Jordan triple product) determined algebra under some additional restrictions on the unital algebra $B$. In Section 2, we show that the answer is "yes" for the triple product if $B$ is any commutative unital algebra even different from $\mathcal{C}$. The Jordan triple product case, treated in Section 3, is more difficult; we only show that the matrix algebra $M_n(F)$, where $F$ is any field with ch$F \neq 2$ (ch$F$ stands for the characteristic of a field) is also zero Jordan triple product determined.

Finally, we end this section by giving an equivalent condition of (b) in the previous definition which is more convenient to use:

(b′) if $x_t, y_t, z_t \in$ A, $t = 1, \ldots, m$, are such that $\sum_{t=1}^{m} x_t y_t z_t = 0$, then $\sum_{t=1}^{m} \{x_t, y_t, z_t\} = 0$.

## 2. Zero Triple Product Determined Matrix Algebras

In this part, we will consider the matrix algebra $M_n(B)$, where $B$ is any commutative unital algebra even different from the (fixed) commutative unital ring $\mathcal{C}$ mentioned in Section 1. By $be_{ij}$, where $b \in B$, we denote the matrix whose $(i, j)$ entry is $b$ and all other entries are 0.

**Theorem 2.1.** *Let $B$ be a commutative unital algebra, then $M_n(B)$ is a zero triple product determined algebra for every $n \geq 3$.*

*Proof.* In order to prove that (a) implies (b), we only prove that (a) implies the equivalent condition (b′). Set $A = M_n(B)$, let $X$ be a $\mathcal{C}$-module, and let $\{\cdot,\cdot,\cdot\}$ be a $\mathcal{C}$-trilinear map such that for all $x, y, z \in A$, $xyz = 0$ implies $\{x, y, z\} = 0$. Throughout the proof, $a$, $b$, and $c$ denote arbitrary elements in $B$ and $i$, $j$, $k$, $l$, $r$, and $h$ denote arbitrary indices.

First, since $ae_{ij}be_{kl}ce_{rh} = 0$ if $j \neq k$ or $r \neq l$, we get

$$\{ae_{ij}, be_{kl}, ce_{rh}\} = 0 \quad \text{if } j \neq k \text{ or } r \neq l. \tag{2.1}$$

Second, $(ae_{ij} + abe_{ik})(be_{jl} - e_{kl})ce_{lh} = 0$, where $j \neq k$, and consequently

$$\{ae_{ij}, be_{jl}, ce_{lh}\} = \{abe_{ik}, e_{kl}, ce_{lh}\}, \tag{2.2}$$

replacing $a$ by $ab$ and $b$ by 1 in (2.2), we have

$$\{ae_{ij}, be_{jl}, ce_{lh}\} = \{abe_{ij}, e_{jl}, ce_{lh}\}, \tag{2.3}$$

and using (2.3) and (2.2), this yields that

$$\{ae_{ij}, be_{jl}, ce_{lh}\} = \{abe_{i1}, e_{1l}, ce_{lh}\}. \tag{2.4}$$

Similarly, from $ae_{ij}(be_{jl} + bce_{jk})(ce_{lh} - e_{kh}) = 0$, where $l \neq k$, we obtain

$$\{ae_{ij}, be_{jl}, ce_{lh}\} = \{ae_{ij}, bce_{j1}, e_{1h}\}. \tag{2.5}$$

Finally, combining (2.5) and (2.4), it follows that

$$\{ae_{ij}, be_{jl}, ce_{lh}\} = \{ae_{ij}, bce_{j1}, e_{1h}\} = \{abce_{i1}, e_{11}, e_{1h}\}. \tag{2.6}$$

Let $x_t,\ y_t,\ z_t \in A$ be such that $\sum_{t=1}^{m} x_t y_t z_t = 0$; then we only need to show

$$\sum_{t=1}^{m} \{x_t, y_t, z_t\} = 0. \tag{2.7}$$

Writing

$$x_t = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{t} e_{ij}, \qquad y_t = \sum_{k=1}^{n} \sum_{l=1}^{n} b_{kl}^{t} e_{kl}, \qquad z_t = \sum_{r=1}^{n} \sum_{h=1}^{n} c_{rh}^{t} e_{rh}, \tag{2.8}$$

it follows, by examining the $(i, h)$ entry of $\sum_{t=1}^{m} x_t y_t z_t$, that for all $i$ and $h$, we have

$$\sum_{t=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{n} a_{ij}^{t} b_{jl}^{t} c_{lh}^{t} = 0. \tag{2.9}$$

Note that

$$\sum_{t=1}^{m} \{x_t, y_t, z_t\} = \sum_{t=1}^{m} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{t} e_{ij}, \sum_{k=1}^{n} \sum_{l=1}^{n} b_{kl}^{t} e_{kl}, \sum_{r=1}^{n} \sum_{h=1}^{n} c_{rh}^{t} e_{rh} \right\}, \tag{2.10}$$

by (2.1), this summation reduces to

$$\sum_{t=1}^{m}\{x_t, y_t, z_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}e_{ij}, b_{jl}^{t}e_{jl}, c_{lh}^{t}e_{lh}\right\}, \tag{2.11}$$

and using (2.6) and (2.9), we obtain

$$\begin{aligned}\sum_{t=1}^{m}\{x_t, y_t, z_t\} &= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}b_{jl}^{t}c_{lh}^{t}e_{i1}, e_{11}, e_{1h}\right\} \\ &= \sum_{i=1}^{n}\sum_{h=1}^{n}\left\{\sum_{t=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{n}a_{ij}^{t}b_{jl}^{t}c_{lh}^{t}e_{i1}, e_{11}, e_{1h}\right\} \\ &= 0.\end{aligned} \tag{2.12}$$

Therefore, the result of this theorem holds. □

## 3. Zero Jordan Triple Product Determined Matrix Algebras

In this part, we only consider the matrix algebra $M_n(F)$, where $F$ is a field with $\mathrm{ch}F \neq 2$, and $\mathcal{C}$ is still the (fixed) commutative unital ring mentioned in Section 1. Let $be_{ij}$, where $b \in F$, be the matrix whose $(i, j)$ entry is $b$ and all other entries are 0.

**Theorem 3.1.** *Let $F$ be a field with $\mathrm{ch}F \neq 2$; then $M_n(F)$ is a zero Jordan triple product determined algebra for every $n \geq 3$.*

*Proof.* Set $A = M_n(F)$, let $X$ be a $\mathcal{C}$-module, and let $\{\cdot, \cdot, \cdot\}$ be a $\mathcal{C}$-trilinear map such that for all $x, y, z \in A$, $x \circ y \circ z = 0$ implies $\{x, y, z\} = 0$. Let $a$, $b$, and $c$ be arbitrary elements from $F$ and $i, j, k, l, r,$ and $h$ denote arbitrary indices.

First, for $j \neq k$, $h \neq k$ or $j \neq k$, $l \neq i$ or $l \neq r$, $h \neq k$ or $l \neq r$, and $l \neq i$, we have $ae_{ij} \circ be_{kl} \circ ce_{rh} = 0$, and so

$$\{ae_{ij}, be_{kl}, ce_{rh}\} = 0 \quad \text{if } j \neq k,\ h \neq k \text{ or } j \neq k,\ l \neq i \text{ or } l \neq r,\ h \neq k \text{ or } l \neq r,\ l \neq i. \tag{3.1}$$

For any $z \in$ A and $i \neq k$, $j \neq k$, we have $(ae_{ij} + abe_{ik}) \circ (be_{jk} - e_{kk}) \circ z = 0$ and $z \circ (ae_{ij} + abe_{ik}) \circ (be_{jk} - e_{kk}) = 0$, which implies

$$\{ae_{ij}, be_{jk}, z\} = \{abe_{ik}, e_{kk}, z\} \quad \text{if } i \neq k,\ j \neq k, \tag{3.2}$$

$$\{z, ae_{ij}, be_{jk}\} = \{z, abe_{ik}, e_{kk}\} \quad \text{if } i \neq k,\ j \neq k. \tag{3.3}$$

Similarly,

$$\{be_{jk}, ae_{ij}, z\} = \{e_{kk}, abe_{ik}, z\} \quad \text{if } i \neq k,\ j \neq k, \tag{3.4}$$

$$\{z, be_{jk}, ae_{ij}\} = \{z, e_{kk}, abe_{ik}\} \quad \text{if } i \neq k,\ j \neq k. \tag{3.5}$$

For any $z \in A$ and $i \neq k$, we have $(ae_{ik}-e_{ii})\circ(abe_{ik}+be_{kk})\circ z = 0$, $(ae_{ik}-e_{kk})\circ(abe_{ik}+be_{ii})\circ z = 0$, $z \circ (ae_{ik} - e_{ii}) \circ (abe_{ik} + be_{kk}) = 0$, and $z \circ (ae_{ik} - e_{kk}) \circ (abe_{ik} + be_{ii}) = 0$, and consequently,

$$\{ae_{ik}, be_{kk}, z\} = \{e_{ii}, abe_{ik}, z\} \quad \text{if } i \neq k, \tag{3.6}$$

$$\{ae_{ik}, be_{ii}, z\} = \{e_{kk}, abe_{ik}, z\} \quad \text{if } i \neq k, \tag{3.7}$$

$$\{z, ae_{ik}, be_{kk}\} = \{z, e_{ii}, abe_{ik}\} \quad \text{if } i \neq k, \tag{3.8}$$

$$\{z, ae_{ik}, be_{ii}\} = \{z, e_{kk}, abe_{ik}\} \quad \text{if } i \neq k. \tag{3.9}$$

Similarly,

$$\{be_{kk}, ae_{ik}, z\} = \{abe_{ik}, e_{ii}, z\} \quad \text{if } i \neq k, \tag{3.10}$$

$$\{be_{ii}, ae_{ik}, z\} = \{abe_{ik}, e_{kk}, z\} \quad \text{if } i \neq k, \tag{3.11}$$

$$\{z, be_{kk}, ae_{ik}\} = \{z, abe_{ik}, e_{ii}\} \quad \text{if } i \neq k, \tag{3.12}$$

$$\{z, be_{ii}, ae_{ik}\} = \{z, abe_{ik}, e_{kk}\} \quad \text{if } i \neq k. \tag{3.13}$$

Since $ae_{ih} \circ (e_{ii} - e_{hh}) \circ (e_{ii} + e_{hh}) = 0$ and $(e_{ii} - e_{hh}) \circ (e_{ii} + e_{hh}) \circ ae_{ih} = 0$, if $h \neq i$, it follows that

$$\{ae_{ih}, e_{ii}, e_{ii}\} = \{ae_{ih}, e_{hh}, e_{hh}\} \quad \text{if } i \neq h, \tag{3.14}$$

$$\{e_{ii}, e_{ii}, ae_{ih}\} = \{e_{hh}, e_{hh}, ae_{ih}\} \quad \text{if } i \neq h. \tag{3.15}$$

Using (3.8), (3.11), (3.15), and (3.14), we arrive at

$$\{e_{ii}, e_{ii}, ae_{ih}\} = \{ae_{ih}, e_{hh}, e_{hh}\} = \{e_{hh}, e_{hh}, ae_{ih}\} = \{ae_{ih}, e_{ii}, e_{ii}\} \quad \text{if } i \neq h. \tag{3.16}$$

Using (3.7) and (3.12), we have

$$\{ae_{ij}, be_{ji}, e_{jj}\} = \{abe_{ij}, e_{ji}, e_{jj}\} \quad \text{if } i \neq j. \tag{3.17}$$

Since $(ae_{ij} + be_{ji}) \circ (ae_{ij} - be_{ji}) \circ e_{ii} = 0$ if $i \neq j$, then

$$\{ae_{ij}, be_{ji}, e_{ii}\} = \{be_{ji}, ae_{ij}, e_{ii}\} \quad \text{if } i \neq j. \tag{3.18}$$

Using (3.18) and (3.17), it follows that

$$\{ae_{ij}, be_{ji}, e_{ii}\} = \{abe_{ji}, e_{ij}, e_{ii}\} \quad \text{if } i \neq j. \tag{3.19}$$

Using (3.6), (3.8), (3.18), and (3.17), we obtain

$$\{e_{ii}, ae_{ij}, be_{ji}\} = \{bae_{ji}, e_{ij}, e_{ii}\} = \{e_{ij}, bae_{ji}, e_{ii}\} \quad \text{if } i \neq j. \tag{3.20}$$

Similarly, using (3.10), (3.9), and (3.17), this yields

$$\{e_{ii}, ae_{ji}, be_{ij}\} = \{abe_{ji}, e_{ij}e_{ii}\} \quad \text{if } i \neq j. \tag{3.21}$$

Further, we claim that

$$\{ae_{ii}, be_{ii}, ce_{ii}\} = \left\{\frac{1}{2}a \circ b \circ ce_{ii}, e_{ii}, e_{ii}\right\}. \tag{3.22}$$

If $i \neq j$, then $((1/2)abe_{ii} + ae_{ij} - (1/2)abe_{jj}) \circ (be_{ji} - e_{ii} + e_{jj}) \circ (ce_{ii} + ce_{jj}) = 0$; consequently,

$$\left\{\frac{1}{2}abe_{ii} + ae_{ij} - \frac{1}{2}abe_{jj},\ be_{ji} - e_{ii} + e_{jj},\ ce_{ii} + ce_{jj}\right\} = 0, \quad \text{if } i \neq j. \tag{3.23}$$

Using (3.8)–(3.13) and (3.16), we get

$$\{ae_{ij}, be_{ji}, ce_{ii} + ce_{jj}\} = \left\{\frac{1}{2}abe_{ii}, e_{ii}, ce_{ii}\right\} + \left\{\frac{1}{2}abe_{jj}, e_{jj}, ce_{jj}\right\}, \quad \text{if } i \neq j. \tag{3.24}$$

From $(ae_{ii} + be_{ji} - be_{ij} - ae_{jj}) \circ (be_{ii} - ae_{ij} + ae_{ji} - be_{jj}) \circ (ce_{ii} + ce_{jj}) = 0$, where $i \neq j$, it follows that

$$\{ae_{ii} + be_{ji} - be_{ij} - ae_{jj}, be_{ii} - ae_{ij} + ae_{ji} - be_{jj}, ce_{ii} + ce_{jj}\} = 0. \tag{3.25}$$

Using (3.3), (3.4), (3.6)–(3.13), and (3.16), we arrive at

$$\{ae_{ii}, be_{ii}, ce_{ii}\} + \{ae_{jj}, be_{jj}, ce_{jj}\} = \{be_{ij}, ae_{ji}, ce_{ii} + ce_{jj}\} + \{be_{ji}, ae_{ij}, ce_{ii} + ce_{jj}\}. \tag{3.26}$$

Then, by (3.24), we have

$$\{abe_{ii}, e_{ii}, ce_{ii}\} + \{abe_{jj}, e_{jj}, ce_{jj}\} = \{ae_{ii}, be_{ii}, ce_{ii}\} + \{ae_{jj}, be_{jj}, ce_{jj}\}, \quad \text{where } i \neq j. \tag{3.27}$$

Since $n \geq 3$, we choose $k$ such that $k \neq i, j$; applying (3.27), we get

$$\begin{aligned}
&\{ae_{ii}, be_{ii}, ce_{ii}\} + \{ae_{jj}, be_{jj}, ce_{jj}\} + \{ae_{kk}, be_{kk}, ce_{kk}\} + \{ae_{ii}, be_{ii}, ce_{ii}\} \\
&\quad = \{abe_{ii}, e_{ii}, ce_{ii}\} + \{abe_{jj}, e_{jj}, ce_{jj}\} + \{abe_{kk}, e_{kk}, ce_{kk}\} + \{abe_{ii}, e_{ii}, ce_{ii}\} \\
&\quad = 2\{abe_{ii}, e_{ii}, ce_{ii}\} + \{ae_{kk}, be_{kk}, ce_{kk}\} + \{ae_{jj}, be_{jj}, ce_{jj}\},
\end{aligned} \tag{3.28}$$

then

$$\{ae_{ii}, be_{ii}, ce_{ii}\} = \{abe_{ii}, e_{ii}ce_{ii}\}. \tag{3.29}$$

Similarly, by $(ce_{ii}+ce_{jj})\circ(be_{ji}-e_{ii}+e_{jj})\circ((1/2)abe_{ii}+ae_{ij}-(1/2)abe_{jj})=0$ and $(ce_{ii}+ce_{jj})\circ(be_{ii}-ae_{ij}+ae_{ji}-be_{jj})\circ(ae_{ii}+be_{ji}-be_{ij}-ae_{jj})=0$, where $i\neq j$, we get

$$\{ce_{ii},be_{ii},ae_{ii}\}=\{ce_{ii},bae_{ii},e_{ii}\}. \tag{3.30}$$

Then, combining (3.29) and (3.30), we have $\{ce_{ii},ae_{ii},be_{ii}\}=\{abce_{ii},e_{ii},e_{ii}\}$. Hence, the claim (3.22) holds.

Finally, we claim that

$$\{ae_{ij},be_{ji},ce_{ij}\}=\{e_{ii},e_{ii},a\circ b\circ ce_{ij}\},\quad \text{where } i\neq j. \tag{3.31}$$

For $i\neq j$, we have $(-(1/2)\ bce_{ij}+b^{-1}ce_{ji}+ce_{jj})\circ(-(1/2)\ abe_{ii}+(1/2)\ abe_{jj}+ae_{ji})\circ(e_{ii}-e_{jj}+be_{ij})=0$, and consequently $\{-(1/2)\ bce_{ij}+b^{-1}ce_{ji}+ce_{jj},-(1/2)\ abe_{ii}+(1/2)\ abe_{jj}+ae_{ji},e_{ii}-e_{jj}+be_{ij}\}=0$. Using (3.8)–(3.13), (3.16), (3.17), and (3.19), this can be reduced to

$$\left\{ab^2ce_{ij},e_{ii},e_{ii}\right\}+\{abce_{ij},e_{ji},e_{jj}\}=\left\{ce_{jj},\frac{1}{2}abe_{jj},e_{jj}\right\}+\left\{\frac{1}{2}bce_{ij},ae_{ji},be_{ij}\right\}. \tag{3.32}$$

Replacing $b$ by 1 and 2 in (3.32), respectively, we have

$$\{ace_{ij},e_{ii},e_{ii}\}+\{ace_{ij},e_{ji},e_{jj}\}=\left\{ce_{jj},\frac{1}{2}ae_{jj},e_{jj}\right\}+\left\{\frac{1}{2}ce_{ij},ae_{ji},e_{ij}\right\}, \tag{3.33}$$

$$\{4ace_{ij},e_{ii},e_{ii}\}+\{2ace_{ij},e_{ji},e_{jj}\}=\{ce_{jj},ae_{jj},e_{jj}\}+\{ce_{ij},ae_{ji},2e_{ij}\}. \tag{3.34}$$

Computing (3.34)–2 (3.33), this yields

$$\{2ace_{ij},e_{ii},e_{ii}\}=\{ce_{ij},ae_{ji},e_{ij}\}, \tag{3.35}$$

then taking (3.35) into (3.33), we derive

$$\{ace_{ij},e_{ji},e_{jj}\}=\left\{ce_{jj},\frac{1}{2}ae_{jj},e_{jj}\right\}, \tag{3.36}$$

replacing $a$ by $ab$ in (3.36), we get

$$\{abce_{ij},e_{ji},e_{jj}\}=\left\{ce_{jj},\frac{1}{2}abe_{jj},e_{jj}\right\}, \tag{3.37}$$

then taking (3.37) into (3.32), it follows that

$$\left\{ab^2ce_{ij},e_{ii},e_{ii}\right\}=\left\{\frac{1}{2}bce_{ij},ae_{ji},be_{ij}\right\}, \tag{3.38}$$

and replacing $b$ by $b+1$ in (3.38), and using (3.38), (3.35), and (3.16), we get

$$\left\{ce_{ij}, ae_{ji}, be_{ij}\right\} = \left\{e_{ii}, e_{ii}, 2abce_{ij}\right\} = \left\{e_{ii}, e_{ii}, a \circ b \circ ce_{ij}\right\}, \quad \text{where } i \neq j. \tag{3.39}$$

Therefore, (3.31) holds.

Let $x_t, y_t,$ and $z_t$ be such that $\sum_{t=1}^{m} x_t \circ y_t \circ z_t = 0$; we have to prove that $\sum_{t=1}^{m}\{x_t, y_t, z_t\} = 0$. Writing

$$x_t = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^t e_{ij}, \quad y_t = \sum_{k=1}^{n}\sum_{l=1}^{n} b_{kl}^t e_{kl}, \quad z_t = \sum_{r=1}^{n}\sum_{h=1}^{n} c_{rh}^t e_{rh}, \tag{3.40}$$

it follows, by examining the $(i,h)$ entry of $\sum_{t=1}^{m} x_t \circ y_t \circ z_t = 0$, that for all $i, h$, we have

$$\sum_{t=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(a_{ij}^t b_{jl}^t c_{lh}^t + c_{ij}^t b_{jl}^t a_{lh}^t\right) = 0. \tag{3.41}$$

First, by (3.1), we get

$$\sum_{t=1}^{m}\left\{x_t, y_t, z_t\right\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k\neq j}^{n}\sum_{r=1}^{n}\left\{a_{ij}^t e_{ij}, b_{ki}^t e_{ki}, c_{rk}^t e_{rk}\right\} \tag{3.42}$$

$$+ \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{h=1}^{n}\left\{a_{ij}^t e_{ij}, b_{jl}^t e_{jl}, c_{lh}^t e_{lh}\right\} \tag{3.43}$$

$$+ \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r\neq i}^{n}\left\{a_{ij}^t e_{ij}, b_{ji}^t e_{ji}, c_{rj}^t e_{rj}\right\}. \tag{3.44}$$

The summation (3.42) can be written as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\sum_{r=1}^{n}\left\{a_{ij}^t e_{ij}, b_{ki}^t e_{ki}, c_{rk}^t e_{rk}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r=1}^{n}\left\{a_{ii}^t e_{ii}, b_{ki}^t e_{ki}, c_{rk}^t e_{rk}\right\}, \tag{3.45}$$

using (3.4), (3.5), and (3.21), the first summation can be rewritten as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\sum_{r=1}^{n}\left\{a_{ij}^t e_{ij}, b_{ki}^t e_{ki}, c_{rk}^t e_{rk}\right\}$$

$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\sum_{r=1}^{n}\left\{e_{jj}, a_{ij}^t b_{ki}^t e_{kj}, c_{rk}^t e_{rk}\right\}$$

$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\sum_{r\neq j}^{n}\left\{e_{jj}, a_{ij}^{t}b_{ki}^{t}e_{kj}, c_{rk}^{t}e_{rk}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\left\{e_{jj}, a_{ij}^{t}b_{ki}^{t}e_{kj}, c_{jk}^{t}e_{jk}\right\}$$
$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\sum_{r\neq j}^{n}\left\{e_{jj}, e_{jj}, a_{ij}^{t}b_{ki}^{t}c_{rk}^{t}e_{rj}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\left\{a_{ij}^{t}b_{ki}^{t}c_{jk}^{t}e_{kj}, e_{jk}, e_{jj}\right\}, \tag{3.46}$$

and using (3.8), (3.12), (3.9), (3.10), (3.4), (3.17), and (3.18), we split the second summation in three parts:

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r=1}^{n}\{a_{ii}^{t}e_{ii}, b_{ki}^{t}e_{ki}, c_{rk}^{t}e_{rk}\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r=1}^{n}\{a_{ii}^{t}b_{ki}^{t}e_{ki}, e_{kk}, c_{rk}^{t}e_{rk}\}$$
$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r\neq k}^{n}\{a_{ii}^{t}b_{ki}^{t}e_{ki}, e_{kk}, c_{rk}^{t}e_{rk}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{a_{ii}^{t}b_{ki}^{t}e_{ki}, e_{kk}, c_{kk}^{t}e_{kk}\}$$
$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r\neq k}^{n}\{a_{ii}^{t}b_{ki}^{t}e_{ki}, c_{rk}^{t}e_{rk}, e_{rr}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{e_{ii}, a_{ii}^{t}b_{ki}^{t}e_{ki}, c_{kk}^{t}e_{kk}\}$$
$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r\neq k,i}^{n}\{a_{ii}^{t}b_{ki}^{t}e_{ki}, c_{rk}^{t}e_{rk}, e_{\mathrm{rr}}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{a_{ii}^{t}b_{ki}^{t}e_{ki}, c_{ik}^{t}e_{ik}, e_{ii}\}$$
$$+ \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{e_{ii}, e_{ii}, a_{ii}^{t}b_{ki}^{t}c_{kk}^{t}e_{ki}\} \tag{3.47}$$
$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r\neq k,i}^{n}\{e_{ii}, a_{ii}^{t}b_{ki}^{t}c_{rk}^{t}e_{ri}, e_{rr}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{a_{ii}^{t}b_{ki}^{t}c_{ik}^{t}e_{ki}, e_{ik}, e_{ii}\}$$
$$+ \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{e_{ii}, e_{ii}, a_{ii}^{t}b_{ki}^{t}c_{kk}^{t}e_{ki}\}$$
$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r\neq k,i}^{n}\{e_{ii}, e_{ii}, a_{ii}^{t}b_{ki}^{t}c_{rk}^{t}e_{ri}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{e_{ik}, a_{ii}^{t}b_{ki}^{t}c_{ik}^{t}e_{ki}, e_{ii}\}$$
$$+ \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{e_{ii}, e_{ii}, a_{ii}^{t}b_{ki}^{t}c_{kk}^{t}e_{ki}\}.$$

Therefore, the summation (3.42) can be rewritten as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\sum_{r\neq j}^{n}\left\{e_{jj}, e_{jj}, a_{ij}^{t}b_{ki}^{t}c_{rk}^{t}e_{rj}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\left\{a_{ij}^{t}b_{ki}^{t}c_{jk}^{t}e_{kj}, e_{jk}, e_{jj}\right\}$$
$$+ \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\sum_{r\neq k,i}^{n}\{e_{ii}, e_{ii}, a_{ii}^{t}b_{ki}^{t}c_{rk}^{t}e_{ri}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{e_{ik}, a_{ii}^{t}b_{ki}^{t}c_{ik}^{t}e_{ki}, e_{ii}\} \tag{3.48}$$
$$+ \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\{e_{ii}, e_{ii}, a_{ii}^{t}b_{ki}^{t}c_{kk}^{t}e_{ki}\}.$$

The summation (3.43) can be written as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}e_{ij},b_{jl}^{t}e_{jl},c_{lh}^{t}e_{lh}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}e_{ij},b_{jj}^{t}e_{jj},c_{jh}^{t}e_{jh}\right\}$$
$$+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}e_{ij},b_{ji}^{t}e_{ji},c_{ih}^{t}e_{ih}\right\}, \tag{3.49}$$

using (3.2), (3.10), (3.8), (3.16), (3.17), (3.18), and (3.6), the first summation can be rewritten as

$$\begin{aligned}
&\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}e_{ij},b_{jl}^{t}e_{jl},c_{lh}^{t}e_{lh}\right\}=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}b_{jl}^{t}e_{il},e_{ll},c_{lh}^{t}e_{lh}\right\}\\
&=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h\neq l}^{n}\left\{a_{ij}^{t}b_{jl}^{t}e_{il},e_{ll},c_{lh}^{t}e_{lh}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\left\{a_{ij}^{t}b_{jl}^{t}e_{il},e_{ll},c_{ll}^{t}e_{ll}\right\}\\
&=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h\neq l}^{n}\left\{a_{ij}^{t}b_{jl}^{t}e_{il},c_{lh}^{t}e_{lh},e_{hh}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},a_{ij}^{t}b_{jl}^{t}e_{il},c_{ll}^{t}e_{ll}\right\}\\
&=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h\neq l,i}^{n}\left\{a_{ij}^{t}b_{jl}^{t}e_{il},c_{lh}^{t}e_{lh},e_{hh}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\left\{a_{ij}^{t}b_{jl}^{t}e_{il},c_{li}^{t}e_{li},e_{ii}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},e_{ii},a_{ij}^{t}b_{jl}^{t}c_{ll}^{t}e_{il}\right\}\\
&=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h\neq l,i}^{n}\left\{a_{ij}^{t}b_{jl}^{t}c_{lh}^{t}e_{ih},e_{hh},e_{hh}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\left\{a_{ij}^{t}b_{jl}^{t}c_{li}^{t}e_{li},e_{il},e_{ii}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},e_{ii},a_{ij}^{t}b_{jl}^{t}c_{ll}^{t}e_{il}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{l\neq i}^{n}\{e_{ii},e_{ii},a_{ii}^{t}b_{il}^{t}c_{ll}^{t}e_{il}\}\\
&=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h\neq l,i,j}^{n}\left\{e_{hh},e_{hh},a_{ij}^{t}b_{jl}^{t}c_{lh}^{t}e_{ih}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{jj},e_{jj},a_{ij}^{t}b_{jl}^{t}c_{lj}^{t}e_{ij}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{a_{ij}^{t}b_{jl}^{t}c_{li}^{t}e_{li},e_{il},e_{ii}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{l\neq i}^{n}\{a_{ii}^{t}b_{il}^{t}c_{li}^{t}e_{li},e_{il},e_{ii}\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},e_{ii},a_{ij}^{t}b_{jl}^{t}c_{ll}^{t}e_{il}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{l\neq i}^{n}\{e_{ii},e_{ii},a_{ii}^{t}b_{il}^{t}c_{ll}^{t}e_{il}\},
\end{aligned} \tag{3.50}$$

and using (3.6), (3.3), (3.8), (3.16), and (3.20), we rewrite the second summation as

$$\begin{aligned}
&\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}e_{ij},b_{jj}^{t}e_{jj},c_{jh}^{t}e_{jh}\right\}=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h=1}^{n}\left\{e_{ii},a_{ij}^{t}b_{jj}^{t}e_{ij},c_{jh}^{t}e_{jh}\right\}\\
&=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h\neq i}^{n}\left\{e_{ii},a_{ij}^{t}b_{jj}^{t}e_{ij},c_{jh}^{t}e_{jh}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii},a_{ij}^{t}b_{jj}^{t}e_{ij},c_{ji}^{t}e_{ji}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h\neq i,j}^{n}\left\{e_{ii}, a_{ij}^{t}b_{jj}^{t}e_{ij}, c_{jh}^{t}e_{jh}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii}, a_{ij}^{t}b_{jj}^{t}e_{ij}, c_{jj}^{t}e_{jj}\right\} \\
&\quad + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij}, a_{ij}^{t}b_{jj}^{t}c_{ji}^{t}e_{ji}, e_{ii}\right\} \\
&= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h\neq i,j}^{n}\left\{e_{ii}, a_{ij}^{t}b_{jj}^{t}c_{jh}^{t}e_{ih}, e_{hh}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii}, e_{ii}, a_{ij}^{t}b_{jj}^{t}c_{jj}^{t}e_{ij}\right\} \\
&\quad + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij}, a_{ij}^{t}b_{jj}^{t}c_{ji}^{t}e_{ji}, e_{ii}\right\} \\
&= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h\neq i,j}^{n}\left\{e_{hh}, e_{hh}, a_{ij}^{t}b_{jj}^{t}c_{jh}^{t}e_{ih}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii}, e_{ii}, a_{ij}^{t}b_{jj}^{t}c_{jj}^{t}e_{ij}\right\} \\
&\quad + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij}, a_{ij}^{t}b_{jj}^{t}c_{ji}^{t}e_{ji}, e_{ii}\right\},
\end{aligned}
\tag{3.51}
$$

and then using (3.3), (3.2), (3.8), (3.6), (3.20), (3.22), and (3.31), the third summation is split in four parts:

$$
\begin{aligned}
&\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{n}\left\{a_{ij}^{t}e_{ij}, b_{ji}^{t}e_{ji}, c_{ih}^{t}e_{ih}\right\} \\
&\quad = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h\neq i}^{n}\left\{a_{ij}^{t}e_{ij}, b_{ji}^{t}e_{ji}, c_{ih}^{t}e_{ih}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{a_{ij}^{t}e_{ij}, b_{ji}^{t}e_{ji}, c_{ii}^{t}e_{ii}\right\} \\
&\quad = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h\neq i,j}^{n}\left\{a_{ij}^{t}e_{ij}, b_{ji}^{t}e_{ji}, c_{ih}^{t}e_{ih}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{a_{ij}^{t}e_{ij}, b_{ji}^{t}e_{ji}, c_{ij}^{t}e_{ij}\right\} \\
&\qquad + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{a_{ij}^{t}e_{ij}, b_{ji}^{t}e_{ji}, c_{ii}^{t}e_{ii}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\left\{a_{ii}^{t}e_{ii}, b_{ii}^{t}e_{ii}, c_{ii}^{t}e_{ii}\right\} \\
&\quad = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h\neq i,j}^{n}\left\{a_{ij}^{t}e_{ij}, b_{ji}^{t}c_{ih}^{t}e_{jh}, e_{hh}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii}, e_{ii}, a_{ij}^{t}\circ b_{ji}^{t}\circ c_{ij}^{t}e_{ij}\right\} \\
&\qquad + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii}, a_{ij}^{t}e_{ij}, b_{ji}^{t}c_{ii}^{t}e_{ji}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\left\{\frac{1}{2}a_{ii}^{t}\circ b_{ii}^{t}\circ c_{ii}^{t}e_{ii}, e_{ii}, e_{ii}\right\} \\
&\quad = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h\neq i,j}^{n}\left\{a_{ij}^{t}b_{ji}^{t}c_{ih}^{t}e_{ih}, e_{hh}, e_{hh}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii}, e_{ii}, a_{ij}^{t}\circ b_{ji}^{t}\circ c_{ij}^{t}e_{ij}\right\} \\
&\qquad + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij}, a_{ij}^{t}b_{ji}^{t}c_{ii}^{t}e_{ji}, e_{ii}\right\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\left\{\frac{1}{2}a_{ii}^{t}\circ b_{ii}^{t}\circ c_{ii}^{t}e_{ii}, e_{ii}, e_{ii}\right\}.
\end{aligned}
\tag{3.52}
$$

Hence, the summation (3.43) can be rewritten as

$$\begin{aligned}
&\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\sum_{h\neq l,i,j}^{n}\left\{e_{hh},e_{hh},a_{ij}^{t}b_{jl}^{t}c_{lh}^{t}e_{ih}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{jj},e_{jj},a_{ij}^{t}b_{jl}^{t}c_{lj}^{t}e_{ij}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{a_{ij}^{t}b_{jl}^{t}c_{li}^{t}e_{li},e_{il},e_{ii}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{l\neq i}^{n}\left\{a_{ii}^{t}b_{il}^{t}c_{li}^{t}e_{li},e_{il},e_{ii}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},e_{ii},a_{ij}^{t}b_{jl}^{t}c_{ll}^{t}e_{il}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{l\neq i}^{n}\left\{e_{ii},e_{ii},a_{ii}^{t}b_{il}^{t}c_{ll}^{t}e_{il}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{h\neq i,j}^{n}\left\{e_{hh},e_{hh},a_{ij}^{t}b_{jj}^{t}c_{jh}^{t}e_{ih}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii},e_{ii},a_{ij}^{t}b_{jj}^{t}c_{jj}^{t}e_{ij}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij},a_{ij}^{t}b_{jj}^{t}c_{ji}^{t}e_{ji},e_{ii}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h\neq i,j}^{n}\left\{e_{hh},e_{hh},a_{ij}^{t}b_{ji}^{t}c_{ih}^{t}e_{ih}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii},e_{ii},a_{ij}^{t}\circ b_{ji}^{t}\circ c_{ij}^{t}e_{ij}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij},a_{ij}^{t}b_{ji}^{t}c_{ii}^{t}e_{ji},e_{ii}\right\}\\
&\quad+\sum_{t=1}^{m}\sum_{i=1}^{n}\left\{\frac{1}{2}a_{ii}^{t}\circ b_{ii}^{t}\circ c_{ii}^{t}e_{ii},e_{ii},e_{ii}\right\}.
\end{aligned}\tag{3.53}$$

Finally, using (3.5), (3.7), (3.8), (3.10), (3.12), (3.16), and (3.21), the summation (3.44) can be written as

$$\begin{aligned}
&\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{r\neq i}^{n}\left\{a_{ij}^{t}e_{ij},b_{ji}^{t}e_{ji},c_{rj}^{t}e_{rj}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{r\neq i}^{n}\left\{a_{ii}^{t}e_{ii},b_{ii}^{t}e_{ii},c_{ri}^{t}e_{ri}\right\}\\
&\quad=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{r\neq i}^{n}\left\{e_{jj},a_{ij}^{t}e_{ij},b_{ji}^{t}c_{rj}^{t}e_{ri}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{r\neq i}^{n}\left\{a_{ii}^{t}e_{ii},b_{ii}^{t}c_{ri}^{t}e_{ri},e_{rr}\right\}\\
&\quad=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{r\neq i,j}^{n}\left\{e_{jj},a_{ij}^{t}e_{ij},b_{ji}^{t}c_{rj}^{t}e_{ri}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{jj},a_{ij}^{t}e_{ij},b_{ji}^{t}c_{jj}^{t}e_{ji}\right\}\\
&\qquad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{r\neq i}^{n}\left\{a_{ii}^{t}b_{ii}^{t}c_{ri}^{t}e_{ri},e_{rr},e_{rr}\right\}\\
&\quad=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{r\neq i,j}^{n}\left\{e_{jj},e_{jj},a_{ij}^{t}b_{ji}^{t}c_{rj}^{t}e_{rj}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{a_{ij}^{t}b_{ji}^{t}c_{jj}^{t}e_{ij},e_{ji},e_{jj}\right\}\\
&\qquad+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{r\neq i}^{n}\left\{e_{rr},e_{rr},a_{ii}^{t}b_{ii}^{t}c_{ri}^{t}e_{ri}\right\}.
\end{aligned}\tag{3.54}$$

Consequently, rewriting the summation (3.44) as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{r\neq i,j}^{n}\left\{e_{jj},e_{jj},a_{ij}^{t}b_{ji}^{t}c_{rj}^{t}e_{rj}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{a_{ij}^{t}b_{ji}^{t}c_{jj}^{t}e_{ij},e_{ji},e_{jj}\right\}$$
$$+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{r\neq i}^{n}\{e_{rr},e_{rr},a_{ii}^{t}b_{ii}^{t}c_{ri}^{t}e_{ri}\}, \tag{3.55}$$

therefore, we get

$$\sum_{t=1}^{m}\left\{x_t,y_t,z_t\right\}=(3.47)+(3.52)+(3.54). \tag{3.56}$$

Next, we only need to prove that (3.47) + (3.52) + (3.54) = 0.

Replacing the indices $r$, $k$, $i$, and $j$ by $i$, $j$, $l$, and $h$, correspondingly, in the first summation of (3.48), and rewriting the summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq j}^{n}\sum_{r\neq j}^{n}\left\{e_{jj},e_{jj},a_{ij}^{t}b_{ki}^{t}c_{rk}^{t}e_{rj}\right\}=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{h\neq i,j,l}^{n}\left\{e_{hh},e_{hh},c_{ij}^{t}b_{jl}^{t}a_{lh}^{t}e_{ih}\right\}, \tag{3.57}$$

then adding (3.57) and the first, the seventh, the tenth summations of (3.53), together, this yields

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{h\neq i,j,l}^{n}\left\{e_{hh},e_{hh},\left(a_{ij}^{t}b_{jl}^{t}c_{lh}^{t}+c_{ij}^{t}b_{jl}^{t}a_{lh}^{t}\right)e_{ih}\right\}. \tag{3.58}$$

Replacing $i$ and $k$ by $l$ and $i$, respectively, in the fifth summation of (3.48), then by (3.16), this summation is further equal to

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{l\neq i}^{n}\{e_{ii},e_{ii},a_{ll}^{t}b_{il}^{t}c_{ii}^{t}e_{il}\}, \tag{3.59}$$

and replacing $r$, $k$, and $i$ by $i$, $j$, and $l$, correspondingly, in the third summation of (3.48), then using (3.16), the summation is further equal to

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},e_{ii},a_{ll}^{t}b_{jl}^{t}c_{ij}^{t}e_{il}\right\}. \tag{3.60}$$

Then adding (3.59), (3.60), and the fifth and the sixth summations of (3.53), together, this yields

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},e_{ii},\left(a_{ij}^{t}b_{jl}^{t}c_{ll}^{t}+a_{ll}^{t}b_{jl}^{t}c_{ij}^{t}\right)e_{il}\right\}. \tag{3.61}$$

Replacing $k$ by $j$ in the fourth summation of (3.48), then adding the 12th summation of (3.53), we get

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij},\left(a_{ij}^{t}b_{ji}^{t}c_{ii}^{t}+a_{ii}^{t}b_{ji}^{t}c_{ij}^{t}\right)e_{ji},e_{ii}\right\}. \tag{3.62}$$

Replacing $i$ and $r$ by $j$ and $i$, correspondingly, in the third summation of (3.55), then adding the 8th summation of (3.53), we have

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ii},e_{ii},\left(a_{ij}^{t}b_{jj}^{t}c_{jj}^{t}+a_{jj}^{t}b_{jj}^{t}c_{ij}^{t}\right)e_{ij}\right\}. \tag{3.63}$$

Replacing $r, j,$ and $i$ by $i, j,$ and $l$, respectively, in the first summation of (3.55), and rewriting this summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{jj},e_{jj},a_{lj}^{t}b_{jl}^{t}c_{ij}^{t}e_{ij}\right\}, \tag{3.64}$$

then adding (3.64) and the second summation of (3.53), together, and by (3.16), we derive

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{e_{ii},e_{ii},\left(a_{ij}^{t}b_{jl}^{t}c_{lj}^{t}+a_{lj}^{t}b_{jl}^{t}c_{ij}^{t}\right)e_{ij}\right\}. \tag{3.65}$$

Replacing $i$ and $l$ by $j$ and $i$, respectively, in the fourth summation of (3.53), then adding the second summation of (3.55), we arrive at

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{\left(a_{ij}^{t}b_{ji}^{t}c_{jj}^{t}+a_{jj}^{t}b_{ji}^{t}c_{ij}^{t}\right)e_{ij},e_{ji},e_{jj}\right\}. \tag{3.66}$$

Replacing $i$, $j$, and $k$ by $l$, $i$, and $j$, respectively, in the second summation of (3.48), and rewriting this summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{a_{li}^{t}b_{jl}^{t}c_{ij}^{t}e_{ji},e_{ij},e_{ii}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{a_{ji}^{t}b_{jj}^{t}c_{ij}^{t}e_{ji},e_{ij},e_{ii}\right\}, \tag{3.67}$$

then using (3.19) and (3.18), the aforementioned summation can be rewritten as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{a_{li}^{t}b_{jl}^{t}c_{ij}^{t}e_{ji},e_{ij},e_{ii}\right\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij},a_{ji}^{t}b_{jj}^{t}c_{ij}^{t}e_{ji},e_{ii}\right\}. \tag{3.68}$$

Adding the second summation of (3.68) and the ninth summation of (3.53), together, we get

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{e_{ij},\left(a_{ij}^{t}b_{jj}^{t}c_{ji}^{t}+a_{ji}^{t}b_{jj}^{t}c_{ij}^{t}\right)e_{ji},e_{ii}\right\}. \tag{3.69}$$

By (3.36), we know that the first summation of (3.68) can be written as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{\mathrm{n}}\sum_{l\neq i,j}^{n}\left\{\frac{1}{2}a_{li}^{t}b_{jl}^{t}c_{ij}^{t}e_{ii},e_{ii},e_{ii}\right\}, \tag{3.70}$$

similarly, using (3.36), the third summation of (3.53) can be written as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{\frac{1}{2}a_{ij}^{t}b_{jl}^{t}c_{li}^{t}e_{ii},e_{ii},e_{ii}\right\}, \tag{3.71}$$

and then adding (3.70) and (3.71), together, we have

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left\{\frac{1}{2}\left(a_{ij}^{t}b_{jl}^{t}c_{li}^{t}+a_{li}^{t}b_{jl}^{t}c_{ij}^{t}\right)e_{ii},e_{ii},e_{ii}\right\}. \tag{3.72}$$

Therefore, we derive

$$\begin{aligned}(3.47)+(3.52)+(3.54) = {} & (3.52)+(3.60)+(3.62)+(3.64)+\text{the 11th summation of (47)}\\ & +(3.61)+(3.65)+(3.68)+(3.71)+\text{the 13th summation of (3.52)}.\end{aligned} \tag{3.73}$$

Our first goal is to show

$$(3.52)+(3.60)+(3.62)+(3.64)+\text{the 11th summation of (3.52)}=0. \tag{3.74}$$

First, notice that by (3.16), the summation (3.61) can be rewritten as

$$\sum_{t=1}^{m}\sum_{i\neq l}^{n}\sum_{j\neq l}^{n}\sum_{l=1}^{n}\left\{e_{ll},e_{ll},\left(a_{ij}^{t}b_{jl}^{t}c_{ll}^{t}+a_{ll}^{t}b_{jl}^{t}c_{ij}^{t}\right)e_{il}\right\}. \tag{3.75}$$

Then replacing $l$ by $h$, we have

$$\sum_{t=1}^{m}\sum_{i\neq h}^{n}\sum_{j\neq h}^{n}\sum_{h=1}^{n}\left\{e_{hh},e_{hh},\left(a_{ij}^{t}b_{jh}^{t}c_{hh}^{t}+a_{hh}^{t}b_{jh}^{t}c_{ij}^{t}\right)e_{ih}\right\}. \tag{3.76}$$

Second, notice that

$$(3.64) + \text{the 11th summation of } (3.52) = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq j}^{n}\left\{e_{ii}, e_{ii}, \left(a_{ij}^{t}b_{jl}^{t}c_{lj}^{t} + c_{ij}^{t}b_{jl}^{t}a_{lj}^{t}\right)e_{ij}\right\}. \quad (3.77)$$

Then, by (3.16), and replacing $j$ by $h$, this yields

$$\sum_{t=1}^{m}\sum_{i\neq h}^{n}\sum_{l\neq h}^{n}\sum_{h=1}^{n}\{e_{hh}, e_{hh}, (a_{ih}^{t}b_{hl}^{t}c_{lh}^{t} + c_{ih}^{t}b_{hl}^{t}a_{lh}^{t})e_{ih}\}. \quad (3.78)$$

Third, using (3.16), and replacing $j$ by $h$ in the summation (3.63), we obtain

$$\sum_{t=1}^{m}\sum_{i\neq h}^{n}\sum_{h=1}^{n}\{e_{hh}, e_{hh}, (a_{ih}^{t}b_{hh}^{t}c_{hh}^{t} + c_{ih}^{t}b_{hh}^{t}a_{hh}^{t})e_{ih}\}. \quad (3.79)$$

Finally, rewriting (3.58) as

$$\sum_{t=1}^{m}\sum_{i\neq h}^{n}\sum_{j\neq h}^{n}\sum_{l\neq h}^{n}\sum_{h=1}^{n}\left\{e_{hh}, e_{hh}, \left(a_{ij}^{t}b_{jl}^{t}c_{lh}^{t} + c_{ij}^{t}b_{jl}^{t}a_{lh}^{t}\right)e_{ih}\right\}, \quad (3.80)$$

then, adding (3.76), (3.78), (3.79), and (3.80), together, it follows that

$$\begin{aligned}&\sum_{t=1}^{m}\sum_{i\neq h}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{h=1}^{n}\left\{e_{hh}, e_{hh}, \left(a_{ij}^{t}b_{jl}^{t}c_{lh}^{t} + c_{ij}^{t}b_{jl}^{t}a_{lh}^{t}\right)e_{ih}\right\}\\ &\quad= \sum_{i\neq h}^{n}\sum_{h=1}^{n}\left\{e_{hh}, e_{hh}, \sum_{t=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(a_{ij}^{t}b_{jl}^{t}c_{lh}^{t} + c_{ij}^{t}b_{jl}^{t}a_{lh}^{t}\right)e_{ih}\right\},\end{aligned} \quad (3.81)$$

and using (3.41), we get (3.74) holds.

Next, we claim that

$$(3.61) + (3.65) + (3.68) + (3.71) + \text{the 13th summation of } (3.52) = 0. \quad (3.82)$$

First, using (3.19) and (3.36), we rewrite the summations (3.62) and (3.69), respectively, as

$$\begin{aligned}&\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{\frac{1}{2}\left(a_{ij}^{t}b_{ji}^{t}c_{ii}^{t} + a_{ii}^{t}b_{ji}^{t}c_{ij}^{t}\right)e_{ii}, e_{ii}, e_{ii}\right\},\\ &\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{\frac{1}{2}\left(a_{ij}^{t}b_{jj}^{t}c_{ji}^{t} + a_{ji}^{t}b_{jj}^{t}c_{ij}^{t}\right)e_{ii}, e_{ii}, e_{ii}\right\},\end{aligned} \quad (3.83)$$

and then, adding (3.83) and (3.72), this yields

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l=1}^{n}\left\{\frac{1}{2}\left(a_{ij}^{t}b_{jl}^{t}c_{li}^{t}+a_{li}^{t}b_{jl}^{t}c_{ij}^{t}\right)e_{ii},e_{ii},e_{ii}\right\}. \tag{3.84}$$

Similarly, by (3.36), the summation (3.66) can be rewritten as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left\{\frac{1}{2}\left(a_{ij}^{t}b_{ji}^{t}c_{jj}^{t}+a_{jj}^{t}b_{ji}^{t}c_{ij}^{t}\right)e_{jj},e_{jj},e_{jj}\right\}. \tag{3.85}$$

Adding (3.85) and the 13th summation of (3.53), it implies

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{1}{2}\left(a_{ij}^{t}b_{ji}^{t}c_{jj}^{t}+a_{jj}^{t}b_{ji}^{t}c_{ij}^{t}\right)e_{jj},e_{jj},e_{jj}\right\}. \tag{3.86}$$

Replacing $i$ and $j$ by $l$ and $i$, respectively, in the summation (3.86), then adding (3.84), and by (3.41), we get

$$\begin{aligned}&\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\left\{\frac{1}{2}\left(a_{ij}^{t}b_{jl}^{t}c_{li}^{t}+a_{li}^{t}b_{jl}^{t}c_{ij}^{t}\right)e_{ii},e_{ii},e_{ii}\right\}\\&\quad=\sum_{i=1}^{n}\left\{\sum_{t=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{1}{2}\left(a_{ij}^{t}b_{jl}^{t}c_{li}^{t}+a_{li}^{t}b_{jl}^{t}c_{ij}^{t}\right)e_{ii},e_{ii},e_{ii}\right\}=0.\end{aligned} \tag{3.87}$$

Therefore, (3.82) holds.

Consequently, we get (3.47) + (3.52) + (3.54) = 0, and using (3.56), we finish the proof of this theorem. □

## Acknowledgments

The authors show great thanks to the referee for his/her valuable comments, which greatly improved the readability of the paper. The work was mainly supported by the National Natural Science Foundation Grants of China (Grant no. 10871056) and the Fundamental Research Funds for the Central Universities (Grant no. HEUCF20111132).

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