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Research Article

Singular Initial Value Problem for a System of Integro-Differential Equations

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Analytical properties like existence, uniqueness, and asymptotic behavior of solutions are studied for the following singular initial value problem: $g_i(t)y_i'(t) = a_i y_i(t)(1 + f_i(t, \mathbf{y}(t), \int_{0^+}^t K_i(t, s, \mathbf{y}(t), \mathbf{y}(s))ds))$, $y_i(0^+) = 0$, $t \in (0, t_0]$, where $\mathbf{y} = (y_1, \dots, y_n)$, $a_i > 0$, $i = 1, \dots, n$ are constants and $t_0 > 0$. An approach which combines topological method of T. Ważewski and Schauder's fixed point theorem is used. Particular attention is paid to construction of asymptotic expansions of solutions for certain classes of systems of integrodifferential equations in a right-hand neighbourhood of a singular point.

1. Introduction and Preliminaries

Singular initial value problem for ordinary differential and integro-differential equations is fairly well studied (see, e.g., [1–16]), but the asymptotic properties of the solutions of such equations are only partially understood. Although the singular initial value problems were widely considered using various methods (see, e.g., [1–13, 16]), our approach to this problem is essentially different from others known in the literature. In particular, we use a combination of the topological method of T. Ważewski [8] and Schauder's fixed point theorem [11]. Our technique leads to the existence and uniqueness of solutions with asymptotic estimates in the right-hand neighbourhood of a singular point. Asymptotic expansions of solutions are constructed for certain classes of systems of integrodifferential equations as well.

Consider the following problem:

$$g_i(t)y_i'(t) = a_i y_i(t)\left(1 + f_i\left(t, \mathbf{y}(t), \int_{0^+}^t K_i(t, s, \mathbf{y}(t), \mathbf{y}(s))ds\right)\right), \tag{1.1}$$

$$y_i(0^+) = 0, \quad t \in (0, t_0], \tag{1.2}$$

where $\mathbf{y} = (y_1, \dots, y_n)$, $a_i > 0$ are constants, $f_i \in C^0(J \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, $K_i \in C^0(J \times J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $J = (0, t_0]$, $t_0 > 0$, $i = 1, \dots, n$.

Denote

(i) $f(t) = O(g(t))$ as $t \to 0^+$ if there is a right-hand neighbourhood $\mathcal{U}(0)$ and a constant $K > 0$ such that $(f(t)/g(t)) \le K$ for $t \in \mathcal{U}(0)$.

(ii) $f(t) = o(g(t))$ as $t \to 0^+$ if there is valid $\lim_{t \to 0^+} f(t)/g(t) = 0$.

(iii) $f(t) \sim g(t)$ as $t \to 0^+$ if there is valid $\lim_{t \to 0^+} f(t)/g(t) = 1$.

Definition 1.1. The sequence of functions $(\phi_n(t))$ is called an asymptotic sequence as $t \to 0^+$ if

$$\phi_{n+1}(t) = o(\phi_n(t)) \quad \text{as } t \to 0^+ \tag{1.3}$$

for all n.

Definition 1.2. The series $\sum c_n \phi_n(t)$, $c_n \in \mathbb{R}$, is called an asymptotic expansion of the function $f(t)$ up to Nth term as $t \to 0^+$ if

(a) $(\phi_n(t))$ is an asymptotic sequence,

(b)

$$\left[f(t) - \sum_{n=1}^{N} c_n \phi_n(t) \right] = o(\phi_N(t)), \quad \text{as } t \to 0^+. \tag{1.4}$$

The functions g_i, f_i, and K_i will be assumed to satisfy the following:

(i) $g_i(t) \in C^1(J)$, $g_i(t) > 0$, $g_i(0^+) = 0$, $g_i'(t) \sim \varphi_i(t) g_i^{\lambda_i}(t)$ as $t \to 0^+$, $\lambda_i > 0$, $\varphi_i(t) g_i^{\tau}(t) = o(1)$ as $t \to 0^+$ for each $\tau > 0$, $i = 1, \dots, n$,

(ii) $|f_i(t, u, v)| \le |u| + |v|$, $|\int_{0^+}^{t} K_i(t, s, \mathbf{y}(t), \mathbf{y}(s)) ds| \le r_i(t)|\mathbf{y}|$, $0 < r_i(t) \in C(J)$, $r_i(t) = \varphi_i(t, C_i) o(1)$ as $t \to 0^+$ where $\varphi_i(t, C_i) = C_i \exp(\int_{t_0}^{t} (a_i / g_i(s)) ds)$ is the general solution of the equation $g_i(t) y_i'(t) = a_i y_i(t)$.

In the text, we will apply topological method of Ważewski and Schauder's theorem. Therefore we give a short summary of them.

Let $f(t, \mathbf{y})$ be a continuous function defined on an open $(t, \mathbf{y})$ set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, Ω^0 an open set of Ω, $\partial\Omega^0$ the boundary of Ω^0, and $\overline{\Omega}^0$ the closure of Ω^0. Consider the following system of ordinary differential equations:

$$\mathbf{y}' = f(t, \mathbf{y}). \tag{1.5}$$

Definition 1.3 (see [17]). The point $(t_0, \mathbf{y}_0) \in \Omega \cap \partial\Omega^0$ is called an egress (or an ingress point) of Ω^0 with respect to system (1.5) if for every fixed solution of the problem $\mathbf{y}(t_0) = \mathbf{y}_0$, there exists an $\epsilon > 0$ such that $(t, \mathbf{y}(t)) \in \Omega^0$ for $t_0 - \epsilon \le t < t_0$ $(t_0 < t \le t_0 + \epsilon)$. An egress point (ingress point) $(t_0, \mathbf{y}_0)$ of Ω^0 is called a strict egress point (strict ingress point) of Ω^0 if $(t, \mathbf{y}(t)) \notin \overline{\Omega}^0$ on interval $t_0 < t \le t_0 + \epsilon_1$ $(t_0 - \epsilon_1 \le t < t_0)$ for an ϵ_1.

Definition 1.4 (see [18]). An open subset Ω^0 of the set Ω is called an (u, v) subset of Ω with respect to system (1.5) if the following conditions are satisfied.

(1) There exist functions $u_i(t, \mathbf{y}) \in C^1(\Omega, \mathbb{R})$, $i = 1, \ldots, m$ and $v_j(t, \mathbf{y}) \in C[\Omega, \mathbb{R}]$ $j = 1, \ldots, n, m + n > 0$ such that

$$\Omega_0 = \{(t, \mathbf{y}) \in \Omega : u_i(t, \mathbf{y}) < 0, v_j(t, \mathbf{y}) < 0 \ \forall i, j\}. \tag{1.6}$$

(2) $\dot{u}_\alpha(t, \mathbf{y}) < 0$ holds for the derivatives of the functions $u_\alpha(t, \mathbf{y})$, $\alpha = 1, \ldots, m$ along trajectories of system (1.5) on the set

$$U_\alpha = \{(t, \mathbf{y}) \in \Omega : u_\alpha(t, \mathbf{y}) = 0, u_i(t, \mathbf{y}) \le 0, v_j(t, \mathbf{y}) \le 0, \forall j \text{ and } i \ne \alpha\}. \tag{1.7}$$

(3) $\dot{v}_\beta(t, \mathbf{y}) > 0$ holds for the derivatives of the functions $v_\beta(t, \mathbf{y})$, $\beta = 1, \ldots, n$ along trajectories of system (1.5) on the set

$$V_\beta = \{(t, \mathbf{y}) \in \Omega : u_\beta(t, \mathbf{y}) = 0, u_i(t, \mathbf{y}) \le 0, v_j(t, y) \le 0, \forall i \text{ and} j \ne \beta\}. \tag{1.8}$$

The set of all points of egress (strict egress) is denoted by Ω_e^0 (Ω_{se}^0).

Lemma 1.5 (see [18]). *Let the set Ω_0 be a (u, v) subset of the set Ω with respect to system* (1.5). *Then*

$$\Omega_{\text{se}}^0 = \Omega_e^0 = \bigcup_{\alpha=1}^{m} U_\alpha \setminus \bigcup_{\beta=1}^{n} V_\beta. \tag{1.9}$$

Definition 1.6 (see [18]). Let X be a topological space and $B \subset X$.

Let $A \subset B$. A function $r \in C(B, A)$ such that $r(a) = a$ for all $a \in A$ is a retraction from B to A in X.

The set $A \subset B$ is a retract of B in X if there exists a retraction from B to A in X.

Theorem 1.7 (Ważewski's theorem [18]). *Let Ω^0 be some (u, v) subset of Ω with respect to system* (1.5). *Let S be a nonempty compact subset of $\Omega^0 \cup \Omega_e^0$ such that the set $S \cap \Omega_e^0$ is not a retract of S but is a retract Ω_e^0. Then there is at least one point $(t_0, \mathbf{y}_0) \in S \cap \Omega_0$ such that the graph of a solution $\mathbf{y}(t)$ of the Cauchy problem $\mathbf{y}(t_0) = \mathbf{y}_0$ for* (1.5) *lies on its right-hand maximal interval of existence.*

Theorem 1.8 (Schauder's theorem [19]). *Let E be a Banach space and S its nonempty convex and closed subset. If P is a continuous mapping of S into itself and PS is relatively compact then the mapping P has at least one fixed point.*

2. Main Results

Theorem 2.1. *Let assumptions (i) and (ii) hold, then for each $C_i \neq 0$ there is one solution $\mathbf{y}(t,\mathbf{C}) = (y_1(t,C_1), y_2(t,C_2),\ldots,y_n(t,C_n))$, $\mathbf{C} = (C_1,\ldots,C_n)$ of initial problem* (1.1) *and* (1.2) *such that*

$$\left|y_i^{(j)}(t,C_i) - \varphi_i^{(j)}(t,C_i)\right| \le \delta\left(\varphi_i^2(t,C_i)\right)^{(j)}, \quad j = 0,1, \tag{2.1}$$

for $t \in (0,t^{\Delta}]$, where $0 < t^{\Delta} \le t_0$, $\delta > 1$ is a constant, and t^{Δ} depends on δ, C_i, $i = 1,\ldots,n$.

Proof. (1) Denote E the Banach space of vector-valued continuous functions $\mathbf{h}(t)$ on the interval $[0,t_0]$ with the norm

$$\|\mathbf{h}(t)\| = \max_{t\in[0,t_0]} |h_i(t)|, \quad i = 1,\ldots,n. \tag{2.2}$$

The subset S of Banach space E will be the set of all functions $h(t)$ from E satisfying the inequality

$$\left|h_i(t) - \varphi_i(t,C_i)\right| \le \delta\varphi_i^2(t,C_i). \tag{2.3}$$

The set S is nonempty, convex, and closed.

(2) Now we will construct the mapping P. Let $\mathbf{h}_0(t) \in S$ be an arbitrary function. Substituting $\mathbf{h}_0(t)$, $\mathbf{h}_0(s)$ instead of $\mathbf{y}(t)$, $\mathbf{y}(s)$ into (1.1), we obtain the following differential equation:

$$g_i(t)y_i'(t) = a_i y_i(t)\left(1 + f_i\left(t, \mathbf{y}(t), \int_{0^+}^{t} K_i(t,s,\mathbf{h}_0(t),\mathbf{h}_0(s))ds\right)\right), \quad i = 1,\ldots,n. \tag{2.4}$$

Put

$$y_i(t) = \varphi_i(t,C_i) + \varphi_i^{(1-\mu)}(t,C_i)Y_{0i}(t), \tag{2.5}$$

$$y_i'(t) = \varphi_i'(t,C) + \frac{1}{g_i(t)}\varphi_i^{(1-\mu)}(t,C_i)Y_{1i}(t), \tag{2.6}$$

where $0 < \mu < 1$ is a constant and new functions $Y_{0i}(t)$, $Y_{1i}(t)$ satisfy the differential equations as

$$g_i(t)Y_{0i}'(t) = (\mu - 1)a_i Y_{0i}(t) + Y_{1i}(t), \quad i = 1,\ldots,n. \tag{2.7}$$

From (2.3), it follows

$$h_{0i}(t) = \varphi_i(t,C_i) + H_{0i}(t), \quad |H_{0i}(t)| \le \delta\varphi_i^2(t,C_i). \tag{2.8}$$

Substituting (2.5), (2.6), and (2.8) into (2.4), we get

$$\begin{aligned}
Y_{1i}(t) = a_i Y_{0i}(t) &+ \left(a_i\varphi_i^{\mu}(t,C_i) + a_i Y_{0i}(t)\right) \\
&\times f_i\Bigg(t, \varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)Y_{01}(t), \dots, \varphi_n(t,C_n) + \varphi_n^{(1-\mu)}(t,C_n)Y_{0n}(t)\ , \\
&\int_{0^+}^{t} K_i\big(t,s,\varphi_1(t,C_1) + H_{01}(t),\dots,\varphi_n(t,C_n) + H_{0n}(t), \varphi_1(s,C_1) \\
&+ H_{01}(s),\dots \varphi_n(s,C_n) + H_{0n}(s)\big)ds\Bigg).
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.7), we get

$$\begin{aligned}
g_i(t)Y'_{0i}(t) = \mu a_i Y_{0i}(t) &+ \left(a_i\varphi_i^{\mu}(t,C_i) + a_i Y_{0i}(t)\right) \\
&\times f_i\Bigg(t, \varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)Y_{01}(t), \dots, \varphi_n(t,C_n) + \varphi_n^{(1-\mu)}(t,C_n)Y_{0n}(t), \\
&\int_{0^+}^{t} K_i\big(t,s,\varphi_1(t,C_1) + H_{01}(t),\dots,\varphi_n(t,C_n) + H_{0n}(t), \varphi_1(s,C_1) \\
&+ H_{01}(s),\dots \varphi_n(s,C_n) + H_{0n}(s)\big)ds\Bigg).
\end{aligned} \tag{2.10}$$

In view of (2.5) and (2.6), it is obvious that a solution of (2.10) determines a solution of (2.4).

Now we use Ważewski's topological method. Consider an open set $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$. Denote $\mathbf{Y}_0 = (Y_{01},\dots,Y_{0n})$. Define an open subset $\Omega_0 \subset \Omega$ as follows:

$$\begin{aligned}
\Omega_0 &= \{(t,\mathbf{Y}_0) : u_i(t,\mathbf{Y}_0) < 0, v(t,\mathbf{Y}_0) < 0, i = 1,\dots,n\}, \\
U_\alpha &= \{(t,\mathbf{Y}_0) : u_\alpha(t,\mathbf{Y}_0) = 0, u_i(t,\mathbf{Y}_0) \le 0, v(t,\mathbf{Y}_0) \le 0, i = 1,\dots,n, i \ne \alpha\}, \\
V_\beta &= V = \left\{(t,\mathbf{Y}_0) : v(t,\mathbf{Y}_0) = 0, u_j(t,\mathbf{Y}_0) \le 0, i = 1,\dots,n\right\},
\end{aligned} \tag{2.11}$$

where

$$u_i(t,\mathbf{Y}_0) = Y_{0i}^2 - \left(\delta\varphi_i^{(1+\mu)}(t,C_i)\right)^2, \quad v(t,\mathbf{Y}_0) = t - t_0, \quad i = 1,\dots,n. \tag{2.12}$$

Calculating the derivatives $\dot{u}_\alpha(t, \mathbf{Y}_0)$, $\dot{v}(t, \mathbf{Y}_0)$ along the trajectories of (2.10) on the set U_α, V, $\alpha = 1, \dots, n$ we obtain

$$\begin{aligned}
\dot{u}_\alpha(t, \mathbf{Y}_0) = \frac{2a_\alpha}{g_\alpha(t)} \Bigg[& \mu Y_{0\alpha}^2(t) + \left(Y_{0\alpha}(t) \varphi_\alpha^\mu(t, C_\alpha) + Y_{0\alpha}^2(t) \right) \\
& \times f_\alpha \Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \dots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \\
& \int_{0^+}^t K_i(t, s, \varphi_1(t, C_1) + H_{01}(t), \dots, \varphi_n(t, C_n) \\
& + H_{0n}(t), \varphi_1(s, C_1) + H_{01}(s), \dots \varphi_n(s, C_n) + H_{0n}(s)) ds \Bigg). \\
& - \delta^2 (1+\mu) \varphi_\alpha^{2(1+\mu)}(t, C_\alpha) \Bigg].
\end{aligned} \tag{2.13}$$

Since

$$\begin{gathered}
\lim_{t \to +0} \varphi_i(t) g_i^\tau(t) = 0 \quad \text{for any } \tau > 0,\ i = 1, \dots, n \\
g_i'(t) \sim \varphi_i(t) g_i^{\lambda_i}(t) \quad \text{as } t \to 0^+,\ \lambda_i > 0,\ i = 1, \dots, n,
\end{gathered} \tag{2.14}$$

then there exists a positive constant M_i such that

$$g_i'(t) < M_i, \quad t \in (0, t_0],\ i = 1, \dots, n. \tag{2.15}$$

Consequently,

$$\int_{t_0}^t \frac{ds}{g_i(s)} < \frac{1}{M_i} \int_{t_0}^t \frac{g_i'(s) dt}{g_i(s)} = \frac{1}{M_i} \ln \frac{g_i(t)}{g_i(t_0)} \longrightarrow -\infty \quad \text{as } t \longrightarrow 0^+,\ i = 1, \dots, n. \tag{2.16}$$

From here $\lim_{t \to 0^+} \varphi_i(t, C_i) = 0$ and by L'Hospital's rule $\varphi_i^\tau(t, C_i) g_i^\sigma(t) = o(1)$, for $t \to 0^+$, $i = 1, \dots, n, \sigma$ is an arbitrary real number. These both identities imply that the powers of $\varphi_i(t, C_i)$ affect the convergence to zero of the terms in (2.13), in a decisive way.

Using the assumptions of Theorem 2.1 and the definition of $Y_0(t)$, $\varphi_i(t, C_i)$, $i = 1, \dots, n$, we get that the first term $\mu Y_{0\alpha}^2(t, C_\alpha)$ in (2.13) has the following form:

$$\mu Y_{0\alpha}^2(t) = \mu \delta^2 \varphi_\alpha^{2(1+\mu)}(t, C_\alpha), \tag{2.17}$$

and the second term

$$\begin{aligned}
&\left(Y_{0\alpha}(t)\varphi_\alpha^\mu(t,C_\alpha)+Y_{0\alpha}^2(t)\right)\\
&\quad\times f_\alpha\Big(t,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)Y_{01}(t),\dots,\varphi_n(t,C_n)\\
&\qquad+\varphi_n^{(1-\mu)}(t,C_n)Y_{0n}(t),\int_{0^+}^t K_\alpha(t,s,\varphi_1(t,C_1)+H_{01}(t),\dots,\varphi_n(t,C_n)\\
&\qquad\qquad+H_{0n}(t),\varphi_1(s,C_1)+H_{01}(s),\dots\varphi_n(s,C_n)\\
&\qquad\qquad+H_{0n}(s))ds\Big).
\end{aligned}\tag{2.18}$$

is bounded by terms with exponents which are greater than $\varphi_\alpha^{2(1+\mu)}(t,C_\alpha)$, $\alpha=1,\dots,n$. From here, we obtain

$$\operatorname{sgn}\dot{u}_\alpha(t,\mathbf{Y}_0)=-\delta^2(1+\mu)\varphi_\alpha^{2(1+\mu)}(t,C_\alpha)=-1\tag{2.19}$$

for sufficiently small t^*, depending on C_α $\alpha=1,\dots,n$, δ, $0<t^*\le t_0$.

It is obvious that sgn $\dot{v}(t,\mathbf{Y}_0)=1$.

Change the orientation of the axis t into opposite. Then, with respect to the new system of coordinates, the set Ω_0 is the (u,v) subset with respect to system (2.10). By Ważewski's topological method, we state that there exists at least one integral curve of (2.10) lying in Ω_0 for $t\in(0,t^*)$. It is obvious that this assertion remains true for an arbitrary function $\mathbf{h}_0(t)\in S$.

Now we prove the uniqueness of a solution of (2.10). Let $\overline{\mathbf{Y}}_0(t)=(\overline{Y}_{01}(t),\dots,\overline{Y}_{0n}(t))$ be also the solution of (2.10). Putting

$$Z_{0i}=Y_{0i}-\overline{Y}_{0i},\quad i=1,\dots,n\tag{2.20}$$

and substituting into (2.10), we obtain

$$\begin{aligned}
g_i(t)Z_{0i}'(t)&=\mu a_iY_{0i}(t)+\left(a_i\varphi_i^\mu(t,C_i)+a_iZ_{0i}(t)\right)\\
&\quad\times f_i\Bigg(t,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)\left(Z_{0i}(t)+\overline{Y}_{01}(t)\right),\dots,\varphi_n(t,C_n)\\
&\qquad+\varphi_n^{(1-\mu)}(t,C_n)\left(Z_{0n}(t)+\overline{Y}_{0n}(t)\right),\\
&\qquad\int_{0^+}^t K_i(t,s,\varphi_1(t,C_1)+H_{01}(t),\dots,\varphi_n(t,C_n)\\
&\qquad\qquad+H_{0n}(t),\varphi_1(s,C_1)+H_{01}(s),\dots\varphi_n(s,C_n)+H_{0n}(s))ds\Bigg).
\end{aligned}\tag{2.21}$$

Define

$$\begin{aligned}\Omega_1(\delta) &= \{(t,\mathbf{Z}_0) : 0 < t < t^*, u_{1i}(t,\mathbf{Z}_0) < 0, v_1(t,\mathbf{Z}_0) < 0, 0 < t < t^*, i = 1,\dots,n\}\\ U_{1\alpha} &= \{(t,\mathbf{Z}_0) : u_{1\alpha}(t,\mathbf{Z}_0) = 0, u_{1i}(t,\mathbf{Z}_0) \le 0, v_1(t,\mathbf{Z}_0) \le 0, i = 1,\dots,n, i \ne \alpha\},\\ V_{1\beta} = V &= \left\{(t,\mathbf{Z}_0) : v_1(t,\mathbf{Z}_0) = 0, u_j(t,\mathbf{Z}_0) \le 0, i = 1,\dots,n\right\},\end{aligned} \tag{2.22}$$

where

$$u_{1i}(t,\mathbf{Z}_0) = Z_{0i}^2 - \left(\delta\varphi_i^{(1+\mu-\gamma)}\right)^2, \quad 0 < \gamma < \mu,\ v_1(t,\mathbf{Z}_0) = t - t^*. \tag{2.23}$$

Using the same method as above, we have

$$\operatorname{sgn} \dot{u}_{1i}(t,\mathbf{Z}_0) = -1, \qquad \operatorname{sgn} \dot{v}_1(t,\mathbf{Z}_0) = 1, \quad i = 1,\dots,n \tag{2.24}$$

for sufficiently small $t^\diamond$, $0 < t^\diamond \le t^*$. It is obvious that $\Omega_0 \subset \Omega_1(\delta)$ for $t \in (0,t^\diamond)$. Let $\overline{\mathbf{Z}}_0(t) = (\overline{Z}_{01}(t),\dots,\overline{Z}_{0n}(t))$ be any nonzero solution of (2.10) such that $(t_1,\overline{\mathbf{Z}_0}(t_1)) \in \Omega_1$ for $0 < t_1 < t^\diamond$. Let $\overline{\delta} \in (0,\delta)$ be such a constant that $(t_1,\overline{\mathbf{Z}_0}(t_1)) \in \partial\Omega_1(\overline{\delta})$. If the curve $\overline{\mathbf{Z}_0}(t)$ lay in $\Omega_1(\overline{\delta})$ for $0 < t < t_1$, then $(t_1,\overline{\mathbf{Z}_0}(t_1))$ would have to be a strict egress point of $\partial\Omega_1(\overline{\delta})$ with respect to the original system of coordinates. This contradicts the relation (2.24). Therefore there exists only the trivial solution $\mathbf{Z}_0(t) \equiv 0$ of (2.21), so $\mathbf{Y}_0 = \overline{\mathbf{Y}_0}(t)$ is the unique solution of (2.10).

From (2.5) we obtain

$$\left|y_i(t,C_i) - \varphi_i(t,C_i)\right| \le \delta\varphi_i^2(t,C_i), \quad i = 1,\dots,n, \tag{2.25}$$

where $(y_1(t,C_1),\dots,y_n(t,C_n))$ is the solution of (2.4) for $t \in (0,t^\diamond]$. Similarly, from (2.6) and (2.9), we have

$$\begin{aligned}\left|y_i'(t,C_i) - \varphi_i'(t,C_i)\right| &= \left|\frac{1}{g_i(t)}\varphi_i^{(1-\mu)}(t,C_i)Y_{1i}(t)\right|\\ &\le \left|\frac{1}{g_i(t)}\varphi_i^{(1-\mu)}(t,C_i)2a_i\delta\varphi_i^{(1-\mu)}(t,C_i)\right| = \delta\left(\varphi_i^2(t,C_i)\right)'.\end{aligned} \tag{2.26}$$

It is obvious (after a continuous extension of $\mathbf{y}(t,\mathbf{C})$ for $t = 0$, $\mathbf{y}(0^+) = 0$) that $P : \mathbf{h}_0 \to \mathbf{y}$ maps S into itself and $PS \subset S$.

(3) We will prove that PS is relatively compact and P is a continuous mapping.

It is easy to see, by (2.25) and (2.26), that PS is the set of uniformly bounded and equicontinuous functions for $t \in [0,t^\diamond]$. By Ascoli's theorem, PS is relatively compact.

Let $\{\mathbf{h}_k(t)\}$ be an arbitrary sequence vector-valued functions in S such that

$$\|\mathbf{h}_k(t) - \mathbf{h}_0(t)\| = \epsilon_k, \qquad \lim_{k\to\infty}\epsilon_k = 0, \quad \mathbf{h}_0(t) \in S. \tag{2.27}$$

The solution $\overline{\mathbf{Y}}_k(t) = (\overline{Y}_{k1}, \ldots, \overline{Y}_{kn})$ of the following equation:

$$\begin{aligned} g_i(t) Y_{0i}'(t) = \mu a_i Y_{0i}(t) + \left(a_i \varphi_i^{\mu}(t, C_i) + a_i Y_{0i}(t) \right) \\ \times f_i \Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \ldots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \\ \int_{0^+}^{t} K_i(t, s, \varphi_1(t, C_1) + H_{01}(t), \ldots, \varphi_n(t, C_n) \\ + H_{0n}(t), \varphi_1(s, C_1) + H_{k1}(s), \ldots \varphi_n(s, C_n) + H_{kn}(s)) ds \Bigg). \end{aligned} \tag{2.28}$$

corresponds to the function $\mathbf{h}_k(t)$ and $\overline{\mathbf{Y}_k}(t) \in \Omega_0$ for $t \in (0, t^{\diamond})$. Similarly, the solution $\overline{\mathbf{Y}_0}(t)$ of (2.10) corresponds to the function $h_0(t)$. We will show that $|\overline{\mathbf{Y}_k}(t) - \overline{\mathbf{Y}_0}(t)| \to 0$ uniformly on $[0, t^{\Delta}]$, where $0 < t^{\Delta} \leq t^{\diamond}$, t^{Δ} is a sufficiently small constant which will be specified later. Consider the following region:

$$\Omega_{0k} = \left\{ (t, \mathbf{Y}_0) : 0 < t < t^{\diamond}, u_{0k_i}(t, \mathbf{Y}_0) < 0, v_0(t, \mathbf{Y}_0) < 0, i = 1, \ldots, n \right\}, \tag{2.29}$$

where

$$\begin{aligned} u_{0k_i}(t, \mathbf{Y}_0) = \left(Y_{0i}(t) - \overline{Y}_{0i}(t) \right)^2 - \left(\epsilon_k \varphi_i^{(1+\mu-\nu)}(t, C_i) \right)^2, \quad 0 < \nu < \alpha,\ i = 1, \ldots, n,\ k \geq 1, \\ v_0(t, \mathbf{Y}_0) = t - t^{\diamond}. \end{aligned} \tag{2.30}$$

There exists sufficiently small constant $t^{\Delta} \leq t^{\diamond}$ such that $\Omega_0 \subset \Omega_{0k}$ for any $k, t \in (0, t^{\Delta})$. Investigate the behaviour of integral curves of (2.28) with respect to the boundary $\partial \Omega_{0k}$, $t \in (0, t^{\Delta}]$. Using the same method as above, we obtain the following trajectory derivatives:

$$\operatorname{sgn} \dot{u}_{0k}(t, \mathbf{Y}_0) = -1, \qquad \operatorname{sgn} \dot{v}_0(t, \mathbf{Y}_0) = 1 \tag{2.31}$$

for $t \in (0, t^{\Delta}]$ and any k. By Ważewski's topological method, there exists at least one solution $\overline{\mathbf{Y}}_k(t)$ lying in Ω_{0k}, $0 < t < t^{\Delta}$. Hence, it follows that

$$\left| \overline{Y}_{ki}(t) - \overline{Y}_{0i}(t) \right| \leq \epsilon_k \varphi_i^{1+\mu-\nu} \leq N_i \epsilon_k, \tag{2.32}$$

$N_i > 0$, $i = 1, \ldots, n$ are constants depending on C_i, t^{Δ}. From (2.5), we obtain

$$\left| y_{ki}(t) - y_{0i}(t) \right| = \varphi_i^{(1-\mu)}(t, C_i) \left| \overline{Y}_{ki}(t) - \overline{Y}_{0i}(t) \right| \leq n_i \epsilon_k, \tag{2.33}$$

where $n_i > 0$, $i = 1, \ldots, n$ are constants depending on t^{Δ}, C_i, N_i. This estimate implies that P is continuous.

We have thus proved that the mapping P satisfies the assumptions of Schauder's fixed point theorem and hence there exists a function $\mathbf{h}(t) \in S$ with $\mathbf{h}(t) = P(\mathbf{h}(t))$. The proof of existence of a solution of (1.1) is complete.

Now we will prove the uniqueness of a solution of (1.1). Substituting (2.5) and (2.6) into (1.1), we get

$$\begin{aligned} Y_{1i}(t) = {} & a_i Y_{0i}(t) + \left(a_i \varphi_i^{\mu}(t, C_i) + a_i Y_{0i}(t)\right) \\ & \times f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \ldots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \\ & \int_{0^+}^{t} K_i\Big(t, s, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \ldots, \varphi_n(t, C_n) \\ & + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \varphi_1(s, C_1) + \varphi_1^{(1-\mu)}(s, C_1) Y_{01}(s), \ldots \varphi_n(s, C_n) \\ & + \varphi_n^{(1-\mu)}(s, C_n) Y_{0n}(s)\Big) ds\Bigg). \end{aligned} \tag{2.34}$$

Equation (2.7) may be written in the following form:

$$\begin{aligned} g_i(t) Y_{0i}'(t) = {} & a_i Y_{0i}(t) + \left(a_i \varphi_i^{\mu}(t, C_i) + a_i Y_{0i}(t)\right) \\ & \times f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \ldots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \\ & \int_{0^+}^{t} K_i\Big(t, s, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \ldots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) \\ & \times Y_{0n}(t), \varphi_1(s, C_1) + \varphi_1^{(1-\mu)}(s, C_1) Y_{01}(s), \ldots \varphi_n(s, C_n) \\ & + \varphi_n^{(1-\mu)}(s, C_n) Y_{0n}(s)\Big) ds\Bigg). \end{aligned} \tag{2.35}$$

Now we know that there exists the solution $\mathbf{y}_0(t) = (y_{01}(t, C_1), \ldots, y_{0n}(t, C_n))$ of (1.1) satisfying (1.2) such that

$$y_{0i}(t, C_i) = \varphi_i(t, C_i) + \varphi_i^{(1-\mu)}(t, C_i) T_{0i}(t), \quad i = 1, \ldots, n, \tag{2.36}$$

where $\mathbf{T}_0(t) = (T_{01}(t), \ldots, T_{0n}(t))$ is the solution of (2.35).

Denote $W_{i0}(t) = Y_{0i}(t) - T_{0i}(t)$, $i = 1, \dots, n$. Substituting $W_{i0}(t)$ into (2.35), we obtain

$$\begin{aligned}
g_i(t)W_{0i}'(t) = a_iW_{0i}(t) &+ \left(a_i\varphi_i^{\mu}(t,C_i) + a_iW_{0i}(t)\right)\\
&\times\Bigg[f_i\Bigg(t,\varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)(W_{01}(t) + T_{01}(t)), \dots, \varphi_n(t,C_n)\\
&\quad + \varphi_n^{(1-\mu)}(t,C_n)\times(W_{0n}(t) + T_{0n}(t)),\\
&\quad \int_{0^+}^{t} K_i\Big(t,s,\varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)(W_{01}(t) + T_{01}(t)), \dots, \varphi_n(t,C_n)\\
&\quad + \varphi_n^{(1-\mu)}(t,C_n)\times(W_{0n}(t) + T_{0n}(t)), \varphi_1(s,C_1) + \varphi_1^{(1-\mu)}(s,C_1)\\
&\quad \times(W_{01}(s) + T_{01}(s)), \dots \varphi_n(s,C_n)\\
&\quad + \varphi_n^{(1-\mu)}(s,C_n)(W_{0n}(s) + T_{0n}(s))\Big)ds\Bigg)\\
&- f_i\Bigg(t,\varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)T_{01}(t), \dots, \varphi_n(t,C_n) + \varphi_n^{(1-\mu)}(t,C_n),\\
&\quad \int_{0^+}^{t} K_i\Big(t,s,\varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)T_{01}(t), \dots, \varphi_n(t,C_n)T_{0n}(t)\\
&\quad + \varphi_n^{(1-\mu)}(t,C_n)T_{0n}(t), \varphi_1(s,C_1) + \varphi_1^{(1-\mu)}(s,C_1)\\
&\quad \times T_{01}(s), \dots \varphi_n(s,C_n) + \varphi_n^{(1-\mu)}(s,C_n)T_{0n}(s)\Big)ds\Bigg)\Bigg].
\end{aligned} \tag{2.37}$$

Let

$$^1\Omega_0 = \left\{(t,\mathbf{W}_0) : 0 < t < t^{\Delta}, u_{1i}(t,\mathbf{W}_0) < 0, v_1(t,\mathbf{W}_0) < 0\right\}, \tag{2.38}$$

where

$$u_{1i}(t,\mathbf{W}_0) = W_{1i}^2 - \left(\delta\varphi_i^{(1+\mu-\rho)}(t,C_i)\right)^2, \quad 0 < \rho < \mu, \quad v_1(t,\mathbf{W}_0) = t - t^{\Delta}, \quad i = 1, \dots, n. \tag{2.39}$$

If (2.37) had only the trivial solution lying in $^1\Omega_0$, then $\mathbf{Y}_0(t) = \mathbf{T}_0(t)$ would be only one solution of (2.37) and from here, by (2.35), $\mathbf{y}_0(t)$ would be only one solution of (1.1) satisfying (1.2) for $t \in (0, t^{\Delta}]$.

We will suppose that there exists nontrivial solution $\overline{\mathbf{W}}_0(t)$ of (2.37) lying in ${}^1\Omega_0$. Substituting $\overline{W}_{0i}(s)$ instead of $W_{0i}(s)$, $i = 1,\dots,n$ into (2.37), we obtain the following differential equation:

$$
\begin{aligned}
g_i(t)W'_{0i}(t) = a_iW_{0i}(t) &+ \left(a_i\varphi_i^{\mu}(t,C_i) + a_iW_{0i}(t)\right) \\
&\times \Bigg[f_i\Bigg(t,\varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)(W_{01}(t)+T_{01}(t)),\dots,\varphi_n(t,C_n) \\
&\qquad + \varphi_n^{(1-\mu)}(t,C_n)\times(W_{0n}(t)+T_{0n}(t)), \\
&\qquad \int_{0^+}^{t} K_i\Big(t,s,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)(W_{01}(t)+T_{01}(t)),\dots,\varphi_n(t,C_n) \\
&\qquad\quad + \varphi_n^{(1-\mu)}(t,C_n)\times(W_{0n}(t)+T_{0n}(t)),\varphi_1(s,C_1)+\varphi_1^{(1-\mu)}(s,C_1) \\
&\qquad\quad \times\left(\overline{W}_{01}(s)+T_{01}(s)\right),\dots\varphi_n(s,C_n) \\
&\qquad\quad + \varphi_n^{(1-\mu)}(s,C_n)\left(\overline{W}_{0n}(s)+T_{0n}(s)\right)\Big)ds\Bigg) \\
&\quad - f_i\Bigg(t,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)T_{01}(t),\dots,\varphi_n(t,C_n)+\varphi_n^{(1-\mu)}(t,C_n), \\
&\qquad \int_{0^+}^{t} K_i\Big(t,s,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)T_{01}(t),\dots,\varphi_n(t,C_n)T_{0n}(t) \\
&\qquad\quad + \varphi_n^{(1-\mu)}(t,C_n)T_{0n}(t),\varphi_1(s,C_1)+\varphi_1^{(1-\mu)}(s,C_1) \\
&\qquad\quad \times T_{01}(s),\dots\varphi_n(s,C_n)+\varphi_n^{(1-\mu)}(s,C_n)T_{0n}(s)\Big)ds\Bigg)\Bigg].
\end{aligned}
\tag{2.40}
$$

Calculating the derivative $\dot{u}_{1i}(t,\mathbf{W}_0)$ along the trajectories of (2.40) on the set $\partial^1\Omega_0$, we get $\operatorname{sgn}\ \dot{u}_{1i}(t,\mathbf{W}_0) = -1$ for $t \in (0,t^{\Delta}]$, $i = 1,\dots,n$.

By the same method as in the case of the existence of a solution of (1.1), we obtain that in ${}^1\Omega_0$ there is only the trivial solution of (2.40). The proof is complete. □

3. Asymptotic Expansions of Solutions

Diblík [3] investigated a singular initial problem for implicit ordinary differential equations and constructed asymptotic expansions of solutions in a right-hand neighbourhood of a singular point. Some results about asymptotic expansions of solutions for integrodifferential equations with separable kernels are given in [3, 10, 12].

The aim of this section is to show that results of paper [2] for ordinary differential equations are possible to extend on certain classes systems integrodifferential equations with a separable kernel in the following form:

$$g(t)y_i' = y_i + \int_{0^+}^{t} \left(\sum_{|\sigma_i|+|\omega_i|=2}^{N_i} u_{\sigma_i\omega_i}(t) v_{\sigma_i\omega_i}(s) \mathbf{y}^{\sigma_i}(t) \mathbf{y}^{\omega_i}(s) \right) ds, \tag{3.1}$$

where $N_i \in \mathbb{N}$, $\sigma_i = (l_{i1}, \ldots, l_{in})$, $\omega_i = (j_{i1}, \ldots, j_{in})$, $l_{ik}, j_{ik} \in \mathbb{N} \cup \{0\}$, $k = 1, \ldots, n$,

$$|\sigma_i| = \sum_{k=1}^{n} l_{ik}, \qquad |\omega_i| = \sum_{k=1}^{n} j_{ik}, \qquad \mathbf{y}^{\sigma_i}(t) = \prod_{k=1}^{n} y_k^{l_{ik}}(t), \qquad \mathbf{y}^{\omega_i}(s) = \prod_{k=1}^{n} y_k^{j_{ik}}(s),$$
$$u_{\sigma_i\omega_i}(t), v_{\sigma_i\omega_i}(t) \in C^0(J), \quad J = (0, t_0], \ i = 1, \ldots, n. \tag{3.2}$$

We will construct the solution of (3.1) in the form of one parametric asymptotic expansions as

$$y_i(t, C) = \sum_{h=1}^{\infty} f_{ih}(t) \phi^h(t, C), \quad i = 1, \ldots, n, \tag{3.3}$$

where $\phi(t, C)$ is the general solution of the differential equation $g(t)y' = y$ so that

$$\phi(t, C) = C \exp\left[\int_{t_0}^{t} \frac{d\tau}{g(\tau)} \right], \tag{3.4}$$

$f_{i1}(t) \equiv 1$, $f_{ih}(t)$, $h \geq 2$, $i = 1, \ldots, n$ are unknown functions, $C \neq 0$ is a constant.

Consider the following differential equation:

$$g(t)y' = qy + p(t). \tag{3.5}$$

Diblík [3] proved asymptotic estimates of the solution of (3.5) which we can be formulated as follows.

Theorem 3.1. *Assume that*

(I) *Let q be a constant, $q < 0$, $g(t) \in C^1(J)$, $g(t) > 0$, $\lim_{t \to t_0^+} g(t) = 0$, $g'(t) \sim \psi_1(t) g^{\lambda_1}(t)$ as $t \to t_0^+$, $\lambda_1 > 0$, $\lim_{t \to t_0^+} \psi_1(t) g^{\tau}(t) = 0$, τ is any positive number.*

(II) $p(t) \in C(J)$, $p(t) = b_0(t) g^{\lambda}(t) + O(b_1(t) g^{\lambda+\epsilon}(t))$, $\epsilon > 0$, $\lim_{t \to t_0^+} b_m(t) g'(t) = 0$, $m = 0, 1$, $b_0(t) \in C(J)$, $b_0(t) \neq 0$, $b_0'(t) \sim \psi_2(t) g^{\lambda_2}(t)$ *as* $t \to t_0^+$, $\lambda_2 + 1 > 0$, $\lim_{t \to t_0^+} \psi_2(t) g^{\tau}(t) = 0$, $\lim_{t \to t_0^+} g^{\tau}(t)(b_0(t))^{-1} = 0$.

Then (3.5) *has a unique solution on* $(0,\bar{t}]$, $0<\bar{t}\le t_0$, *satisfying asymptotic estimates*

$$y(x)=\frac{-1}{q}b_0(x)g^{\lambda}(x)+O(g^{\nu}(x)),\qquad y'(x)=O\Big(g^{\nu-1}(x)\Big),\tag{3.6}$$

where $\nu\in(\lambda,\lambda+\min\{\lambda_1,\lambda_2+1,\epsilon\})$.

Now we will show the results of Theorem 3.1. regarding only differential equation (3.5) we can apply to system of integrodifferential equations (3.1).

Substituting (3.3) into (3.1) and comparing the terms with the same powers of $\phi(t,C)$, we obtain the following system of recurrence equations:

$$g(t)f_{ih}'=(1-h)f_{ih}+\phi^{-h}(t,C)\int_{0^+}^{t}R_{ih}(t,s)ds,\tag{3.7}$$

$h\ge 2$, $i=1,\dots,n$ and

$$\begin{aligned}R_{ih}(t,s)=R_{ih}\big[&f_{11}(t),\dots,f_{ih-1}(t),\dots,f_{n1}(t),\dots,f_{nh-1}(t),\\&f_{11}(s),\dots,f_{1h-1}(s),\dots,f_{n1}(s),\dots,f_{nh-1}(s)\big].\end{aligned}\tag{3.8}$$

Denote

$$p_{ih}(t)=\phi^{-h}(t,C)\int_{0^+}^{t}R_{ih}(t,s)ds,\tag{3.9}$$

then it is obvious that the recurrence equations

$$g(t)f_{ih}'=(1-h)f_{ih}+p_{ih}(t)\tag{3.10}$$

$h\ge 2$, $i=1,\dots,n$ have the same form as (3.5) with the constant $q=1-h$. Hence we can apply Theorem 3.1, after the modification of assumption (II) of Theorem 3.1 for indices $h\ge 2$, $i=1,\dots,n$, to recurrence (3.10) which we will demonstrate with the following example.

Example 3.2. Consider the following system of integrodifferential equations:

$$\begin{aligned}t^2y_1'&=y_1+\int_{0^+}^{t}\frac{1}{t^3}y_1(s)y_2(s)ds,\\t^2y_2'&=y_2+\int_{0^+}^{t}\sqrt{t}y_1(t)y_2(s)ds.\end{aligned}\tag{3.11}$$

System (3.11) has the form of system (3.1) for

$$\begin{aligned}&\sigma_1\omega_1=(0,0,1,1),\qquad u_{\sigma_1\omega_1}(t)=\frac{1}{t^3},\qquad v_{\sigma_1\omega_1}(s)=1,\qquad N_1=2\\&\sigma_2\omega_2=(1,0,0,1),\qquad u_{\sigma_2\omega_2}(t)=\sqrt{t},\qquad v_{\sigma_2\omega_2}(s)=1,\qquad N_2=2.\end{aligned}\tag{3.12}$$

We will construct a solution of system (3.11) in the following form:

$$y_1 = \sum_{k=1}^{\infty} f_{1k}(t)\phi^k(t,C), \qquad y_2 = \sum_{k=1}^{\infty} f_{2k}(t)\phi^k(t,C), \tag{3.13}$$

where $\phi(t,C)$ is the general solution of the equation $t^2y' = y$. We will demonstrate the calculation of coefficients f_{ih} for $h = 3$. Substituting (3.13) in (3.11) and comparing the terms with the same powers of $\phi(t,C)$, we obtain the following system of recurrence equations:

$$\begin{aligned} \phi^1(t,C)&\colon 1 = 1, \\ &\quad 1 = 1. \end{aligned} \tag{3.14}$$

$$\begin{aligned} \phi^2(t,C)&\colon t^2 f_{12}' = -f_{12} + \phi^{-2}(t,C)\int_{0^+}^{t} \frac{1}{t^3}\phi^2(s,C)ds, \\ &\quad t^2 f_{22}' = -f_{22} + \phi^{-2}(t,C)\int_{0^+}^{t} \sqrt{t}\ \phi(t,C)\phi(s,C)ds. \end{aligned} \tag{3.15}$$

$$\begin{aligned} \phi^3(t,C)&\colon t^2 f_{13}' = -2f_{13} + \phi^{-3}(t,C)\int_{0^+}^{t} \frac{1}{t^3}\left[f_{12}(s) + f_{22}(s)\right]\phi^3(s,C)ds, \\ &\quad t^2 f_{23}' = -2f_{23} + \phi^{-3}(t,C)\int_{0^+}^{t} \sqrt{t}\Big[f_{12}(t)\phi^2(t,C)\phi(s,C) \\ &\qquad + f_{22}(s)\phi^2(s,C)\phi(t,C)\Big]ds. \end{aligned} \tag{3.16}$$

Put

$$u_1 = \phi^{-2}(t,C)\int_{0^+}^{t}\phi^2(s,C)ds, \qquad u_2 = \phi^{-1}(t,C)\int_{0^+}^{t}\phi(s,C)ds. \tag{3.17}$$

Differentiating both equations (3.17), we obtain the following differential equations:

$$t^2u_1' = -2u_1 + t^2, \tag{3.18}$$

$$t^2u_2' = -u_2 + t^2. \tag{3.19}$$

Equation (3.18) satisfies assumptions of Theorem 3.1. with following functions and coefficients:

$$\begin{aligned} &a = -2, \quad b_0(t) = 1, \quad g^{\lambda}(t) = \left(t^2\right)^1 \Rightarrow \lambda = 1, \quad b_1(t) = 0, \\ &g'(t) = \left(t^2\right)' = 2\left(g(t)\right)^{1/2} \Rightarrow \lambda_1 = \frac{1}{2}, \quad b_0'(t) = 0\cdot g^{\lambda_2}(t). \end{aligned} \tag{3.20}$$

Hence we can choose a constant $\lambda_2 + 1 > 1/2$ and similarly $\epsilon > 1/2$. By Theorem 3.1., we have

$$u_1 = \frac{1}{2}t^2 + O\left(t^{2\nu_1}\right), \qquad \nu_1 \in \left(1, \frac{3}{2}\right). \tag{3.21}$$

Second equation (3.19) is different from (3.18) only in the constant $a = -1$. Thus

$$u_2 = t^2 + O\left(t^{2\nu_2}\right), \qquad \nu_2 \in \left(1, \frac{3}{2}\right). \tag{3.22}$$

Substituting solutions (3.21) and (3.22) into (3.15) instead of integral terms, we obtain for unknown coefficients f_{12}, f_{22} the following differential equations:

$$t^2 f_{12}' = -f_{12} + \frac{1}{2t} + O\left(t^{2\nu_1 - 3}\right), \tag{3.23}$$

$$t^2 f_{22}' = -f_{22} + t^{5/2} + O\left(t^{2\nu_2 + 1/2}\right). \tag{3.24}$$

For (3.23), we can put

$$\begin{aligned} a &= -1, \qquad b_0(t) = \frac{1}{2}, \quad g^{\lambda}(t) = \left(t^2\right)^{-1/2}, \qquad \lambda = -\frac{1}{2}, \quad b_1(t) = 1, \\ \epsilon &= \nu_1 - 1, \qquad g'(t) = \left(t^2\right)' = 2\left(g(t)\right)^{1/2} \Rightarrow \lambda_1 = \frac{1}{2}, \qquad b_0'(t) = 0 \cdot g^{\lambda_2}(t). \end{aligned} \tag{3.25}$$

Then we can choose a constant $\lambda_2 + 1 > 1/2$. By Theorem 3.1., we get

$$f_{12}(t) = \frac{1}{2t} + O\left(t^{2\nu_{12}}\right), \qquad f_{12}'(t) = O\left(t^{2\nu_{12} - 2}\right), \qquad \nu_{12} \in \left(-\frac{1}{2}, 0\right). \tag{3.26}$$

Similarly for (3.24), we can put $a = -1$, $b_0(t) = 1$, $g^{\lambda}(t) = (t^2)^{5/4}$, $\lambda = 5/4$, $b_1(t) = 1$, $\epsilon = \nu_2 - 1$,

$$g'(t) = \left(t^2\right)' = 2\left(g(t)\right)^{1/2} \Longrightarrow \lambda_1 = \frac{1}{2}, \qquad b_0'(t) = 0 \cdot g^{\lambda_2}(t). \tag{3.27}$$

Then we can choose a constant $\lambda_2 + 1 > 1/2$. By Theorem 3.1., we have

$$f_{22}(t) = t^{5/2} + O\left(t^{2\nu_{22}}\right), \qquad f_{22}'(t) = O\left(t^{2\nu_{22} - 2}\right), \qquad \nu_{22} \in \left(\frac{5}{4}, \frac{7}{4}\right). \tag{3.28}$$

Substituting coefficients f_{12}, f_{22} into (3.16) and using the same method as in the calculation of coefficients f_{12}, f_{22}, we have

$$f_{13}(t) = \frac{1}{12t^2} + O\left(t^{2\nu_{13}}\right), \quad f_{13}'(t) = O\left(t^{2\nu_{13}-1}\right), \quad \nu_{13} \in \left(-1, -\frac{1}{2}\right),$$
$$f_{23}(t) = \frac{1}{4}t^{3/2} + O\left(t^{2\nu_{23}}\right), \quad f_{23}'(t) = O\left(t^{2\nu_{23}-1}\right), \quad \nu_{23} \in \left(\frac{3}{4}, \frac{5}{4}\right). \tag{3.29}$$

Thus the solution of system (3.11) has for $h = 3$ the following asymptotic expansions:

$$y_1 \approx \phi(t,C) + \left[\frac{1}{2t} + O\left(t^{2\nu_{12}}\right)\right]\phi^2(t,C) + \left[\frac{1}{12t^2} + O\left(t^{2\nu_{13}}\right)\right]\phi^3(t,C),$$
$$y_2 \approx \phi(t,C) + \left[t^{5/2} + O\left(t^{2\nu_{22}}\right)\right]\phi^2(t,C) + \left[\frac{1}{4}t^{3/2} + O\left(t^{2\nu_{23}}\right)\right]\phi^3(t,C). \tag{3.30}$$

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