

Research Article

Hybrid Extragradient Iterative Algorithms for Variational Inequalities, Variational Inclusions, and Fixed-Point Problems

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We investigate the problem of finding a common solution of a general system of variational inequalities, a variational inclusion, and a fixed-point problem of a strictly pseudocontractive mapping in a real Hilbert space. Motivated by Nadezhkina and Takahashi's hybrid-extragradient method, we propose and analyze new hybrid-extragradient iterative algorithm for finding a common solution. It is proven that three sequences generated by this algorithm converge strongly to the same common solution under very mild conditions. Based on this result, we also construct an iterative algorithm for finding a common fixed point of three mappings, such that one of these mappings is nonexpansive, and the other two mappings are strictly pseudocontractive mappings.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H , and let P_C be the metric projection from H onto C . Let $S : C \rightarrow C$ be a self-mapping on C . We denote by $\text{Fix}(S)$ the set of fixed points of S and by \mathbf{R} the set of all real numbers. A mapping $A : C \rightarrow H$ is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \quad (1.1)$$

A mapping $A : C \rightarrow H$ is called L -Lipschitz continuous if there exists a constant $L > 0$, such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \quad (1.2)$$

For a given mapping $A : C \rightarrow H$, we consider the following variational inequality (VI) of finding $x^* \in C$, such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (1.3)$$

The solution set of the VI (1.3) is denoted by $VI(C, A)$. The variational inequality was first discussed by Lions [1] and now is well known. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, and equilibrium problems; see, for example, [2–4]. To construct a mathematical model which is as close as possible to a real complex problem, we often have to use more than one constraint. Solving such problems, we have to obtain some solution which is simultaneously the solution of two or more subproblem or the solution of one subproblem on the solution set of another subproblem. Actually, these subproblems can be given by problems of different types. For example, Antipin considered a finite-dimensional variant of the variational inequality, where the solution should satisfy some related constraint in inequality form [5] or some system of constraints in inequality and equality form [6]. Yamada [7] considered an infinite-dimensional variant of the solution of the variational inequality on the fixed-point set of some mapping.

A mapping $A : C \rightarrow H$ is called α -inverse strongly monotone if there exists a constant $\alpha > 0$, such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C; \quad (1.4)$$

see [8, 9]. It is obvious that an α -inverse strongly monotone mapping A is monotone and Lipschitz continuous. A self-mapping $S : C \rightarrow C$ is called k -strictly pseudocontractive if there exists a constant $k \in [0, 1)$, such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C; \quad (1.5)$$

see [10]. In particular, if $k = 0$, then S is called a nonexpansive mapping; see [11].

A set-valued mapping M with domain $D(M)$ and range $R(M)$ in H is called monotone if its graph $G(M) = \{(x, f) \in H \times H : x \in D(M), f \in Mx\}$ is a monotone set in $H \times H$; that is, M is monotone if and only if

$$(x, f), (y, g) \in G(M) \implies \langle x - y, f - g \rangle \geq 0. \quad (1.6)$$

A monotone set-valued mapping M is called maximal if its graph $G(M)$ is not properly contained in the graph of any other monotone mapping in H .

Let Φ be a single-valued mapping of C into H , and let M be a multivalued mapping with $D(M) = C$. Consider the following variational inclusion: find $x^* \in C$, such that

$$0 \in \Phi(x^*) + Mx^*. \quad (1.7)$$

We denote by Ω the solution set of the variational inclusion (1.7). In particular, if $\Phi = M = 0$, then $\Omega = C$.

In 1998, Huang [12] studied problem (1.7) in the case where M is maximal monotone, and Φ is strongly monotone and Lipschitz continuous with $D(M) = C = H$. Subsequently, Zeng et al. [13] further studied problem (1.7) in the case which is more general than Huang's one [12]. Moreover, the authors [13] obtained the same strong convergence conclusion as in Huang's result [12]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions.

In 2003, for finding an element of $\text{Fix}(S) \cap \text{VI}(C, A)$ when $C \subset H$ is nonempty, closed, and convex, $S : C \rightarrow C$ is nonexpansive, and $A : C \rightarrow H$ is α -inverse strongly monotone. Takahashi and Toyoda [14] introduced the following iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0, \quad (1.8)$$

where $x_0 \in C$ chosen arbitrarily, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $\text{Fix}(S) \cap \text{VI}(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z \in \text{Fix}(S) \cap \text{VI}(C, A)$. In 2006, to solve this problem (i.e., to find an element of $\text{Fix}(S) \cap \text{VI}(C, A)$), Nadezhkina and Takahashi [15] introduced an iterative algorithm by a hybrid method. Generally speaking, the suggested algorithm is based on two well-known types of methods, that is, on the extragradient-type method due to Korpelevich [16] for solving variational inequality and so-called hybrid or outer-approximation method due to Haugazeau (see [15]) for solving fixed point problem. It is worth emphasizing that the idea of "hybrid" or "outer-approximation" types of methods was successfully generalized and extended in many papers; see, for example, [17–23]. In addition, the idea of the extragradient iterative algorithm introduced by Korpelevich [16] was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, for example, [24–29].

Theorem NT (see [15, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be a monotone and k -Lipschitz-continuous mapping, and let $S : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(S) \cap \text{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \quad (1.9)$$

where $x_0 \in C$ is chosen arbitrarily, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$, and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $P_{\text{Fix}(S) \cap \text{VI}(C, A)} x_0$.

It is easy to see that the class of α -inverse strongly monotone mappings in the above mentioned problem of Takahashi and Toyoda [14] is the quite important class of mappings in various classes of well-known mappings. It is also easy to see that while α -inverse strongly monotone mappings are tightly connected with the important class of nonexpansive mappings, α -inverse strongly monotone mappings are also tightly connected with the more

general and also quite important class of strictly pseudocontractive mappings. That is, if a mapping $S : C \rightarrow C$ is nonexpansive, then the mapping $I - S$ is $(1/2)$ -inverse strongly monotone; moreover, $\text{Fix}(S) = \text{VI}(C, I - S)$ (see, e.g., [14]). The construction of fixed points of nonexpansive mappings via Mann's algorithm has extensively been investigated in the literature (see, e.g., [30, 31] and references therein). At the same time, if a mapping $S : C \rightarrow C$ is k -strictly pseudocontractive, then the mapping $I - S$ is $(1 - k)/2$ -inverse strongly monotone and $2/(1 - k)$ -Lipschitz continuous.

Let $B_1, B_2 : C \rightarrow H$ be two mappings. Recently, Ceng et al. [32] introduced and considered the following problem of finding $(x^*, y^*) \in C \times C$, such that

$$\begin{aligned} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C, \end{aligned} \quad (1.10)$$

which is called a general system of variational inequalities (GSVI), where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. The set of solutions of problem (1.10) is denoted by $\text{GSVI}(C, B_1, B_2)$. In particular, if $B_1 = B_2 = A$, then problem (1.10) reduces to the new system of variational inequalities (NSVI), introduced and studied by Verma [33]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to the VI (1.3).

In particular, if $B_1 = A$ and $B_2 = 0$, then the GSVI (1.10) is equivalent to the VI (1.3).

Indeed, in this case, the GSVI (1.10) is equivalent to the following problem of finding $(x^*, y^*) \in C \times C$, such that

$$\begin{aligned} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\ \langle y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C. \end{aligned} \quad (1.11)$$

Thus we must have $x^* = y^*$. As a matter of fact, if $x^* \neq y^*$, then by setting $x = x^*$ we have

$$0 > -\|x^* - y^*\|^2 = \langle y^* - x^*, x^* - y^* \rangle \geq 0, \quad (1.12)$$

which hence leads to a contradiction. Therefore, the GSVI (1.10) coincides with the VI (1.3).

Recently, Ceng et al. [32] transformed problem (1.10) into a fixed-point problem in the following way.

Lemma 1.1 (see [32]). *For given $\bar{x}, \bar{y} \in C$, (\bar{x}, \bar{y}) is a solution of problem (1.10) if and only if \bar{x} is a fixed point of the mapping $G : C \rightarrow C$ defined by*

$$G(x) = P_C [P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C, \quad (1.13)$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

In particular, if the mapping $B_i : C \rightarrow H$ is β_i -inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i]$ for $i = 1, 2$.

Utilizing Lemma 1.1, they introduced and studied a relaxed extragradient method for solving the GSVI (1.10). Throughout this paper, the set of fixed points of the mapping G is denoted by Ξ . Based on the relaxed extragradient method and viscosity approximation

method, Yao et al. [34] proposed and analyzed an iterative algorithm for finding a common solution of the GSVI (1.10) and the fixed point problem of a strictly pseudocontractive mapping $S : C \rightarrow C$.

Subsequently, Ceng et al. [35] further presented and analyzed an iterative scheme for finding a common element of the solution set of the VI (1.3), the solution set of the GSVI (1.10), and the fixed point set of a strictly pseudo-contractive mapping $S : C \rightarrow C$.

Theorem CGY (see [35, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be α -inverse strongly monotone, and let $B_i : C \rightarrow H$ be β_i -inverse strongly monotone for $i = 1, 2$. Let $S : C \rightarrow C$ be a k -strictly pseudocontractive mapping such that $\text{Fix}(S) \cap \Xi \cap \text{VI}(C, A) \neq \emptyset$. Let $Q : C \rightarrow C$ be a ρ -contraction with $\rho \in [0, 1/2)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= P_C(x_n - \lambda_n A x_n), \\ y_n &= \alpha_n Q x_n + (1 - \alpha_n) P_C [P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (1.14)$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\lambda_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$, such that

- (i) $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n$, for all $n \geq 0$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ and $\liminf_{n \rightarrow \infty} \delta_n > 0$;
- (iv) $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$;
- (v) $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2\alpha$ and $\lim_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ generated by (1.14) converges strongly to $\bar{x} = P_{\text{Fix}(S) \cap \Xi \cap \text{VI}(C, A)} Q \bar{x}$, and (\bar{x}, \bar{y}) is a solution of the GSVI (1.10), where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

On the other hand, let $A : C \rightarrow H$ be a monotone, and let L -Lipschitz-continuous mapping, $\Phi : C \rightarrow H$ be an α -inverse strongly monotone mapping. Let M be a maximal monotone mapping with $D(M) = C$, and let $S : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(S) \cap \Omega \cap \text{VI}(C, A) \neq \emptyset$. Motivated Nadezhkina and Takahashi's hybrid-extragradient algorithm (1.9), Ceng et al. [36, Theorem 3.1] introduced another modified hybrid-extragradient algorithm

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n A x_n), \\ t_n &= P_C(x_n - \lambda_n A y_n), \\ \hat{t}_n &= J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n &= (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n \hat{t}_n + \hat{\alpha}_n S \hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \quad (1.15)$$

where $J_{M,\mu_n} = (I + \mu_n M)^{-1}$, $x_0 \in C$ chosen arbitrarily, $\{\lambda_n\} \subset (0, 1/L)$, $\{\mu_n\} \subset (0, 2\alpha]$, and $\{\alpha_n\}, \{\hat{\alpha}_n\} \subset (0, 1]$ such that $\alpha_n + \hat{\alpha}_n \leq 1$. It was proven in [36] that under very mild conditions three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (1.15) converge strongly to the same point $P_{\text{Fix}(S) \cap \Omega \cap \text{VI}(C,A)} x_0$.

Inspired by the research going on this area, we propose and analyze the following hybrid extragradient iterative algorithm for finding a common element of the solution set Ξ of the GSVI (1.10), the solution set Ω of the variational inclusion (1.7), and the fixed point set $\text{Fix}(S)$ of a strictly pseudo-contractive mapping $S : C \rightarrow C$.

Algorithm 1.2. Assume that $\text{Fix}(S) \cap \Omega \cap \Xi \neq \emptyset$. Let $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\mu_n\} \subset (0, 2\alpha]$, and $\{\sigma_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$, for all $n \geq 0$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences generated by the hybrid extragradient iterative scheme

$$\begin{aligned} y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\ \hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{1.16}$$

where $J_{M,\mu_n} = (I + \mu_n M)^{-1}$, for all $n \geq 0$.

Under very appropriate assumptions, it is proven that all the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\bar{x} = P_{\text{Fix}(S) \cap \Omega \cap \Xi} x_0$. Furthermore, (\bar{x}, \bar{y}) is a solution of the GSVI (1.10), where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Let $T : C \rightarrow C$ be a k -strictly pseudocontractive mapping, let $\Gamma : C \rightarrow C$ be a κ -strictly pseudocontractive mapping, and let $S : C \rightarrow C$ be a nonexpansive mapping. Putting $B_1 = I - T$, $B_2 = 0$, $\Phi = I - \Gamma$, $M = 0$, and $\sigma_n = 0$, for all $n \geq 0$ in Algorithm 1.2, we consider and analyze the following hybrid extragradient iterative algorithm for finding a common fixed point of three mappings S , Γ , and T .

Algorithm 1.3. Assume that $\text{Fix}(S) \cap \text{Fix}(\Gamma) \cap \text{Fix}(T) \neq \emptyset$. Let $\mu_1 \in (0, 1 - k)$, $\{\mu_n\} \subset (0, 1 - \kappa]$, and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$, for all $n \geq 0$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences generated by the hybrid extragradient iterative scheme

$$\begin{aligned} y_n &= x_n - \mu_1(x_n - T x_n), \\ t_n &= y_n - \mu_1(y_n - T y_n), \\ \hat{t}_n &= t_n - \mu_n(t_n - \Gamma t_n), \end{aligned}$$

$$\begin{aligned}
z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\
C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0.
\end{aligned} \tag{1.17}$$

Under quite mild conditions, it is shown that all the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $P_{\text{Fix}(S) \cap \Omega \cap \text{Fix}(\Gamma) \cap \text{Fix}(T)} x_0$.

Observe that Ceng et al. [36, Theorem 3.1] considered the problem of finding an element of $\text{Fix}(S) \cap \Omega \cap \text{VI}(C, A)$ where $S : C \rightarrow C$ is nonexpansive, Nadezhkina and Takahashi [15, Theorem 3.1] studied the problem of finding an element of $\text{Fix}(S) \cap \text{VI}(C, A)$ where $S : C \rightarrow C$ is nonexpansive, and Ceng et al. [35, Theorem 3.1] investigated the problem of finding an element of $\text{Fix}(S) \cap \Xi \cap \text{VI}(C, A)$ where $S : C \rightarrow C$ is strictly pseudocontractive. It is clear that every one of these three problems is very different from our problem of finding an element of $\text{Fix}(S) \cap \Omega \cap \Xi$ where $S : C \rightarrow C$ is strictly pseudocontractive. Hence there is no doubt that the strong convergence results for solving our problem are very interesting and quite valuable. Because our hybrid extragradient iterative algorithms involve two inverse strongly monotone mappings B_1 and B_2 , a strictly pseudo-contractive self-mapping S , and several parameter sequences, they are more flexible and more subtle than the corresponding ones in [36, Theorem 3.1] and [15, Theorem 3.1], respectively. Furthermore, the relaxed extragradient iterative scheme in Yao et al. [34, Theorem 3.2] is extended to develop our hybrid extragradient iterative algorithms. In our results, the hybrid extragradient iterative algorithms drop the requirements that $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ and $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$ in [34, Theorem 3.2] and [35, Theorem 3.1]. Therefore, our results represent the modification, supplementation, extension, and improvement of [36, Theorem 3.1], [15, Theorem 3.1], [34, Theorem 3.2], and [35, Theorem 3.1] to a great extent.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H . We write \rightarrow to indicate that the sequence $\{x_n\}$ converges strongly to x and \rightharpoonup to indicate that the sequence $\{x_n\}$ converges weakly to x . Moreover, we use $\omega_w(x_n)$ to denote the weak ω -limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}. \tag{2.1}$$

For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.2}$$

P_C is called the metric projection of H onto C . We know that P_C is a firmly nonexpansive mapping of H onto C ; that is, there holds the following relation

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.3)$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \quad (2.4)$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad (2.5)$$

for all $x \in H, y \in C$; see [11, 37] for more details. Let $A : C \rightarrow H$ be a monotone mapping. In the context of the variational inequality, this implies that

$$x \in \text{VI}(C, A) \iff x = P_C(x - \lambda Ax), \quad \forall \lambda > 0. \quad (2.6)$$

It is also known that the norm of every Hilbert space H satisfies the weak lower semicontinuity [4]. That is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\| \quad (2.7)$$

holds.

Recall that a set-valued mapping $M : D(M) \subset H \rightarrow 2^H$ is called maximal monotone if M is monotone and $(I + \lambda M)D(M) = H$ for each $\lambda > 0$, where I is the identity mapping of H . We denote by $G(M)$ the graph of M . It is known that a monotone mapping M is maximal if and only if, for $(x, f) \in H \times H, \langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$. Here the following example illustrates the concept of maximal monotone mappings in the setting of Hilbert spaces.

Let $A : C \rightarrow H$ be a monotone, L -Lipschitz-continuous mapping, and let $N_C v$ be the normal cone to C at $v \in C$, that is,

$$N_C v = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \quad (2.8)$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \text{VI}(C, A)$; see [38].

Assume that $M : D(M) \subset H \rightarrow 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with M , the resolvent operator $J_{M, \lambda}$ can be defined as

$$J_{M, \lambda} x = (I + \lambda M)^{-1} x, \quad \forall x \in H. \quad (2.9)$$

In terms of Huang [12] (see also [13]), there holds the following property for the resolvent operator $J_{M, \lambda} : H \rightarrow H$.

Lemma 2.1. $J_{M,\lambda}$ is single valued and firmly nonexpansive, that is,

$$\langle J_{M,\lambda}x - J_{M,\lambda}y, x - y \rangle \geq \|J_{M,\lambda}x - J_{M,\lambda}y\|^2, \quad \forall x, y \in H. \quad (2.10)$$

Consequently, $J_{M,\lambda}$ is nonexpansive and monotone.

Lemma 2.2 (see [39]). *There holds the relation:*

$$\|\lambda x + \mu y + \nu z\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 - \lambda\mu\|x - y\|^2 - \mu\nu\|y - z\|^2 - \lambda\nu\|x - z\|^2, \quad (2.11)$$

for all $x, y, z \in H$ and $\lambda, \mu, \nu \in [0, 1]$ with $\lambda + \mu + \nu = 1$.

Lemma 2.3 (see [36]). *Let M be a maximal monotone mapping with $D(M) = C$. Then for any given $\lambda > 0$, $x^* \in C$ is a solution of problem (1.7) if and only if $x^* \in C$ satisfies*

$$x^* = J_{M,\lambda}(x^* - \lambda\Phi(x^*)). \quad (2.12)$$

Lemma 2.4 (see [13]). *Let M be a maximal monotone mapping with $D(M) = C$, and let $V : C \rightarrow H$ be a strong monotone, continuous, and single-valued mapping. Then for each $z \in H$, the equation $z \in Vx + \lambda Mx$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.5 (see [36]). *Let M be a maximal monotone mapping with $D(M) = C$, and let $A : C \rightarrow H$ be a monotone, continuous, and single-valued mapping. Then $(I + \lambda(M + A))C = H$ for each $\lambda > 0$. In this case, $M + A$ is maximal monotone.*

It is clear that, in a real Hilbert space H , $S : C \rightarrow C$ is k -strictly pseudo-contractive if and only if there holds the following inequality:

$$\langle Sx - Sy, x - y \rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in C. \quad (2.13)$$

This immediately implies that if S is a k -strictly pseudocontractive mapping, then $I - S$ is $(1 - k)/2$ -inverse strongly monotone; for further detail, we refer to [10] and the references therein. It is well known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings.

Lemma 2.6 (see [10, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $S : C \rightarrow C$ be a mapping.*

(i) *If S is a k -strict pseudo-contractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C. \quad (2.14)$$

(ii) *If S is a k -strict pseudo-contractive mapping, then the mapping $I - S$ is semiclosed at 0; that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightarrow \tilde{x}$ weakly and $(I - S)x_n \rightarrow 0$ strongly, then $(I - S)\tilde{x} = 0$.*

(iii) If S is k -quasistrict pseudo-contraction, then the fixed point set $\text{Fix}(S)$ of S is closed and convex, so that the projection $P_{\text{Fix}(S)}$ is well defined.

Lemma 2.7 (see [34]). Let C be a nonempty closed convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a k -strictly pseudo-contractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)k \leq \gamma$. Then

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C. \quad (2.15)$$

The following lemma is well known to us.

Lemma 2.8 (see [11]). Every Hilbert space H has the Kadec-Klee property; that is, for given $x \in H$ and $\{x_n\} \subset H$, we have

$$\left. \begin{array}{l} x_n \rightharpoonup x \\ \|x_n\| \rightarrow \|x\| \end{array} \right\} \implies x_n \rightarrow x. \quad (2.16)$$

3. Main Results

In this section, we first prove the strong convergence of the sequences generated by our hybrid extragradient iterative algorithm for finding a common solution of a general system of variational inequalities, a variational inclusion, and a fixed problem of a strictly pseudocontractive self-mapping.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $B_i : C \rightarrow H$ be β_i -inverse strongly monotone for $i = 1, 2$, let $\Phi : C \rightarrow H$ be an α -inverse strongly monotone mapping, let M be a maximal monotone mapping with $D(M) = C$, and let $S : C \rightarrow C$ be a k -strictly pseudocontractive mapping such that $\text{Fix}(S) \cap \Omega \cap \Xi \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{aligned} y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\ \hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \quad (3.1)$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\mu_n\} \subset [e, 2\alpha]$ for some $e \in (0, 2\alpha)$, and $\{\sigma_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\{\sigma_n\} \subset [0, c]$ for some $c \in [0, 1)$, $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1)$, $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n$, for all $n \geq 0$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\bar{x} = P_{\text{Fix}(S) \cap \Omega \cap \Xi} x_0$ if and only if $\|S \hat{t}_n - \hat{t}_n\| \rightarrow 0$. Furthermore, (\bar{x}, \bar{y}) is a solution of the GSVI (1.10), where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n = 0, 1, 2, \dots$. As

$$C_n = \left\{ z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z \rangle \leq 0 \right\}, \quad (3.2)$$

we also know that C_n is convex for every $n = 0, 1, 2, \dots$. As

$$Q_n = \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \quad (3.3)$$

we have $\langle x_n - z, x - x_n \rangle \geq 0$, for all $z \in Q_n$, and hence $x_n = P_{Q_n}x_0$ by (2.4).

First of all, assume that the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\bar{x} = P_{\text{Fix}(S) \cap \Omega \cap \Xi}x_0$. Then it is clear that $\|x_n - y_n\| \rightarrow 0$ and $\|x_n - z_n\| \rightarrow 0$. Observe that from the nonexpansiveness of the mappings $P_C(I - \mu_1 B_1)$ and $P_C(I - \mu_2 B_2)$ (due to $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$), we have

$$\begin{aligned} \|y_n - t_n\| &= \|P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\ &\quad - P_C[P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)]]\| \\ &= \|P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)x_n - P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)y_n\| \\ &\leq \|P_C(I - \mu_2 B_2)x_n - P_C(I - \mu_2 B_2)y_n\| \\ &\leq \|x_n - y_n\|. \end{aligned} \quad (3.4)$$

Hence, we conclude that $\|y_n - t_n\| \rightarrow 0$ and $t_n \rightarrow \bar{x}$. Since $\bar{x} \in \text{Fix}(S) \cap \Omega \cap \Xi$, we obtain that $S\bar{x} = \bar{x}$ and $\bar{x} = J_{M, \mu_n}(\bar{x} - \mu_n \Phi(\bar{x}))$. Thus, from the nonexpansiveness of the mapping $J_{M, \mu_n}(I - \mu_n \Phi)$, we have

$$\begin{aligned} \|\hat{t}_n - t_n\| &= \|\sigma_n t_n + (1 - \sigma_n)J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) - t_n\| \\ &= (1 - \sigma_n)\|J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) - t_n\| \\ &\leq \|J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) - \bar{x}\| + \|\bar{x} - t_n\| \\ &= \|J_{M, \mu_n}(I - \mu_n \Phi)t_n - J_{M, \mu_n}(I - \mu_n \Phi)\bar{x}\| + \|t_n - \bar{x}\| \\ &\leq \|t_n - \bar{x}\| + \|t_n - \bar{x}\| = 2\|t_n - \bar{x}\|. \end{aligned} \quad (3.5)$$

So, we deduce that $\|\hat{t}_n - t_n\| \rightarrow 0$ and $\hat{t}_n \rightarrow \bar{x}$. Note that

$$\begin{aligned} \|S\hat{t}_n - \hat{t}_n\| &\leq \|S\hat{t}_n - \bar{x}\| + \|\bar{x} - \hat{t}_n\| \\ &= \|S\hat{t}_n - S\bar{x}\| + \|\bar{x} - \hat{t}_n\| \\ &\leq \left(\frac{1+k}{1-k} + 1 \right) \|\hat{t}_n - \bar{x}\| = \frac{2}{1-k} \|\hat{t}_n - \bar{x}\|. \end{aligned} \quad (3.6)$$

This implies that $\|S\hat{t}_n - \hat{t}_n\| \rightarrow 0$ as $n \rightarrow \infty$.

For the remainder of the proof, we divide it into several steps.

Step 1. We claim that $\text{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \dots$

Indeed, take a fixed $p \in \text{Fix}(S) \cap \Omega \cap \Xi$ arbitrarily. Then $Sp = p$, $J_{M, \mu_n}(p - \mu_n \Phi(p)) = p$, for all $n \geq 0$, and

$$p = P_C [P_C(p - \mu_2 B_2 p) - \mu_1 B_1 P_C(p - \mu_2 B_2 p)]. \quad (3.7)$$

For simplicity, we write $q = P_C(p - \mu_2 B_2 p)$, $\tilde{x}_n = P_C(x_n - \mu_2 B_2 x_n)$, and $\tilde{y}_n = P_C(y_n - \mu_2 B_2 y_n)$,

$$\begin{aligned} y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] = P_C(\tilde{x}_n - \mu_1 B_1 \tilde{x}_n), \\ t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] = P_C(\tilde{y}_n - \mu_1 B_1 \tilde{y}_n), \end{aligned} \quad (3.8)$$

for each $n \geq 0$. Since $B_i : C \rightarrow H$ is β_i -inverse strongly monotone, and $0 < \mu_i < 2\beta_i$ for $i = 1, 2$, we know that for all $n \geq 0$,

$$\begin{aligned} & \|y_n - p\|^2 \\ &= \|P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] - p\|^2 \\ &= \|P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\ &\quad - P_C [P_C(p - \mu_2 B_2 p) - \mu_1 B_1 P_C(p - \mu_2 B_2 p)]\|^2 \\ &\leq \| [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\ &\quad - [P_C(p - \mu_2 B_2 p) - \mu_1 B_1 P_C(p - \mu_2 B_2 p)] \|^2 \\ &= \| [P_C(x_n - \mu_2 B_2 x_n) - P_C(p - \mu_2 B_2 p)] - \mu_1 [B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(p - \mu_2 B_2 p)] \|^2 \\ &\leq \| P_C(x_n - \mu_2 B_2 x_n) - P_C(p - \mu_2 B_2 p) \|^2 \\ &\quad - \mu_1 (2\beta_1 - \mu_1) \| B_1 P_C(x_n - \mu_2 B_2 x_n) - B_1 P_C(p - \mu_2 B_2 p) \|^2 \\ &\leq \| (x_n - \mu_2 B_2 x_n) - (p - \mu_2 B_2 p) \|^2 - \mu_1 (2\beta_1 - \mu_1) \| B_1 \tilde{x}_n - B_1 q \|^2 \\ &= \| (x_n - p) - \mu_2 (B_2 x_n - B_2 p) \|^2 - \mu_1 (2\beta_1 - \mu_1) \| B_1 \tilde{x}_n - B_1 q \|^2 \\ &\leq \| x_n - p \|^2 - \mu_2 (2\beta_2 - \mu_2) \| B_2 x_n - B_2 p \|^2 - \mu_1 (2\beta_1 - \mu_1) \| B_1 \tilde{x}_n - B_1 q \|^2 \leq \| x_n - p \|^2. \end{aligned} \quad (3.9)$$

Repeating the same argument, we can obtain that for all $n \geq 0$,

$$\begin{aligned} \|t_n - p\|^2 &\leq \|y_n - p\|^2 - \mu_2 (2\beta_2 - \mu_2) \|B_2 y_n - B_2 p\|^2 \\ &\quad - \mu_1 (2\beta_1 - \mu_1) \|B_1 \tilde{y}_n - B_1 q\|^2 \leq \|y_n - p\|^2. \end{aligned} \quad (3.10)$$

Furthermore, by Lemma 2.1 we derive from (3.9) and (3.10)

$$\begin{aligned}
\|\widehat{t}_n - p\|^2 &= \|\sigma_n(t_n - p) + (1 - \sigma_n)(J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)) - p)\|^2 \\
&\leq \sigma_n\|t_n - p\|^2 + (1 - \sigma_n)\|J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)) - p\|^2 \\
&= \sigma_n\|t_n - p\|^2 + (1 - \sigma_n)\|J_{M,\mu_n}(t_n - \mu_n\Phi(t_n)) - J_{M,\mu_n}(p - \mu_n\Phi(p))\|^2 \\
&\leq \sigma_n\|t_n - p\|^2 + (1 - \sigma_n)\|(t_n - \mu_n\Phi(t_n)) - (p - \mu_n\Phi(p))\|^2 \\
&\leq \sigma_n\|t_n - p\|^2 + (1 - \sigma_n)\left[\|t_n - p\|^2 + \mu_n(\mu_n - 2\alpha)\|\Phi(t_n) - \Phi(p)\|^2\right] \\
&\leq \|t_n - p\|^2 \tag{3.11} \\
&\leq \|y_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2y_n - B_2p\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1\tilde{y}_n - B_1q\|^2 \\
&\leq \|x_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2x_n - B_2p\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1\tilde{x}_n - B_1q\|^2 \\
&\quad - \mu_2(2\beta_2 - \mu_2)\|B_2y_n - B_2p\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1\tilde{y}_n - B_1q\|^2 \\
&= \|x_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\left(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2\right) \\
&\quad - \mu_1(2\beta_1 - \mu_1)\left(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2\right).
\end{aligned}$$

Since $(\gamma_n + \delta_n)k \leq \gamma_n$, for all $n \geq 0$, utilizing Lemmas 2.2 and 2.7, we get from (3.11)

$$\begin{aligned}
\|z_n - p\|^2 &= \left\| \beta_n(x_n - p) + \gamma_n(\widehat{t}_n - p) + \delta_n(S\widehat{t}_n - p) \right\|^2 \\
&= \left\| \beta_n(x_n - p) + (\gamma_n + \delta_n) \frac{1}{\gamma_n + \delta_n} \left[\gamma_n(\widehat{t}_n - p) + \delta_n(S\widehat{t}_n - p) \right] \right\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n) \left\| \frac{1}{\gamma_n + \delta_n} \left[\gamma_n(\widehat{t}_n - p) + \delta_n(S\widehat{t}_n - p) \right] \right\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\|\widehat{t}_n - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n) \left\{ \|x_n - p\|^2 - \mu_2(2\beta_2 - \mu_2)\left(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2\right) \right. \\
&\quad \left. - \mu_1(2\beta_1 - \mu_1)\left(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2\right) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \|x_n - p\|^2 - (\gamma_n + \delta_n) \left\{ \mu_2(2\beta_2 - \mu_2) \left(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2 \right) \right. \\
&\quad \left. + \mu_1(2\beta_1 - \mu_1) \left(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2 \right) \right\} \\
&\leq \|x_n - p\|^2,
\end{aligned} \tag{3.12}$$

for every $n = 0, 1, 2, \dots$, and hence $p \in C_n$. So, $\text{Fix}(S) \cap \Omega \cap \Xi \subset C_n$ for every $n = 0, 1, 2, \dots$. Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\text{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \dots$. For $n = 0$, we have $Q_0 = C$. Hence we obtain $\text{Fix}(S) \cap \Omega \cap \Xi \subset C_0 \cap Q_0$. Suppose that x_n is given and $\text{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$ for some integer $n \geq 0$. Since $\text{Fix}(S) \cap \Omega \cap \Xi$ is nonempty, $C_n \cap Q_n$ is a nonempty closed convex subset of C . So, there exists a unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n} x_0$. It is also obvious that there holds $\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geq 0$ for $z \in \text{Fix}(S) \cap \Omega \cap \Xi$, and hence $\text{Fix}(S) \cap \Omega \cap \Xi \subset Q_{n+1}$. Therefore, we derive $\text{Fix}(S) \cap \Omega \cap \Xi \subset C_{n+1} \cap Q_{n+1}$.

Step 2. We claim that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \tag{3.13}$$

Indeed, let $l_0 = P_{\text{Fix}(S) \cap \Omega \cap \Xi} x_0$. From $x_{n+1} = P_{C_n \cap Q_n} x_0$, and $l_0 \in \text{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - l_0\| \tag{3.14}$$

for every $n = 0, 1, 2, \dots$. Therefore, $\{x_n\}$ is bounded. From (3.9)–(3.12), we also obtain that $\{\tilde{x}_n\}$, $\{y_n\}$, $\{\tilde{y}_n\}$, $\{t_n\}$, $\{\hat{t}_n\}$, and $\{z_n\}$ all are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n} x_0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \tag{3.15}$$

for every $n = 0, 1, 2, \dots$. Therefore, there exists $\lim_{n \rightarrow \infty} \|x_n - x_0\|$. Since $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, utilizing (2.5), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \tag{3.16}$$

for every $n = 0, 1, 2, \dots$. This implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$, and hence

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2\|x_{n+1} - x_n\| \quad (3.18)$$

for every $n = 0, 1, 2, \dots$. From $\|x_{n+1} - x_n\| \rightarrow 0$ it follows that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (3.19)$$

Step 3. We claim that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{n \rightarrow \infty} \|x_n - t_n\| = \lim_{n \rightarrow \infty} \|\hat{t}_n - t_n\| = 0. \quad (3.20)$$

Indeed, for $p \in \text{Fix}(S) \cap \Omega \cap \Xi$, we obtain from (3.12)

$$\begin{aligned} \|z_n - p\|^2 &\leq \|x_n - p\|^2 \\ &\quad - (\gamma_n + \delta_n) \left\{ \mu_2(2\beta_2 - \mu_2) \left(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2 \right) \right. \\ &\quad \left. + \mu_1(2\beta_1 - \mu_1) \left(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2 \right) \right\}. \end{aligned} \quad (3.21)$$

Therefore, we have

$$\begin{aligned} &(\gamma_n + \delta_n) \left\{ \mu_2(2\beta_2 - \mu_2) \left(\|B_2x_n - B_2p\|^2 + \|B_2y_n - B_2p\|^2 \right) \right. \\ &\quad \left. + \mu_1(2\beta_1 - \mu_1) \left(\|B_1\tilde{x}_n - B_1q\|^2 + \|B_1\tilde{y}_n - B_1q\|^2 \right) \right\} \\ &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &= (\|x_n - p\| - \|z_n - p\|) (\|x_n - p\| + \|z_n - p\|) \\ &\leq \|x_n - z_n\| (\|x_n - p\| + \|z_n - p\|). \end{aligned} \quad (3.22)$$

Since $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, $\|x_n - z_n\| \rightarrow 0$, and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we deduce that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|B_2 x_n - B_2 p\| &= \lim_{n \rightarrow \infty} \|B_2 y_n - B_2 p\| = \lim_{n \rightarrow \infty} \|B_1 \tilde{x}_n - B_1 q\| \\ &= \lim_{n \rightarrow \infty} \|B_1 \tilde{y}_n - B_1 q\| = 0. \end{aligned} \quad (3.23)$$

On the other hand, by firm nonexpansiveness of P_C , we have

$$\begin{aligned} \|\tilde{x}_n - q\|^2 &= \|P_C(x_n - \mu_2 B_2 x_n) - P_C(p - \mu_2 B_2 p)\|^2 \\ &\leq \langle (x_n - \mu_2 B_2 x_n) - (p - \mu_2 B_2 p), \tilde{x}_n - q \rangle \\ &= \frac{1}{2} \left[\|x_n - p - \mu_2 (B_2 x_n - B_2 p)\|^2 + \|\tilde{x}_n - q\|^2 \right. \\ &\quad \left. - \|(x_n - p) - \mu_2 (B_2 x_n - B_2 p) - (\tilde{x}_n - q)\|^2 \right] \\ &\leq \frac{1}{2} \left[\|x_n - p\|^2 + \|\tilde{x}_n - q\|^2 - \|(x_n - \tilde{x}_n) - \mu_2 (B_2 x_n - B_2 p) - (p - q)\|^2 \right] \quad (3.24) \\ &= \frac{1}{2} \left[\|x_n - p\|^2 + \|\tilde{x}_n - q\|^2 - \|x_n - \tilde{x}_n - (p - q)\|^2 \right. \\ &\quad \left. + 2\mu_2 \langle x_n - \tilde{x}_n - (p - q), B_2 x_n - B_2 p \rangle - \mu_2^2 \|B_2 x_n - B_2 p\|^2 \right] \\ &\leq \frac{1}{2} \left[\|x_n - p\|^2 + \|\tilde{x}_n - q\|^2 - \|x_n - \tilde{x}_n - (p - q)\|^2 \right. \\ &\quad \left. + 2\mu_2 \|x_n - \tilde{x}_n - (p - q)\| \|B_2 x_n - B_2 p\| \right]; \end{aligned}$$

that is,

$$\|\tilde{x}_n - q\|^2 \leq \|x_n - p\|^2 - \|x_n - \tilde{x}_n - (p - q)\|^2 + 2\mu_2 \|x_n - \tilde{x}_n - (p - q)\| \|B_2 x_n - B_2 p\|. \quad (3.25)$$

Repeating the same argument, we can also obtain

$$\|\tilde{y}_n - q\|^2 \leq \|y_n - p\|^2 - \|y_n - \tilde{y}_n - (p - q)\|^2 + 2\mu_2 \|y_n - \tilde{y}_n - (p - q)\| \|B_2 y_n - B_2 p\|. \quad (3.26)$$

Moreover, using the argument technique similar to the above one, we derive

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(\tilde{x}_n - \mu_1 B_1 \tilde{x}_n) - P_C(q - \mu_1 B_1 q)\|^2 \\
&\leq \langle (\tilde{x}_n - \mu_1 B_1 \tilde{x}_n) - (q - \mu_1 B_1 q), y_n - p \rangle \\
&= \frac{1}{2} \left[\|\tilde{x}_n - q - \mu_1 (B_1 \tilde{x}_n - B_1 q)\|^2 + \|y_n - p\|^2 \right. \\
&\quad \left. - \|(\tilde{x}_n - q) - \mu_1 (B_1 \tilde{x}_n - B_1 q) - (y_n - p)\|^2 \right] \\
&\leq \frac{1}{2} \left[\|\tilde{x}_n - q\|^2 + \|y_n - p\|^2 - \|(\tilde{x}_n - y_n) - \mu_1 (B_1 \tilde{x}_n - B_1 q) + (p - q)\|^2 \right] \quad (3.27) \\
&= \frac{1}{2} \left[\|\tilde{x}_n - q\|^2 + \|y_n - p\|^2 - \|\tilde{x}_n - y_n + (p - q)\|^2 \right. \\
&\quad \left. + 2\mu_1 \langle \tilde{x}_n - y_n + (p - q), B_1 \tilde{x}_n - B_1 q \rangle - \mu_1^2 \|B_1 \tilde{x}_n - B_1 q\|^2 \right] \\
&\leq \frac{1}{2} \left[\|\tilde{x}_n - q\|^2 + \|y_n - p\|^2 - \|\tilde{x}_n - y_n + (p - q)\|^2 \right. \\
&\quad \left. + 2\mu_1 \|\tilde{x}_n - y_n + (p - q)\| \|B_1 \tilde{x}_n - B_1 q\| \right];
\end{aligned}$$

that is,

$$\|y_n - p\|^2 \leq \|\tilde{x}_n - q\|^2 - \|\tilde{x}_n - y_n + (p - q)\|^2 + 2\mu_1 \|\tilde{x}_n - y_n + (p - q)\| \|B_1 \tilde{x}_n - B_1 q\|. \quad (3.28)$$

Repeating the same argument, we can also obtain

$$\|t_n - p\|^2 \leq \|\tilde{y}_n - q\|^2 - \|\tilde{y}_n - t_n + (p - q)\|^2 + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\|. \quad (3.29)$$

Utilizing (3.11), (3.25)–(3.29), we have

$$\begin{aligned}
\|z_n - p\|^2 &= \left\| \beta_n (x_n - p) + \gamma_n (\hat{t}_n - p) + \delta_n (S\hat{t}_n - p) \right\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (\gamma_n + \delta_n) \|\hat{t}_n - p\|^2 \\
&= \beta_n \|x_n - p\|^2 + (\gamma_n + \delta_n) \|t_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 \\
&\quad + (\gamma_n + \delta_n) \left[\|\tilde{y}_n - q\|^2 - \|\tilde{y}_n - t_n + (p - q)\|^2 \right. \\
&\quad \left. + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\| \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \beta_n \|x_n - p\|^2 \\
&\quad + (\gamma_n + \delta_n) \left[\|y_n - p\|^2 - \|y_n - \tilde{y}_n - (p - q)\|^2 \right. \\
&\quad\quad + 2\mu_2 \|y_n - \tilde{y}_n - (p - q)\| \|B_2 y_n - B_2 p\| \\
&\quad\quad \left. - \|\tilde{y}_n - t_n + (p - q)\|^2 + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\| \right] \\
&\leq \beta_n \|x_n - p\|^2 \\
&\quad + (\gamma_n + \delta_n) \left\{ \|\tilde{x}_n - q\|^2 - \|\tilde{x}_n - y_n + (p - q)\|^2 \right. \\
&\quad\quad + 2\mu_1 \|\tilde{x}_n - y_n + (p - q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
&\quad\quad - \|y_n - \tilde{y}_n - (p - q)\|^2 + 2\mu_2 \|y_n - \tilde{y}_n - (p - q)\| \|B_2 y_n - B_2 p\| \\
&\quad\quad \left. - \|\tilde{y}_n - t_n + (p - q)\|^2 + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\| \right\} \\
&\leq \beta_n \|x_n - p\|^2 \\
&\quad + (\gamma_n + \delta_n) \left\{ \|x_n - p\|^2 - \|x_n - \tilde{x}_n - (p - q)\|^2 \right. \\
&\quad\quad + 2\mu_2 \|x_n - \tilde{x}_n - (p - q)\| \|B_2 x_n - B_2 p\| \\
&\quad\quad - \|\tilde{x}_n - y_n + (p - q)\|^2 + 2\mu_1 \|\tilde{x}_n - y_n + (p - q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
&\quad\quad - \|y_n - \tilde{y}_n - (p - q)\|^2 + 2\mu_2 \|y_n - \tilde{y}_n - (p - q)\| \|B_2 y_n - B_2 p\| \\
&\quad\quad \left. - \|\tilde{y}_n - t_n + (p - q)\|^2 + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\| \right\} \\
&\leq \|x_n - p\|^2 + 2\mu_2 \|x_n - \tilde{x}_n - (p - q)\| \|B_2 x_n - B_2 p\| \\
&\quad + 2\mu_1 \|\tilde{x}_n - y_n + (p - q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
&\quad + 2\mu_2 \|y_n - \tilde{y}_n - (p - q)\| \|B_2 y_n - B_2 p\| + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\| \\
&\quad - (\gamma_n + \delta_n) \left[\|x_n - \tilde{x}_n - (p - q)\|^2 + \|\tilde{x}_n - y_n + (p - q)\|^2 \right. \\
&\quad\quad \left. + \|y_n - \tilde{y}_n - (p - q)\|^2 + \|\tilde{y}_n - t_n + (p - q)\|^2 \right], \tag{3.30}
\end{aligned}$$

which hence implies that

$$\begin{aligned}
&(\gamma_n + \delta_n) \left[\|x_n - \tilde{x}_n - (p - q)\|^2 + \|\tilde{x}_n - y_n + (p - q)\|^2 \right. \\
&\quad \left. + \|y_n - \tilde{y}_n - (p - q)\|^2 + \|\tilde{y}_n - t_n + (p - q)\|^2 \right] \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2\mu_2 \|x_n - \tilde{x}_n - (p - q)\| \|B_2 x_n - B_2 p\|
\end{aligned}$$

$$\begin{aligned}
& + 2\mu_1 \|\tilde{x}_n - y_n + (p - q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
& + 2\mu_2 \|y_n - \tilde{y}_n - (p - q)\| \|B_2 y_n - B_2 p\| \\
& + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\| \\
\leq & \|x_n - z_n\| (\|x_n - p\| + \|z_n - p\|) \\
& + 2\mu_2 \|x_n - \tilde{x}_n - (p - q)\| \|B_2 x_n - B_2 p\| \\
& + 2\mu_1 \|\tilde{x}_n - y_n + (p - q)\| \|B_1 \tilde{x}_n - B_1 q\| \\
& + 2\mu_2 \|y_n - \tilde{y}_n - (p - q)\| \|B_2 y_n - B_2 p\| \\
& + 2\mu_1 \|\tilde{y}_n - t_n + (p - q)\| \|B_1 \tilde{y}_n - B_1 q\|.
\end{aligned} \tag{3.31}$$

Since $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1)$, $\|x_n - z_n\| \rightarrow 0$, and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$, and $\{t_n\}$ all are bounded, it follows from (3.23) that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|x_n - \tilde{x}_n - (p - q)\| &= \lim_{n \rightarrow \infty} \|\tilde{x}_n - y_n + (p - q)\| = 0, \\
\lim_{n \rightarrow \infty} \|y_n - \tilde{y}_n - (p - q)\| &= \lim_{n \rightarrow \infty} \|\tilde{y}_n - t_n + (p - q)\| = 0.
\end{aligned} \tag{3.32}$$

Consequently, it immediately follows that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0, \quad \lim_{n \rightarrow \infty} \|y_n - t_n\| = 0. \tag{3.33}$$

This shows that

$$\lim_{n \rightarrow \infty} \|x_n - t_n\| = 0. \tag{3.34}$$

Also, note that

$$\begin{aligned}
\|z_n - x_n\|^2 &= \left\| \gamma_n (\hat{t}_n - x_n) + \delta_n (S\hat{t}_n - x_n) \right\|^2 \\
&= \left\| (\gamma_n + \delta_n) \frac{1}{\gamma_n + \delta_n} \left[\gamma_n (\hat{t}_n - x_n) + \delta_n (S\hat{t}_n - x_n) \right] \right\|^2 \\
&= (\gamma_n + \delta_n)^2 \left[\frac{\gamma_n}{\gamma_n + \delta_n} \|\hat{t}_n - x_n\|^2 + \frac{\delta_n}{\gamma_n + \delta_n} \|S\hat{t}_n - x_n\|^2 \right. \\
&\quad \left. - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} \|\hat{t}_n - S\hat{t}_n\|^2 \right] \\
&= (\gamma_n + \delta_n) \left[\gamma_n \|\hat{t}_n - x_n\|^2 + \delta_n \|S\hat{t}_n - x_n\|^2 \right] - \gamma_n \delta_n \|\hat{t}_n - S\hat{t}_n\|^2.
\end{aligned} \tag{3.35}$$

Thus we have

$$\begin{aligned}
 d^2 \left\| \widehat{S}t_n - x_n \right\|^2 &\leq (\gamma_n + \delta_n) \left[\gamma_n \left\| \widehat{t}_n - x_n \right\|^2 + \delta_n \left\| S\widehat{t}_n - x_n \right\|^2 \right] \\
 &= \|z_n - x_n\|^2 + \gamma_n \delta_n \left\| \widehat{t}_n - S\widehat{t}_n \right\|^2 \\
 &\leq \|z_n - x_n\|^2 + \left\| \widehat{t}_n - S\widehat{t}_n \right\|^2.
 \end{aligned} \tag{3.36}$$

This together with $\|z_n - x_n\| \rightarrow 0$ and $\|\widehat{t}_n - S\widehat{t}_n\| \rightarrow 0$ implies that

$$\lim_{n \rightarrow \infty} \left\| S\widehat{t}_n - x_n \right\| = 0, \quad \lim_{n \rightarrow \infty} \left\| \widehat{t}_n - x_n \right\| = 0. \tag{3.37}$$

Consequently, from (3.34) we immediately derive

$$\lim_{n \rightarrow \infty} \left\| \widehat{t}_n - t_n \right\| = 0. \tag{3.38}$$

Step 4. We claim that $\omega_w(x_n) \subset \text{Fix}(S) \cap \Omega \cap \Xi$.

Indeed, as $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $u \in \omega_w(x_n)$. We can obtain that $u \in \text{Fix}(S) \cap \Omega \cap \Xi$. First, it is clear from Lemma 2.6(ii) that $u \in \text{Fix}(S)$. Now let us show that $u \in \Xi$. We note that

$$\begin{aligned}
 \|x_n - G(x_n)\| &= \|x_n - P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)]\| \\
 &= \|x_n - y_n\| \rightarrow 0 \quad (n \rightarrow \infty),
 \end{aligned} \tag{3.39}$$

where $G : C \rightarrow C$ is defined as that in Lemma 1.1. According to Lemma 2.6(ii) we obtain $u \in \Xi$. Further, let us show that $u \in \Omega$. As a matter of fact, since Φ is α -inverse strongly monotone, and M is maximal monotone, by Lemma 2.5 we know that $M + \Phi$ is maximal monotone. Take a fixed $(y, g) \in G(M + \Phi)$ arbitrarily. Then we have $g \in My + \Phi(y)$. So, we have $g - \Phi(y) \in My$. Since

$$\widehat{t}_{n_i} = \sigma_{n_i} t_{n_i} + (1 - \sigma_{n_i}) J_{M, \mu_{n_i}}(t_{n_i} - \mu_{n_i} \Phi(t_{n_i})) \tag{3.40}$$

implies

$$\frac{1}{\mu_{n_i}} (t_{n_i} - s_{n_i} - \mu_{n_i} \Phi(t_{n_i})) \in Ms_{n_i}, \tag{3.41}$$

where $s_{n_i} = t_{n_i} + (\widehat{t}_{n_i} - t_{n_i}) / (1 - \sigma_{n_i})$, we have

$$\left\langle y - s_{n_i}, g - \Phi(y) - \frac{1}{\mu_{n_i}} (t_{n_i} - s_{n_i} - \mu_{n_i} \Phi(t_{n_i})) \right\rangle \geq 0, \tag{3.42}$$

which hence yields

$$\begin{aligned}
\langle \mathbf{y} - s_{n_i}, g \rangle &\geq \left\langle \mathbf{y} - s_{n_i}, \Phi(\mathbf{y}) + \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i} - \mu_{n_i}\Phi(t_{n_i})) \right\rangle \\
&= \langle \mathbf{y} - s_{n_i}, \Phi(\mathbf{y}) - \Phi(t_{n_i}) \rangle + \left\langle \mathbf{y} - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i}) \right\rangle \\
&\geq \alpha \|\Phi(\mathbf{y}) - \Phi(s_{n_i})\|^2 + \langle \mathbf{y} - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i}) \rangle + \left\langle \mathbf{y} - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i}) \right\rangle \\
&\leq \langle \mathbf{y} - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i}) \rangle + \left\langle \mathbf{y} - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i}) \right\rangle.
\end{aligned} \tag{3.43}$$

Observe that

$$\begin{aligned}
&\left| \langle \mathbf{y} - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i}) \rangle + \left\langle \mathbf{y} - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i}) \right\rangle \right| \\
&\leq \|\mathbf{y} - s_{n_i}\| \|\Phi(s_{n_i}) - \Phi(t_{n_i})\| + \|\mathbf{y} - s_{n_i}\| \left\| \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i}) \right\| \\
&\leq \frac{1}{\alpha} \|\mathbf{y} - s_{n_i}\| \|s_{n_i} - t_{n_i}\| + \frac{1}{\epsilon} \|\mathbf{y} - s_{n_i}\| \|t_{n_i} - s_{n_i}\| \\
&= \left(\frac{1}{\alpha} + \frac{1}{\epsilon} \right) \|\mathbf{y} - s_{n_i}\| \|t_{n_i} - s_{n_i}\|.
\end{aligned} \tag{3.44}$$

From $\|s_{n_i} - t_{n_i}\| = (1/(1 - \sigma_{n_i}))\|\widehat{t}_{n_i} - t_{n_i}\| \leq (1/(1 - c))\|\widehat{t}_{n_i} - t_{n_i}\| \rightarrow 0$, it follows that

$$\lim_{i \rightarrow \infty} \left| \langle \mathbf{y} - s_{n_i}, \Phi(s_{n_i}) - \Phi(t_{n_i}) \rangle + \left\langle \mathbf{y} - s_{n_i}, \frac{1}{\mu_{n_i}}(t_{n_i} - s_{n_i}) \right\rangle \right| = 0. \tag{3.45}$$

Since $\|x_n - t_n\| \rightarrow 0$, $\|\widehat{t}_n - t_n\| \rightarrow 0$, and $x_{n_i} \rightarrow u$, we derive $s_{n_i} = t_{n_i} + ((\widehat{t}_{n_i} - t_{n_i})/(1 - \sigma_{n_i})) \rightarrow u$, and hence by letting $i \rightarrow \infty$ we get from (3.43)

$$\langle \mathbf{y} - u, g \rangle \geq 0. \tag{3.46}$$

This shows that $0 \in \Phi(u) + Mu$. Thus, $u \in \Omega$. Therefore, $u \in \text{Fix}(S) \cap \Omega \cap \Xi$.

Step 5. We claim that

$$\lim_{n \rightarrow \infty} \|x_n - l_0\| = \lim_{n \rightarrow \infty} \|y_n - l_0\| = \lim_{n \rightarrow \infty} \|z_n - l_0\| = 0, \tag{3.47}$$

where $l_0 = P_{\text{Fix}(S) \cap \Omega \cap \Xi} x_0$.

Indeed, Since $l_0 = P_{\text{Fix}(S) \cap \Omega \cap \Xi} x_0$, and $u \in \text{Fix}(S) \cap \Omega \cap \Xi$, from (3.14) we have

$$\|l_0 - x_0\| \leq \|u - x_0\| \leq \liminf_{i \rightarrow \infty} \|x_{n_i} - x_0\| \leq \limsup_{i \rightarrow \infty} \|x_{n_i} - x_0\| \leq \|l_0 - x_0\|. \tag{3.48}$$

So, we obtain

$$\lim_{i \rightarrow \infty} \|x_{n_i} - x_0\| = \|u - x_0\|. \quad (3.49)$$

From $x_{n_i} - x_0 \rightharpoonup u - x_0$, we have $x_{n_i} - x_0 \rightarrow u - x_0$ (due to the Kadec-Klee property of Hilbert spaces [37]), and hence $x_{n_i} \rightarrow u$. Since $x_n = P_{Q_n}x_0$, and $l_0 \in \text{Fix}(S) \cap \Omega \cap \Xi \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x_0 \rangle + \langle l_0 - x_{n_i}, x_0 - l_0 \rangle \geq \langle l_0 - x_{n_i}, x_0 - l_0 \rangle. \quad (3.50)$$

As $i \rightarrow \infty$, we obtain $-\|l_0 - x_{n_i}\|^2 \geq \langle l_0 - u, x_0 - l_0 \rangle \geq 0$ by $l_0 = P_{\text{Fix}(S) \cap \Omega \cap \Xi}x_0$, and $u \in \text{Fix}(S) \cap \Omega \cap \Xi$. Hence we have $u = l_0$. This implies that $x_n \rightarrow l_0$. It is easy to see that $y_n \rightarrow l_0$ and $z_n \rightarrow l_0$. This completes the proof. \square

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $B_i : C \rightarrow H$ be β_i -inverse strongly monotone for $i = 1, 2$, let $\Phi : C \rightarrow H$ be an α -inverse strongly monotone mapping, let M be a maximal monotone mapping with $D(M) = C$, and let $S : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(S) \cap \Omega \cap \Xi \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned} y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\ \hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)), \\ z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \quad (3.51)$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\mu_n\} \subset [e, 2\alpha]$ for some $e \in (0, 2\alpha]$, and $\{\sigma_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\} \subset [0, 1]$ such that $\{\sigma_n\} \subset [0, c]$ for some $c \in [0, 1)$, $\{\gamma_n\}$, $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1)$, and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\bar{x} = P_{\text{Fix}(S) \cap \Omega \cap \Xi}x_0$. Furthermore, (\bar{x}, \bar{y}) is a solution of the GSVI (1.10), where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. Since S is a nonexpansive mapping, S must be a k -strictly pseudocontractive mapping with $k = 0$. Take a fixed $p \in \text{Fix}(S) \cap \Omega \cap \Xi$ arbitrarily. Note that in Step 1 for the proof of Theorem 3.1, we have obtained that $\{x_n\}$ is bounded and the relation holds

$$\|\hat{t}_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0 \quad (3.52)$$

(due to (3.11)). Moreover, in Step 2 for the proof of Theorem 3.1, we have proven that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \quad (3.53)$$

Now, utilizing Lemma 2.2, from the nonexpansiveness of S we deduce that

$$\begin{aligned}
\|z_n - p\|^2 &= \left\| \beta_n(x_n - p) + \gamma_n(\hat{t}_n - p) + \delta_n(S\hat{t}_n - p) \right\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + \gamma_n\|\hat{t}_n - p\|^2 + \delta_n\|S\hat{t}_n - p\|^2 - \gamma_n\delta_n\|S\hat{t}_n - \hat{t}_n\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + \gamma_n\|\hat{t}_n - p\|^2 + \delta_n\|\hat{t}_n - p\|^2 - \gamma_n\delta_n\|S\hat{t}_n - \hat{t}_n\|^2 \\
&= \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\|\hat{t}_n - p\|^2 - \gamma_n\delta_n\|S\hat{t}_n - \hat{t}_n\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\|x_n - p\|^2 - \gamma_n\delta_n\|S\hat{t}_n - \hat{t}_n\|^2 \\
&= \|x_n - p\|^2 - \gamma_n\delta_n\|S\hat{t}_n - \hat{t}_n\|^2.
\end{aligned} \tag{3.54}$$

This together with $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ implies that

$$\begin{aligned}
d^2\|S\hat{t}_n - \hat{t}_n\|^2 &\leq \gamma_n\delta_n\|S\hat{t}_n - \hat{t}_n\|^2 \leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned} \tag{3.55}$$

So, we immediately derive

$$\lim_{n \rightarrow \infty} \|S\hat{t}_n - \hat{t}_n\| = 0. \tag{3.56}$$

It is easy to see that all the conditions of Theorem 3.1 are satisfied. Therefore, in terms of Theorem 3.1 we obtain the desired result. \square

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $B_i : C \rightarrow H$ be β_i -inverse strongly monotone for $i = 1, 2$, and let $S : C \rightarrow C$ be a k -strictly pseudocontractive mapping such that $\text{Fix}(S) \cap \Xi \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\
z_n &= \beta_n x_n + \gamma_n P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] \\
&\quad + \delta_n S P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\
C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.57}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1)$, $\beta_n + \gamma_n + \delta_n = 1$, and $(\gamma_n + \delta_n)k \leq \gamma_n$ for all $n \geq 0$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge

strongly to the same point $\bar{x} = P_{\text{Fix}(S) \cap \Xi} x_0$. Furthermore, (\bar{x}, \bar{y}) is a solution of the GSVI (1.10), where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. Putting $\Phi = M = 0$ in Theorem 3.1, we have $\Omega = C$ and $\text{Fix}(S) \cap \Omega \cap \Xi = \text{Fix}(S) \cap \Xi$. Let α be any positive number in the interval $(0, \infty)$, and take any sequence $\{\sigma_n\} \subset [0, c]$ for some $c \in [0, 1)$ and any sequence $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$. Then Φ is α -inverse strongly monotone, and we have

$$\begin{aligned} y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\ t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\ \hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) = \sigma_n t_n + (1 - \sigma_n)(I + \mu_n M)^{-1} t_n = t_n, \\ z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{3.58}$$

which is just equivalent to (3.57). In this case, we have

$$z_n = \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n = \beta_n x_n + \gamma_n t_n + \delta_n S t_n. \tag{3.59}$$

Note that in Steps 2 and 3 for the proof of Theorem 3.1, we have proven that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0, \quad \lim_{n \rightarrow \infty} \|t_n - x_n\| = 0, \tag{3.60}$$

respectively. Thus, we have

$$\|\delta_n (S t_n - x_n)\| \leq \|z_n - x_n\| + \gamma_n \|t_n - x_n\| \rightarrow 0. \tag{3.61}$$

Consequently, it follows from $\{\delta_n\} \subset [d, 1]$ that $\|S t_n - x_n\| \rightarrow 0$, and hence $\|S t_n - t_n\| \rightarrow 0$. This shows that $\|S \hat{t}_n - \hat{t}_n\| \rightarrow 0$. Utilizing Theorem 3.1, we obtain the desired result. \square

Remark 3.4. Our Theorems 3.1 improves, extends, and develops [36, Theorem 3.1], [15, Theorem 3.1], [34, Theorem 3.2], and [35, Theorem 3.1] in the following aspects.

- (i) Compared with the relaxed extragradient iterative algorithm in [34, Theorem 3.2] and the hybrid extragradient iterative algorithm in [35, Theorem 3.1], our hybrid extragradient iterative algorithms remove the requirements that $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ and $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$.

- (ii) The problem of finding an element of $\text{Fix}(S) \cap \Omega \cap \Xi$ in our Theorem 3.1 is more general than the corresponding ones in [36, Theorem 3.1], [15, Theorem 3.1], and [34, Theorem 3.2] to a great extent. Thus, beyond question our results are very interesting and quite valuable.
- (iii) The relaxed extragradient method for finding an element of $\text{Fix}(S) \cap \Xi$ in [34, Theorem 3.2] is extended to develop our hybrid extragradient iterative algorithms for finding an element of $\text{Fix}(S) \cap \Omega \cap \Xi$.
- (iv) The proof of our results are very different from that of [15, Theorem 3.1] because our argument technique depends on two inverse strongly monotone mappings B_1 and B_2 , the property of strict pseudocontractions (see Lemmas 2.6 and 2.7), and the properties of the resolvent $J_{M,\lambda}$ to a great extent.
- (v) Because our iterative algorithms involve two inverse strongly monotone mappings B_1 and B_2 , a k -strictly pseudocontractive self-mapping S , and several parameter sequences, they are more flexible and more subtle than the corresponding ones in [36, Theorem 3.1], [15, Theorem 3.1], and [34, Theorem 3.2], respectively.

4. Applications

Utilizing Theorem 3.1, we prove some strong convergence theorems in a real Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $B_i : C \rightarrow H$ be β_i -inverse strongly monotone for $i = 1, 2$, let $\Phi : C \rightarrow H$ be an α -inverse strongly monotone mapping, and let M be a maximal monotone mapping with $D(M) = C$ such that $\Omega \cap \Xi \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
 y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\
 t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\
 z_n &= \beta_n x_n + (1 - \beta_n) [\sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n))], \\
 C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
 Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
 x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
 \end{aligned} \tag{4.1}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\sigma_n\}, \{\beta_n\} \subset [0, 1]$ such that $\{\sigma_n\} \subset [0, c]$ for some $c \in [0, 1)$, and $\{\beta_n\} \subset [0, d]$ for some $d \in [0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the same point $\bar{x} = P_{\Omega \cap \Xi} x_0$. Furthermore, (\bar{x}, \bar{y}) is a solution of the GSVI (1.10), where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. In Corollary 3.2, putting $S = I$, we have

$$\begin{aligned}
 y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)], \\
 t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)], \\
 \hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)), \\
 z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n = \beta_n x_n + (1 - \beta_n) \hat{t}_n, \\
 C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
 Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
 x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
 \end{aligned} \tag{4.2}$$

which is just equivalent to (4.1). In this case, we know that $\text{Fix}(S) \cap \Omega \cap \Xi = \Omega \cap \Xi$. Therefore, by Corollary 3.2 we obtain desired result. \square

Theorem 4.2 (see [15, Theorem 4.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $S : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(S)$ is nonempty. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
 z_n &= (1 - \delta_n)x_n + \delta_n Sx_n, \\
 C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
 Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
 x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
 \end{aligned} \tag{4.3}$$

where $\{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{\text{Fix}(S)} x_0$.

Proof. Putting $B_1 = B_2 = \Phi = M = 0$ in Corollary 3.2, we let β_1, β_2 , and α be any positive numbers in the interval $(0, \infty)$, and take any numbers $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$ and any sequence $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$. Then $B_i : C \rightarrow H$ is β_i -inverse strongly monotone for $i = 1, 2$, and $\Phi : C \rightarrow H$ is α -inverse strongly monotone. In this case, we know that $\text{Fix}(S) \cap \Omega \cap \Xi = \text{Fix}(S)$ and

$$\begin{aligned}
 y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] = x_n, \\
 t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] = y_n,
 \end{aligned}$$

$$\begin{aligned}
\hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) = t_n, \\
z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n = (1 - \delta_n) x_n + \delta_n S x_n, \\
C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{4.4}$$

which is just equivalent to (4.3). Therefore, by Corollary 3.2 we obtain the desired result. \square

Remark 4.3. Originally Theorem 4.2 is the result of Nakajo and Takahashi [22].

Theorem 4.4. *Let H be a real Hilbert space. Let $A : H \rightarrow H$ be a λ -inverse strongly monotone mapping, let $\Phi : H \rightarrow H$ be an α -inverse strongly monotone mapping, let $M : H \rightarrow 2^H$ be a maximal monotone mapping, and let $S : H \rightarrow H$ be a nonexpansive mapping such that $\text{Fix}(S) \cap \Omega \cap A^{-1}0 \neq \emptyset$. For given $x_0 \in H$ arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned}
t_n &= x_n - \mu [Ax_n + A(x_n - \mu Ax_n)], \\
z_n &= \beta_n x_n + \gamma_n J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) + \delta_n S J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)), \\
C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{4.5}$$

where $\mu \in (0, 2\lambda)$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{\text{Fix}(S) \cap \Omega \cap A^{-1}0} x_0$.

Proof. Putting $C = H$, $B_1 = A$, $B_2 = 0$, $\mu_1 = \mu$, and $\sigma_n = 0$, for all $n \geq 0$ in Corollary 3.2, we know that $P_C = P_H = I$ and the GSVI (1.10) coincides with the VI (1.3). Hence we have $A^{-1}0 = \text{VI}(H, A) = \Xi$. In this case, we conclude that $\text{Fix}(S) \cap \Omega \cap \Xi = \text{Fix}(S) \cap \Omega \cap A^{-1}0$ and

$$\begin{aligned}
y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] = x_n - \mu A x_n, \\
t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] = x_n - \mu A x_n - \mu A(x_n - \mu A x_n), \\
\hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) = J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)), \\
z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\
C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0.
\end{aligned} \tag{4.6}$$

Therefore, by Corollary 3.2 we obtain the desired result. \square

Let $B : H \rightarrow 2^H$ be a maximal monotone mapping. Then, for any $x \in H$ and $r > 0$, consider $J_{B,r}x = (I + rB)^{-1}x$. It is known that such a $J_{B,r}$ is the resolvent of B .

Theorem 4.5. *Let H be a real Hilbert space. Let $A : H \rightarrow H$ be a λ -inverse strongly monotone mapping, let $\Phi : H \rightarrow H$ be an α -inverse strongly monotone mapping, and let $B, M : H \rightarrow 2^H$ be two maximal monotone mappings such that $A^{-1}0 \cap B^{-1}0 \cap \Omega \neq \emptyset$. Let $J_{B,r}$ be the resolvent of B for each $r > 0$. For given $x_0 \in H$ arbitrarily, let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{aligned} t_n &= x_n - \mu [Ax_n + A(x_n - \mu Ax_n)], \\ z_n &= \beta_n x_n + \gamma_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) + \delta_n J_{B,r} J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{4.7}$$

where $\mu \in (0, 2\lambda)$, $\{\mu_n\} \subset [\epsilon, 2\alpha]$ for some $\epsilon \in (0, 2\alpha]$, and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{A^{-1}0 \cap B^{-1}0 \cap \Omega} x_0$.

Proof. Putting $S = J_{B,r}$ in Theorem 4.4, we know that $\text{Fix}(S) = \text{Fix}(J_{B,r}) = B^{-1}0$. In this case, (4.5) is coincident with (4.7). Therefore, by Theorem 4.4 we obtain the desired result. \square

It is well known that a mapping $T : C \rightarrow C$ is called pseudocontractive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$, for all $x, y \in C$. It is easy to see that this definition is equivalent to the one that a mapping $T : C \rightarrow C$ is called pseudocontractive if $\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2$, for all $x, y \in C$; see [8]. In the meantime, we also know one more definition of a k -strictly pseudocontractive mapping, which is equivalent to the definition given in the introduction. A mapping $T : C \rightarrow C$ is called k -strictly pseudocontractive if there exists a constant $k \in [0, 1)$, such that

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1-k}{2} \|(I - T)x - (I - T)y\|^2, \tag{4.8}$$

for all $x, y \in C$. It is clear that in this case the mapping $I - T$ is $(1 - k)/2$ -inverse strongly monotone. From [10], we know that if T is a k -strictly pseudocontractive mapping, then T is Lipschitz continuous with constant $(1+k)/(1-k)$, such that $\text{Fix}(T) = \text{VI}(C, I - T)$ (see, e.g., the proof of Theorem 4.6). It is obvious that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings and the class of pseudocontractions strictly includes the class of strict pseudocontractions.

In the following theorem we introduce an iterative algorithm that converges strongly to a common fixed point of three mappings: one of which is nonexpansive, and the other two ones are strictly pseudocontractive mappings.

Theorem 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a k -strictly pseudocontractive mapping, let $\Gamma : C \rightarrow C$ be a κ -strictly pseudocontractive mapping,*

and let $S : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \cap \text{Fix}(S) \cap \text{Fix}(\Gamma) \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{aligned} y_n &= x_n - \mu_1(x_n - Tx_n), \\ t_n &= y_n - \mu_1(y_n - Ty_n), \\ \hat{t}_n &= t_n - \mu_n(t_n - \Gamma t_n), \\ z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \tag{4.9}$$

where $\mu_1 \in (0, 1 - k)$, $\{\mu_n\} \subset [\epsilon, 1 - \kappa]$ for some $\epsilon \in (0, 1 - \kappa]$, and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\{\gamma_n\}, \{\delta_n\} \subset [d, 1]$ for some $d \in (0, 1]$, and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $P_{\text{Fix}(T) \cap \text{Fix}(S) \cap \text{Fix}(\Gamma)} x_0$.

Proof. Putting $B_1 = I - T$, $B_2 = 0$, $\Phi = I - \Gamma$, $M = 0$, and $\sigma_n = 0$, for all $n \geq 0$ in Corollary 3.2, we know that B_1 is β_1 -inverse strongly monotone with $\beta_1 = (1 - k)/2$ and Φ is α -inverse strongly monotone with $\alpha = (1 - \kappa)/2$. Moreover, we have $\Xi = \text{VI}(C, B_1) = \text{VI}(C, I - T)$. Noticing $\mu_1 \in (0, 1 - k)$ and $k \in [0, 1)$, we know that $\mu_1 \in (0, 1)$, and hence $(1 - \mu_1)x_n + \mu_1 Tx_n \in C$. Also, noticing $\{\mu_n\} \subset [\epsilon, 1 - \kappa] \subset (0, 1 - \kappa]$, we know that $\{\mu_n\} \subset (0, 1]$, and hence $(1 - \mu_n)t_n + \mu_n \Gamma t_n \in C$. This implies that

$$\begin{aligned} y_n &= P_C [P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)] \\ &= P_C((1 - \mu_1)x_n + \mu_1 Tx_n) = x_n - \mu_1(x_n - Tx_n), \\ t_n &= P_C [P_C(y_n - \mu_2 B_2 y_n) - \mu_1 B_1 P_C(y_n - \mu_2 B_2 y_n)] \\ &= P_C((1 - \mu_1)y_n + \mu_1 Ty_n) = y_n - \mu_1(y_n - Ty_n), \\ \hat{t}_n &= \sigma_n t_n + (1 - \sigma_n) J_{M, \mu_n}(t_n - \mu_n \Phi(t_n)) = t_n - \mu_n(t_n - \Gamma t_n), \\ z_n &= \beta_n x_n + \gamma_n \hat{t}_n + \delta_n S \hat{t}_n, \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{aligned} \tag{4.10}$$

Now let us show $\text{Fix}(T) = \text{VI}(C, B_1)$. In fact, we have, for $\lambda > 0$,

$$\begin{aligned}
 u \in \text{VI}(C, B_1) &\iff \langle B_1 u, y - u \rangle \geq 0, \quad \forall y \in C \\
 &\iff \langle u - \lambda B_1 u - u, u - y \rangle \geq 0, \quad \forall y \in C \\
 &\iff u = P_C(u - \lambda B_1 u) \\
 &\iff u = P_C(u - \lambda u + \lambda T u) \\
 &\iff \langle u - \lambda u + \lambda T u - u, u - y \rangle \geq 0, \quad \forall y \in C \\
 &\iff \langle u - T u, u - y \rangle \leq 0, \quad \forall y \in C \\
 &\iff u = T u \\
 &\iff u \in \text{Fix}(T).
 \end{aligned} \tag{4.11}$$

Next let us show $\Omega = \text{Fix}(\Gamma)$. In fact, noticing that $M = 0$ and $\Phi = I - \Gamma$, we have

$$u \in \Omega \iff 0 \in \Phi(u) + Mu \iff 0 = \Phi(u) = u - \Gamma u \iff u \in \text{Fix}(\Gamma). \tag{4.12}$$

Consequently,

$$\text{Fix}(S) \cap \Omega \cap \Xi = \text{Fix}(S) \cap \text{Fix}(\Gamma) \cap \text{VI}(C, B_1) = \text{Fix}(S) \cap \text{Fix}(\Gamma) \cap \text{Fix}(T). \tag{4.13}$$

Therefore, by Theorem 3.1 we obtain the desired result. \square

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