Research Article

A New Extended Jacobi Elliptic Function Expansion Method and Its Application to the Generalized Shallow Water Wave Equation

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With the aid of symbolic computation, a new extended Jacobi elliptic function expansion method is presented by means of a new ansatz, in which periodic solutions of nonlinear evolution equations, which can be expressed as a finite Laurent series of some 12 Jacobi elliptic functions, are very effective to uniformly construct more new exact periodic solutions in terms of Jacobi elliptic function solutions of nonlinear partial differential equations. As an application of the method, we choose the generalized shallow water wave (GSWW) equation to illustrate the method. As a result, we can successfully obtain more new solutions. Of course, more shock wave solutions or solitary wave solutions can be gotten at their limit condition.

1. Introduction

In recent years, the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, and so forth. With the development of soliton theory, There has been a great amount of activities aiming to find methods for exact solutions of nonlinear differential equations, such as Bäcklund transformation, Darboux transformation, Cole-Hopf transformation, similarity reduction method, variable separation approach, Exp-function method, homogeneous balance method, varied tanh methods, and varied Jacobi elliptic function methods [1–21].

Among those, the direct ansätz method [8–21] provides a straightforward and effective algorithm to obtain such particular solutions for a large number of nonlinear partial differential equations, in which the starting point is the ansätz that the solution sought is

expressible as a finite series of special function, such as tanh function, sech function, tan function, sec function, sine-cosine function, Weierstrass elliptic function, theta function, and Jacobi elliptic function.

In this paper, a new Jacobi elliptic function expansion method is presented by means of a new general ansätz and is more powerful to uniformly construct more new exact doubly-periodic solutions in terms of Jacobi elliptic functions of nonlinear partial differential equations. The algorithm and its application are demonstrated later.

This paper is organized as follows. In Section 2, we summarize the extended Jacobi elliptic function expansion method. In Section 3, we apply the extended method to the generalized shallow water wave (GSWW) equation and bring out more new solutions. Conclusions will be presented in finally.

2. Summary of the Extended Jacobi Elliptic Function Expansion Method

In the following, we would like to outline the main steps of our extended method.

Step 1. For a given nonlinear partial differential equation with some physical fields $u(x, y, t)$ in three variables x, y, t,

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, u_{xy}, \ldots) = 0, \qquad (2.1)$$

by using the wave transformation

$$u(x, y, t) = u(\xi), \quad \xi = k(x + ly - ct), \qquad (2.2)$$

where k, l, and c are constants to be determined later. Then the nonlinear partial differential equation (2.1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G(u, u', u'', \ldots) = 0. \qquad (2.3)$$

Step 2. We introduce some new ansätz in terms of finite Jacobi elliptic function expansion in the following forms:

(1) $\mathrm{sn}'\xi/\mathrm{sn}\xi$ expansion:

$$u = a_0 + \sum_{i=1}^{n} a_i \left(\frac{\mathrm{sn}'(\xi)}{\mathrm{sn}(\xi)} \right)^i + b_i \left(\frac{\mathrm{sn}'(\xi)}{\mathrm{sn}(\xi)} \right)^{-i}, \qquad (2.4a)$$

(2) $\mathrm{cn}'\xi/\mathrm{cn}\xi$ expansion:

$$u = a_0 + \sum_{i=1}^{n} a_i \left(\frac{\mathrm{cn}'(\xi)}{\mathrm{cn}(\xi)} \right)^i + b_i \left(\frac{\mathrm{cn}'(\xi)}{\mathrm{cn}(\xi)} \right)^{-i}, \qquad (2.4b)$$

(3) $dn'\xi/dn\xi$ expansion:

$$u = a_0 + \sum_{i=1}^{n} a_i \left(\frac{dn'(\xi)}{dn(\xi)} \right)^i + b_i \left(\frac{dn'(\xi)}{dn(\xi)} \right)^{-i}, \tag{2.4c}$$

where $sn\xi$, $cn\xi$, $dn\xi$ are Jacobi elliptic sine function, the Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind, which are periodic and possess the following properties:

(1) properties of triangular functions

$$cn^2\xi + sn^2\xi = dn^2\xi + m^2 sn^2\xi = 1, \tag{2.5}$$

(2) derivatives of the Jacobi elliptic functions

$$sn'\xi = cn\xi dn\xi, \qquad cn'\xi = -sn\xi dn\xi, \qquad dn'\xi = -m^2 sn\xi cn\xi, \tag{2.6}$$

where m is a modulus. The Jacobi-Glaisher functions for elliptic function can be found in [22, 23]. It is necessary to point out that above combinations only require solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed. Therefore, other Jacobi elliptic functions can be chosen to combine as new ansätz. For simplicity, we omit them here.

Step 3. In order to obtain the value of n in (2.4a)–(2.4c), we define the degree of $u(\xi)$ as $D[u(\xi)] = n$, which gives rise to the degree of other expressions as

$$D\left[u^{(\alpha)}\right] = n + \alpha, \qquad D\left[u^\beta \left(u^{(\alpha)}\right)^s\right] = n\beta + (\alpha + n)s. \tag{2.7}$$

Therefore, we can get the value of n in (2.4a)–(2.4c). If n is a nonnegative integer, then we first make the transformation $u = \omega^n$.

Step 4. Respectively substitute three cases of (2.4a)–(2.4c) into (2.3) along with (2.5) and (2.6) and then respectively set all coefficients of $sn^i\xi cn^j\xi dn^k\xi$ $(i = 0, 1, 2, \ldots; j = 0, 1; k = 0, 1)$ to be zero to get an overdetermined system of nonlinear algebraic equations with respect to a_0, a_i, b_i, k, l, c $(i = 0, 1, 2, \ldots)$.

Step 5. By use of the *Maple* software package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [24], solving the overdetermined algebraic equations, we would end up with the explicit expressions for a_0, a_i, b_i, k, l, c $(i = 0, 1, 2, \ldots)$. In this way, we can get periodic solutions with Jacobi elliptic function.

Since

$$\lim_{m \to 1} sn\xi = \tanh\xi, \qquad \lim_{m \to 1} cn\xi = sech\xi, \qquad \lim_{m \to 1} dn\xi = sech\xi,$$

$$\lim_{m \to 0} sn\xi = \sin\xi, \qquad \lim_{m \to 0} cn\xi = \cos\xi, \qquad \lim_{m \to 0} dn\xi = 1, \tag{2.8}$$

u degenerate respectively as in the following form:

(1) solitary wave solutions:

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \text{sech}^i \xi \text{csch}^i \xi + b_i \cosh^i \xi \sinh^i \xi, \tag{2.9a}$$

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i(-\tanh \xi)^i + b_i(-\coth \xi)^i, \tag{2.9b}$$

(2) triangular function formal solutions:

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \cot^i \xi + b_i \tan^i \xi, \tag{2.10a}$$

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i(-\tan \xi)^i + b_i(-\cot \xi)^i. \tag{2.10b}$$

So the new extended Jacobi elliptic function expansion method can obtain more new solutions, which contain solitary wave solutions, singular solitary solutions, and triangular function solutions. These new solutions cannot be obtained by other Jacobi elliptic function methods, such as Fu et al. [13], Fan [14], Yan [15–17], and Chen et al. [18–21].

3. Exact Solutions of the GSWW Equation

The Boussinesq approximation theory [25] for generalized classical shallow water wave leads to the GSWW equation:

$$v_{xxxt} + \alpha v_x v_{xt} + \beta v_t v_{xx} - v_{xt} - v_{xx} = 0, \tag{3.1}$$

where α and β are nonzero constants. Using $v_x = u$ in [26], (3.1) can be simplified to

$$u_{xxt} + \alpha u u_t - \beta u_x \partial_x^{-1} u_t - u_t - u_x = 0, \tag{3.2}$$

which has recently attracted many attentions from researchers under the following cases.
 (1) For the case when $\alpha = 2\beta$, the following AKNS-SWW equation:

$$u_{xxt} + 2\beta u u_t - \beta u_x \partial_x^{-1} u_t - u_t - u_x = 0 \tag{3.3}$$

was discussed by Ablowitz et al. [27].
 (2) For the case when $\alpha = \beta$, the following equation:

$$u_{xxt} + \beta u u_t - \beta u_x \partial_x^{-1} u_t - u_t - u_x = 0 \tag{3.4}$$

was discussed by Hirota and Satsuma [28].

The GSWW equation (3.1) in potential form was studied by Clarkson and Mansfield [26, 29] who gave a complete catalog of classical and nonclassical symmetry reductions. The necessary conditions of Painlevé tests were given by Weiss et al. [30] and the complete integrability of (3.1) for the case when $\alpha = 2\beta$ and $\alpha = \beta$ was established by Ablowitz et al. [31]. It has been proven by Hietarinta [32] that the GSWW equation (3.2) can be expressed in Hirota's bilinear form and when $\alpha = 2\beta$, (3.2) can be reduced to (3.3) or to (3.4) when $\alpha = \beta$. In other words, both (3.3) and (3.4) are solvable by using Hirota's bilinear method [33]. The N-soliton solutions for (3.3) and (3.4) had also been found using this technique [28]. Although there is no scaling transformation that can reduce (3.3) to (3.4), the classical methods of Lie, the nonclassical method of Bluman and Cole, and the direct method of Clarkson and Kruskal can be applied to solve the GSWW equation (3.2) to obtain some kinds of symmetry reductions [34, 35]. Recently, Elwakil found a lot of exact solutions by using the modified extended tanh-function method [36]. Yang and Hon employed a rational expansion to generalize Fan's method for exact travelling wave solutions for the generalized shallow water wave (GSWW) equation [37].

According to the above method, to seek travelling wave solutions of (3.1), we make the following transformation:

$$u(x,t) = U(\xi), \quad \xi = k(x - ct), \tag{3.5}$$

where k and c are constants to be determined later. Thus (3.1) becomes

$$-ck^2 U''' + c(\beta - \alpha)UU' + (c - 1)U' = 0. \tag{3.6}$$

Equation (3.6) is integrated at once yielding

$$-ck^2 U'' + \frac{c(\beta - \alpha)}{2}U^2 + (c - 1)U = 0. \tag{3.7}$$

For the following, without loss of generality, we can set the integration constant equal to zero.

Now we consider the system (3.7) in the above three cases, so that (2.4a)–(2.4c). According to **Step 2** and **Step 3** in **Section 2**, by balancing the nonlinear term U^2 with the highest-order differential term U'' in (3.7), we suppose that (3.7) has the following formal solutions.

3.1. $\mathrm{sn}'\xi/\mathrm{sn}\xi$ *Expansion*

Now we consider the ansätz (2.4a). For (3.7), the ansätz (2.4a) becomes

$$u = a_0 + a_1 \left(\frac{\mathrm{sn}'(\xi)}{\mathrm{sn}(\xi)} \right) + a_2 \left(\frac{\mathrm{sn}'(\xi)}{\mathrm{sn}(\xi)} \right)^2 + b_1 \left(\frac{\mathrm{sn}'(\xi)}{\mathrm{sn}(\xi)} \right)^{-1} + b_2 \left(\frac{\mathrm{sn}'(\xi)}{\mathrm{sn}(\xi)} \right)^{-2}, \tag{3.8}$$

where $a_0, a_1, a_2, b_1, b_2, k, l$ and c are constants to be determined later.

With the aid of Maple, substituting (3.8) along with (2.5) and (2.6) into (3.7), yields a set of algebraic equations for $\mathrm{sn}^i\xi\mathrm{cn}^j\xi\mathrm{dn}^k\xi$ ($i = 0, 1, 2, \ldots; j = 0, 1; k = 0, 1$). Setting the

coefficients of these terms $sn^i \xi cn^j \xi dn^k \xi$ to be zero yields a set of overdetermined algebraic equations with respect to a_0, a_1, a_2, b_1, b_2, k, l, and c.

By use of the Maple software package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [24], solving the overdetermined algebraic equations, we get the following results.

Case 1.

$$a_0 = -\frac{1 - 48ck^2 - c + 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)}, \quad a_1 = a_2 = b_1 = 0,$$

$$b_2 = \frac{3\left[64ck^2\left(-28ck^2 + \sqrt{768c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{4c^2k^2(\alpha - \beta)}, \tag{3.9}$$

$$m = \pm\frac{\sqrt{c\left[-28ck^2 + \sqrt{768c^2k^4 + (c-1)^2}\right]}}{2ck}.$$

Case 2.

$$a_0 = -\frac{1 - 48ck^2 - c - 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)}, \quad a_1 = a_2 = b_1 = 0,$$

$$b_2 = \frac{3\left[64ck^2\left(-28ck^2 - \sqrt{768c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{4c^2k^2(\alpha - \beta)}, \tag{3.10}$$

$$m = \pm\frac{\sqrt{c\left[-28ck^2 - \sqrt{768c^2k^4 + (c-1)^2}\right]}}{2ck}.$$

Case 3.

$$a_0 = -\frac{1 - 48ck^2 - c + 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{12k^2}{\alpha - \beta},$$

$$\tag{3.11}$$

$$m = \pm\frac{\sqrt{c\left[32ck^2 + \sqrt{768c^2k^4 + (c-1)^2}\right]}}{2ck}.$$

Case 4.

$$a_0 = -\frac{1 - 48ck^2 - c - 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{12k^2}{\alpha - \beta},$$

(3.12)

$$m = \pm\frac{\sqrt{c\left[32ck^2 - \sqrt{768c^2k^4 + (c-1)^2}\right]}}{2ck}.$$

Case 5.

$$a_0 = -\frac{2(1 + 12ck^2 - c) + \sqrt{-192c^2k^4 + (c-1)^2}}{2c(\alpha - \beta)}, \quad a_1 = b_1 = 0, \quad a_2 = -\frac{12k^2}{\alpha - \beta},$$

$$b_2 = \frac{3\left[16ck^2\left(8ck^2 + \sqrt{-192c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{64c^2k^2(\alpha - \beta)},$$

(3.13)

$$m = \pm\frac{\sqrt{c\left[8ck^2 + \sqrt{-192c^2k^4 + (c-1)^2}\right]}}{4ck}.$$

Case 6.

$$a_0 = -\frac{2(1 + 12ck^2 - c) - \sqrt{-192c^2k^4 + (c-1)^2}}{2c(\alpha - \beta)}, \quad a_1 = b_1 = 0, \quad a_2 = -\frac{12k^2}{\alpha - \beta},$$

$$b_2 = \frac{3\left[16ck^2\left(8ck^2 - \sqrt{-192c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{64c^2k^2(\alpha - \beta)},$$

(3.14)

$$m = \pm\frac{\sqrt{c\left[-28ck^2 - \sqrt{-192c^2k^4 + (c-1)^2}\right]}}{4ck}.$$

From (3.8) and Cases 1–6, we obtain the following solutions for (3.1).

Family 1. From (3.9), we obtain the following sn′ξ/snξ expansion solutions for the GSWW equation as follows:

$$
\begin{aligned}
u_1 = &-\frac{1 - 48ck^2 - c + 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} \\
&+ \frac{3\left[64ck^2\left(-28ck^2 + \sqrt{768c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{4c^2k^2(\alpha - \beta)}\mathrm{sc}^2(\xi)\mathrm{nd}^2(\xi),
\end{aligned}
\tag{3.15}
$$

where $m = \pm\sqrt{c[-28ck^2 + \sqrt{768c^2k^4 + (c-1)^2}]}/2ck$ and c and k are arbitrary constants.

Family 2. From (3.10), we obtain the following sn′ξ/snξ expansion solutions for the GSWW equation as follows:

$$
\begin{aligned}
u_2 = &-\frac{1 - 48ck^2 - c - 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} \\
&+ \frac{3\left[64ck^2\left(-28ck^2 - \sqrt{768c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{4c^2k^2(\alpha - \beta)}\mathrm{sc}^2(\xi)\mathrm{nd}^2(\xi),
\end{aligned}
\tag{3.16}
$$

where $m = \pm\sqrt{c[-28ck^2 - \sqrt{768c^2k^4 + (c-1)^2}]}/2ck$ and c and k are arbitrary constants.

Family 3. From (3.11), we obtain the following sn′ξ/snξ expansion solutions for the GSWW equation as follows:

$$
u_3 = -\frac{1 - 48ck^2 - c + 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\mathrm{cs}^2(\xi)\mathrm{dn}^2(\xi),
\tag{3.17}
$$

where $m = \pm\sqrt{c[32ck^2 + \sqrt{768c^2k^4 + (c-1)^2}]}/2ck$ and c and k are arbitrary constants.

Family 4. From (3.12), we obtain the following sn′ξ/snξ expansion solutions for the GSWW equation as follows:

$$
u_4 = -\frac{1 - 48ck^2 - c - 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\mathrm{cs}^2(\xi)\mathrm{dn}^2(\xi),
\tag{3.18}
$$

where $m = \pm\sqrt{c[32ck^2 - \sqrt{768c^2k^4 + (c-1)^2}]}/2ck$ and c and k are arbitrary constants.

Family 5. From (3.13), we obtain the following sn'ξ/snξ expansion solutions for the GSWW equation as follows:

$$u_5 = -\frac{2(1 + 12ck^2 - c) + \sqrt{-192c^2k^4 + (c-1)^2}}{2c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\text{cs}^2(\xi)\text{dn}^2(\xi)$$

$$+ \frac{3\left[16ck^2\left(8ck^2 + \sqrt{-192c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{64c^2k^2(\alpha - \beta)}\text{sc}^2(\xi)\text{nd}^2(\xi),$$

(3.19)

where $m = \pm\sqrt{c[8ck^2 + \sqrt{-192c^2k^4 + (c-1)^2}]}/4ck$ and c and k are arbitrary constants.

Family 6. From (3.14), we obtain the following sn'ξ/snξ expansion solutions for the GSWW equation as follows:

$$u_6 = -\frac{2(1 + 12ck^2 - c) - \sqrt{-192c^2k^4 + (c-1)^2}}{2c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\text{cs}^2(\xi)\text{dn}^2(\xi)$$

$$+ \frac{3\left[16ck^2\left(8ck^2 - \sqrt{-192c^2k^4 + (c-1)^2}\right) - (c-1)^2\right]}{64c^2k^2(\alpha - \beta)}\text{sc}^2(\xi)\text{nd}^2(\xi),$$

(3.20)

where $m = \pm\sqrt{c[-28ck^2 - \sqrt{-192c^2k^4 + (c-1)^2}]}/4ck$ and c and k are arbitrary constants.

Remark 3.1. Here we find that the modulus m of the Jacobi elliptic functions has relations with c and k in Cases 1–6. To further analyze their relations, we take the solutions (3.9), (3.11), and (3.13) as samples by three figures (see Figures 1, 2, and 3).

3.2. cn'ξ/cnξ *Expansion*

Now we consider the ansätz (2.4b). For (3.7), the ansätz (2.4b) becomes

$$u = a_0 + a_1\left(\frac{\text{cn}'(\xi)}{\text{cn}(\xi)}\right) + a_2\left(\frac{\text{cn}'(\xi)}{\text{cn}(\xi)}\right)^2 + b_1\left(\frac{\text{cn}'(\xi)}{\text{cn}(\xi)}\right)^{-1} + b_2\left(\frac{\text{cn}'(\xi)}{\text{cn}(\xi)}\right)^{-2},$$

(3.21)

where a_0, a_1, a_2, b_1, b_2, k, l, and c are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following cn'ξ/cnξ expansion solutions.
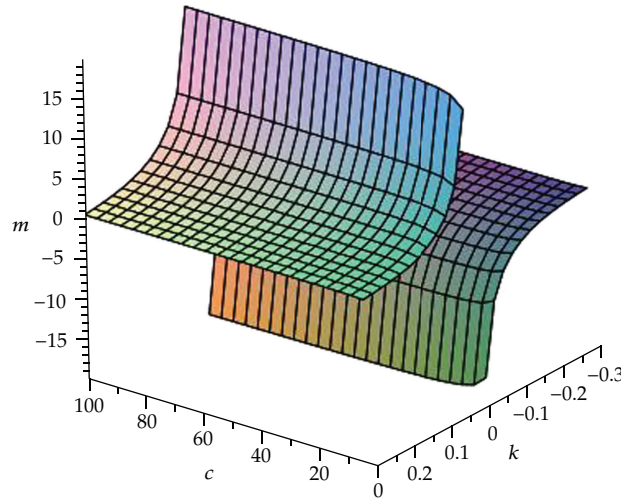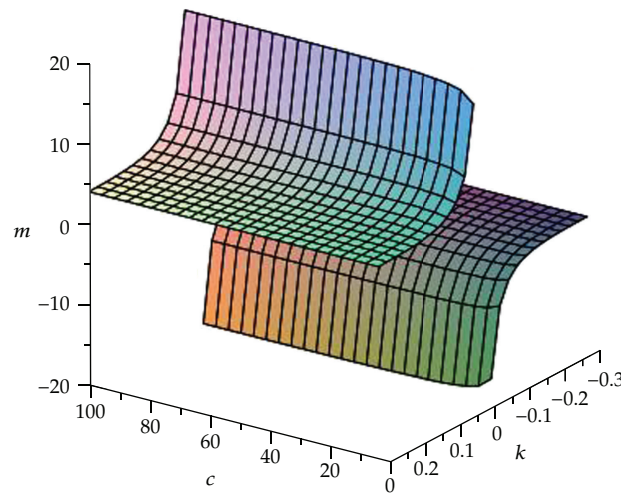
Figure 1: Figure of m in (3.9).

Figure 2: Figure of m in (3.11).

Family 7.

$$u_7 = \frac{c - 1 + \sqrt{48c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\mathrm{cs}^2(\xi)\mathrm{nd}^2(\xi), \tag{3.22}$$

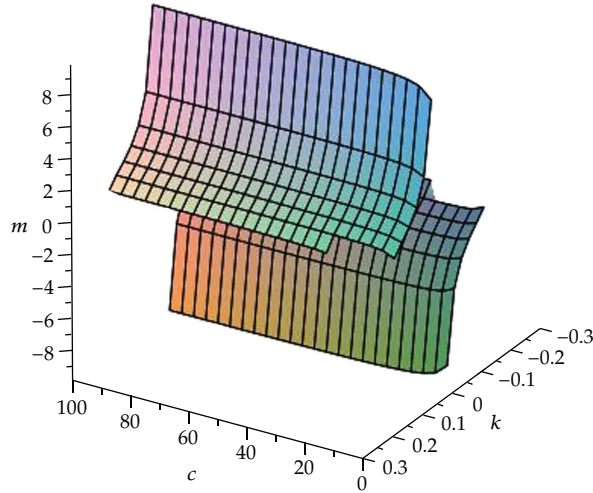where $m = \pm\sqrt{c(8ck^2 + \sqrt{48c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.

Figure 3: Figure of m in (3.13).

Family 8.

$$u_8 = \frac{c - 1 - \sqrt{48c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\mathrm{cs}^2(\xi)\mathrm{nd}^2(\xi), \tag{3.23}$$

where $m = \pm\sqrt{c(8ck^2 - \sqrt{48c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.

Family 9.

$$u_9 = \frac{c - 1 + \sqrt{48c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\mathrm{sc}^2(\xi)\mathrm{dn}^2(\xi), \tag{3.24}$$

where $m = \pm\sqrt{c(8ck^2 + \sqrt{48c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.

Family 10.

$$u_{10} = \frac{c - 1 - \sqrt{48c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\mathrm{sc}^2(\xi)\mathrm{dn}^2(\xi), \tag{3.25}$$

where $m = \pm\sqrt{c(8ck^2 - \sqrt{48c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.

Family 11.

$$u_{11} = \frac{c - 1 + \sqrt{-192c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\left(\text{sc}^2(\xi)\text{dn}^2(\xi) + \text{cs}^2(\xi)\text{nd}^2(\xi)\right), \qquad (3.26)$$

where $m = \pm\sqrt{c(8ck^2 + \sqrt{-192c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.

Family 12.

$$u_{12} = \frac{c - 1 - \sqrt{-192c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12k^2}{\alpha - \beta}\left(\text{sc}^2(\xi)\text{dn}^2(\xi) + \text{cs}^2(\xi)\text{nd}^2(\xi)\right), \qquad (3.27)$$

where $m = \pm\sqrt{c(8ck^2 - \sqrt{-192c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.

3.3. $\text{dn}'\xi/\text{dn}\xi$ *Expansion*

Now we consider the ansätz (2.4c). For (3.7), the ansätz (2.4c) becomes

$$u = a_0 + a_1\left(\frac{\text{dn}'(\xi)}{\text{dn}(\xi)}\right) + a_2\left(\frac{\text{dn}'(\xi)}{\text{dn}(\xi)}\right)^2 + b_1\left(\frac{\text{dn}'(\xi)}{\text{dn}(\xi)}\right)^{-1} + b_2\left(\frac{\text{dn}'(\xi)}{\text{dn}(\xi)}\right)^{-2}, \qquad (3.28)$$

where a_0, a_1, b_1, a_2, b_2, k, l, and c are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following $\text{dn}'\xi/\text{dn}\xi$ expansion solutions.

Family 13.

$$u_{13} = \frac{3\left[-1792c^2k^4 - (c-1)^2 - 64ck^2\sqrt{768c^2k^4 + (c-1)^2}\right]}{4m^4c^2k^2(\alpha - \beta)}\text{ds}^2(\xi)\text{nc}^2(\xi)$$

$$+ \frac{c - 1 - 48ck^2 - 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)}, \qquad (3.29)$$

where $m = \pm\sqrt{c(32ck^2 + \sqrt{768c^2k^4 + (c-1)^2})}/2ck$ and c and k are arbitrary constants.

Family 14.

$$
u_{14} = \frac{3\left[-1792c^2k^4 - (c-1)^2 + 64ck^2\sqrt{768c^2k^4 + (c-1)^2}\right]}{4m^4c^2k^2(\alpha - \beta)} \mathrm{ds}^2(\xi)\mathrm{nc}^2(\xi)
$$

$$
+ \frac{c - 1 - 48ck^2 + 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)},
$$

(3.30)

where $m = \pm\sqrt{c(32ck^2 - \sqrt{768c^2k^4 + (c-1)^2})}/2ck$ and c and k are arbitrary constants.

Family 15.

$$
u_{15} = \frac{c - 1 - 48ck^2 - 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12m^4k^2}{\alpha - \beta}\mathrm{sd}^2(\xi)\mathrm{cn}^2(\xi),
$$

(3.31)

where $m = \pm\sqrt{c(32ck^2 + \sqrt{768c^2k^4 + (c-1)^2})}/2ck$ and c and k are arbitrary constants.

Family 16.

$$
u_{16} = \frac{c - 1 - 48ck^2 + 2\sqrt{768c^2k^4 + (c-1)^2}}{c(\alpha - \beta)} - \frac{12m^4k^2}{\alpha - \beta}\mathrm{sd}^2(\xi)\mathrm{cn}^2(\xi),
$$

(3.32)

where $m = \pm\sqrt{c(32ck^2 - \sqrt{768c^2k^4 + (c-1)^2})}/2ck$ and c and k are arbitrary constants.

Family 17.

$$
u_{17} = \frac{2c - 2 + 24ck^2 - \sqrt{-192c^2k^4 + (c-1)^2}}{2c(\alpha - \beta)} - \frac{12m^4k^2}{\alpha - \beta}\mathrm{sd}^2(\xi)\mathrm{cn}^2(\xi)
$$

$$
+ \frac{3\left[-(c-1)^2 + 128c^2k^4 - 16ck^2\sqrt{-192c^2k^4 + (c-1)^2}\right]}{64m^4c^2k^2(\alpha - \beta)} \mathrm{ds}^2(\xi)\mathrm{nc}^2(\xi),
$$

(3.33)

where $m = \pm\sqrt{c(8ck^2 + \sqrt{-192c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.
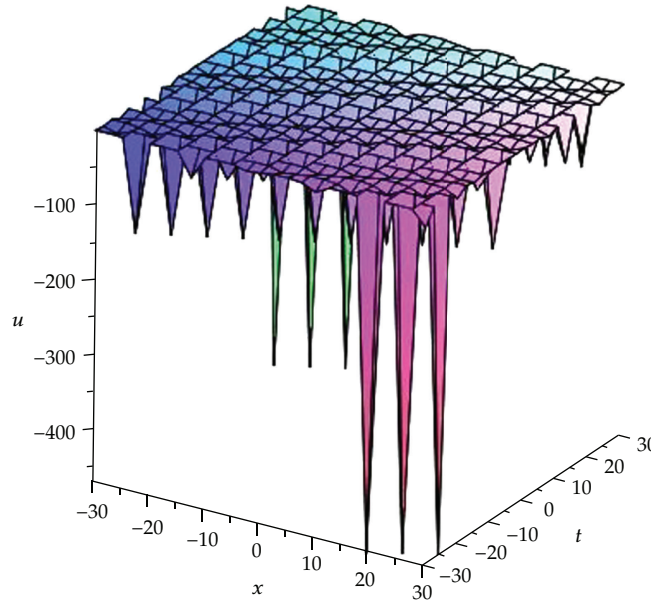
Figure 4: The solution u_1 of GSWW equation with $k = 0.3$, $c = 2$, $\alpha = 3$, $\beta = 1$.

Family 18.

$$u_{18} = \frac{2c - 2 + 24ck^2 + \sqrt{-192c^2k^4 + (c-1)^2}}{2c(\alpha - \beta)} - \frac{12m^4k^2}{\alpha - \beta}\mathrm{sd}^2(\xi)\mathrm{cn}^2(\xi)$$

$$+ \frac{3\left[-(c-1)^2 + 128c^2k^4 + 16ck^2\sqrt{-192c^2k^4 + (c-1)^2}\right]}{64m^4c^2k^2(\alpha - \beta)}\mathrm{ds}^2(\xi)\mathrm{nc}^2(\xi), \tag{3.34}$$

where $m = \pm\sqrt{c(8ck^2 - \sqrt{-192c^2k^4 + (c-1)^2})}/4ck$ and c and k are arbitrary constants.

Remark 3.2. The solutions obtained here, to our knowledge, are all new families of periodic solution of the GSWW equation.

Remark 3.3. In order to understand the significance of these solutions in Families 1 and 18, here we take the solutions (3.15), (3.17), (3.19), (3.22), (3.24), (3.26), (3.29), (3.31), and (3.33) as samples to further analyze their properties by some figures (see Figures 4, 5, 6, 7,8, 9, 10, 11, and 12).

4. Conclusion and More General Form

In short, we have presented the new extended Jacobi elliptic function expansion method. The GSWW equation is chosen to illustrate the method such that many families of new Jacobi
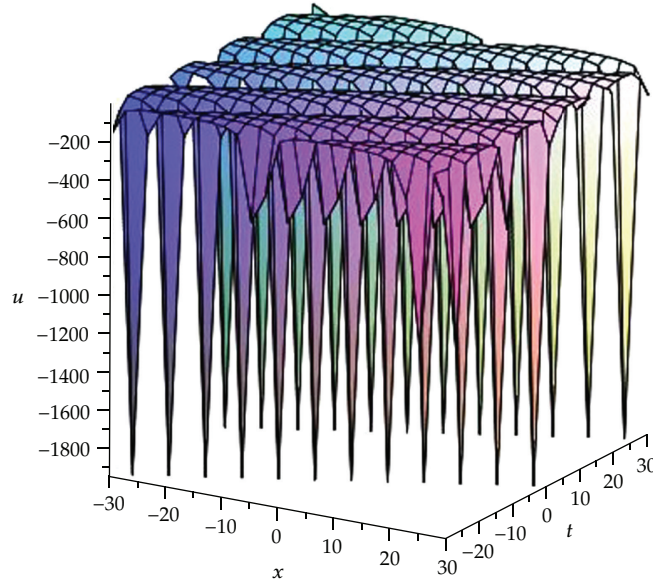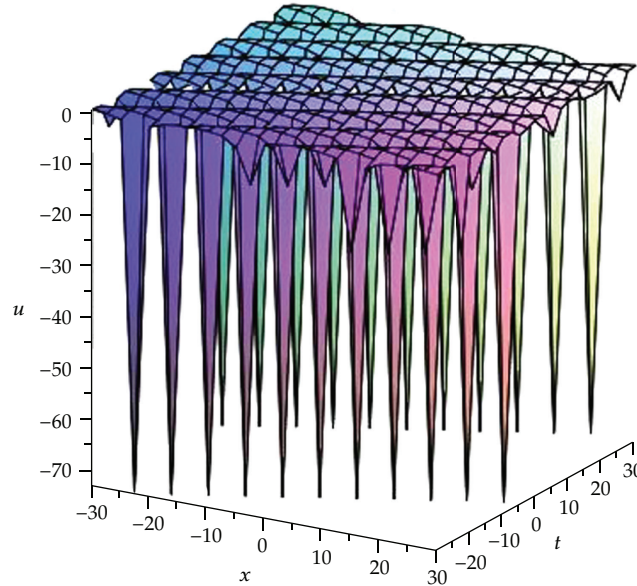
Figure 5: The solution u_3 of GSWW equation with $k = 0.3$, $c = 2$, $\alpha = 3$, $\beta = 1$.
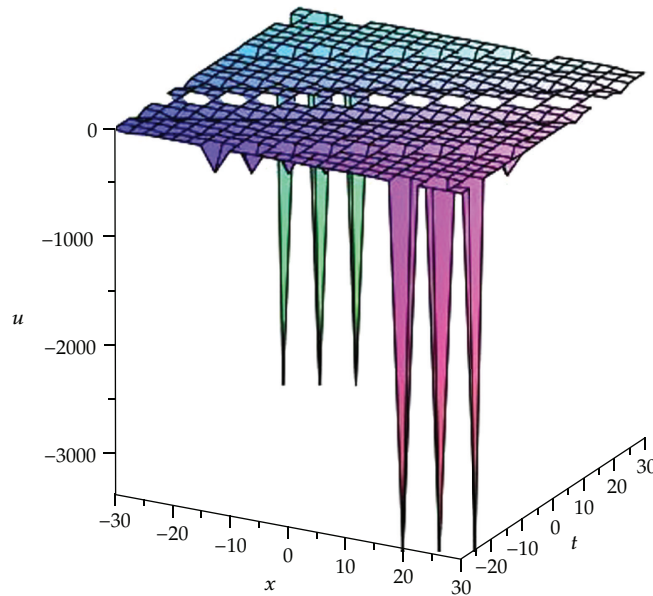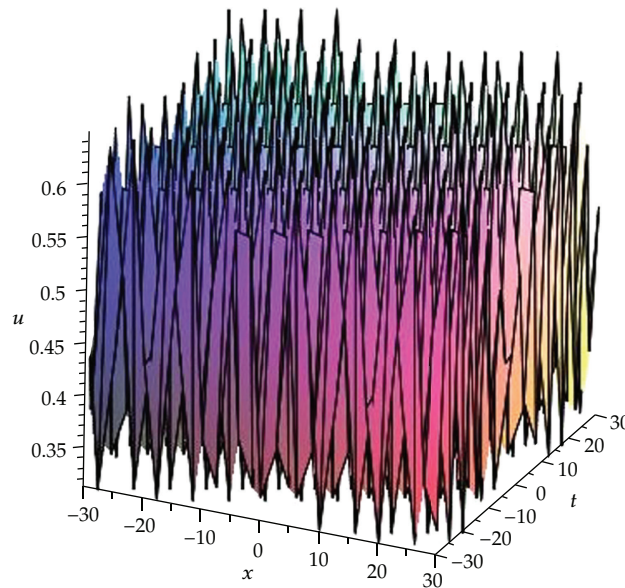
Figure 6: The solution u_5 of GSWW equation with $k = 0.1$, $c = 2$, $\alpha = 3$, $\beta = 1$.

elliptic function solutions are obtained. When the modulus $m \rightarrow 1$ or 0, some of these obtained solutions degenerate as solitary solutions or trigonometric function solutions. The algorithm can be applied to many nonlinear differential equations in mathematical physics. Other types of solutions of (2.1) need to be studied further.

It is easy to see that above-mentioned method is only applied to these nonlinear ODEs with constant coefficients or nonlinear partial differential equations, which can be reduced to

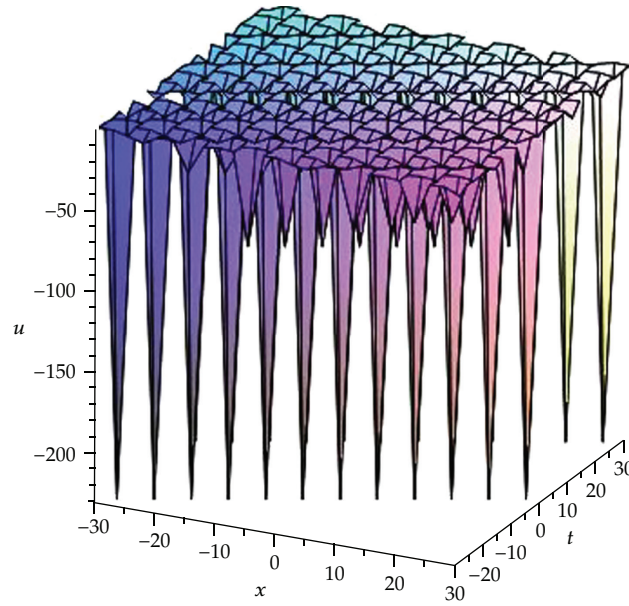**Figure 7:** The solution u_7 of GSWW equation with $k = 0.3$, $c = 2$, $\alpha = 3$, $\beta = 1$.

Figure 8: The solution u_9 of GSWW equation with $k = 0.3$, $c = 2$, $\alpha = 3$, $\beta = 1$.

the corresponding nonlinear ODEs with constant coefficients by using some transformations; otherwise, the method will not work. In order to overcome the disadvantage of the method, we change the method into a general form as follows.

If we do not reduce (2.1) to a nonlinear ODE (2.3) with constant coefficients, then we directly assume that (2.1) has the following solutions:
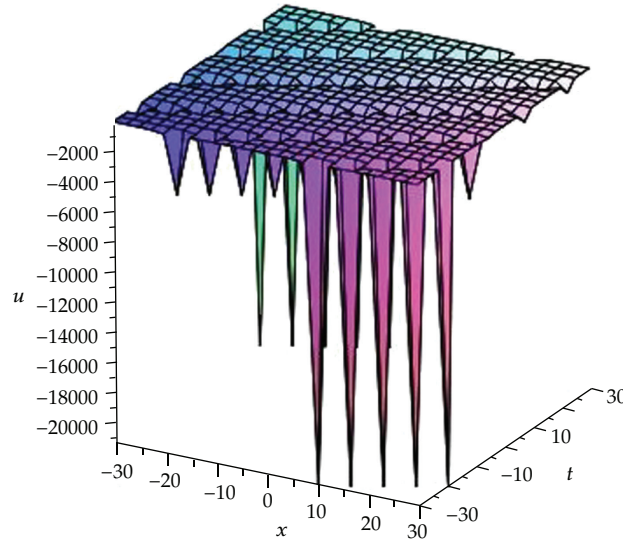
Figure 9: The solution u_{11} of GSWW equation with $k = 0.1$, $c = 2$, $\alpha = 3$, $\beta = 1$.
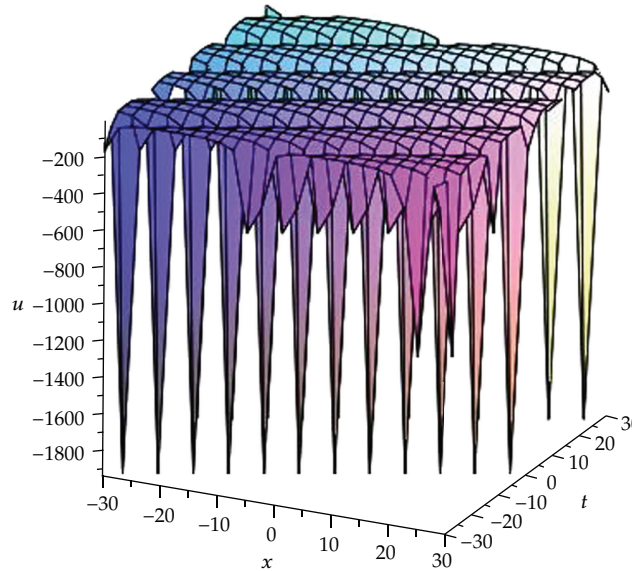
Figure 10: The solution u_{13} of GSWW equation with $k = 0.3$, $c = 2$, $\alpha = 3$, $\beta = 1$.

(1) $\mathrm{sn}'(\xi)/\mathrm{sn}(\xi)$ expansion:

$$u(x,t) = a_0(x,t) + \sum_{i=1}^{n}\left[a_i(x,t)\left(\frac{\mathrm{sn}'(\varphi(x,t))}{\mathrm{sn}(\varphi(x,t))}\right)^i + b_i(x,t)\left(\frac{\mathrm{sn}'(\varphi(x,t))}{\mathrm{sn}(\varphi(x,t))}\right)^{-i}\right], \qquad (4.1a)$$

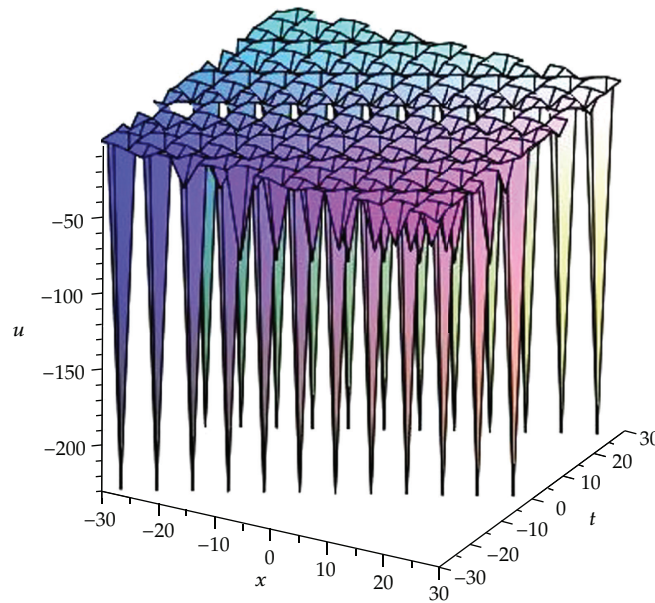**Figure 11:** The solution u_{15} of GSWW equation with $k = 0.3$, $c = 2$, $\alpha = 3$, $\beta = 1$.

Figure 12: The solution u_{17} of GSWW equation with $k = 0.1$, $c = 2$, $\alpha = 3$, $\beta = 1$.

(2) $\mathrm{cn}'(\xi)/\mathrm{cn}(\xi)$ expansion:

$$u(x,t) = a_0(x,t) + \sum_{i=1}^{n} \left[a_i(x,t) \left(\frac{\mathrm{cn}'(\varphi(x,t))}{\mathrm{cn}(\varphi(x,t))} \right)^{i} + b_i(x,t) \left(\frac{\mathrm{cn}'(\varphi(x,t))}{\mathrm{cn}(\varphi(x,t))} \right)^{-i} \right], \tag{4.1b}$$

(3) $\mathrm{dn}'\xi/\mathrm{dn}\xi$ expansion:

$$u(x,t) = a_0(x,t) + \sum_{i=1}^{n}\left[a_i(x,t)\left(\frac{\mathrm{dn}'(\varphi(x,t))}{\mathrm{dn}(\varphi(x,t))}\right)^i + b_i(x,t)\left(\frac{\mathrm{dn}'(\varphi(x,t))}{\mathrm{dn}(\varphi(x,t))}\right)^{-i}\right], \qquad (4.1c)$$

where $a_0(x,t)$, $a_i(x,t)$, $b_i(x,t)$ $(i = 1,2,\ldots,n)$, and $\varphi(x,t)$ are functions to be determined later. Substituting (4.1a), (4.1b), and (4.1c) with (2.5) and (2.6) into (2.1) yields a set of nonlinear partial differential equations with respect to these unknown functions $a_0(x,t)$, $a_i(x,t)$, $b_i(x,t)$ $(i = 1,2,\ldots,n)$, and $\varphi(x,t)$, respectively. If we can solve these functions from the obtained set of nonlinear partial differential equations, then we may obtain more types of doubly periodic solutions of (2.1). The more general form is better applied to nonlinear differential equations with variable coefficients. Particularly, if we only deduce the conclusions that all these function $a_0(x,t)$, $a_i(x,t)$, $b_i(x,t)$ $(i = 1,2,\ldots)$ are all constants and $\varphi(x,t)$ is of the form $k(x - ct)$ (k, c constants), then the obtained results are the same as the ones found by using the method presented in Section 2. About the applications of the more general form, we will give some examples in future.

Conflict of Interests

The authors declare that they have no conflict of interests.

Acknowledgments

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