**Research Article**

**Existence of Solutions for the $p(x)$-Laplacian Problem with the Critical Sobolev-Hardy Exponent**

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Received 18 February 2012; Accepted 11 July 2012

Academic Editor: Norimichi Hirano

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This paper deals with the $p(x)$-Laplacian equation involving the critical Sobolev-Hardy exponent. Firstly, a principle of concentration compactness in $W_{0}^{1,p(x)}(\Omega)$ space is established, then by applying it we obtain the existence of solutions for the following $p(x)$-Laplacian problem:

$$-\text{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p_*(x)-2}u = (h(x)|u|^{p(x)}2u/|x|^{q(x)}) + f(x, u), \quad x \in \Omega, \quad u = 0, \quad x \in \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 \in \Omega$, $1 < p^- \leq p(x) \leq p^+ < N$, and $f(x, u)$ satisfies $p(x)$-growth conditions.

1. Introduction

In this paper we are concerned with the following $p(x)$-Laplacian problem:

$$-\text{div} \left( |\nabla u|^{p(x)-2}\nabla u \right) + |u|^{p(x)-2}u = \frac{h(x)|u|^{p_*(x)-2}u}{|x|^{q(x)}} + f(x, u), \quad x \in \Omega, \quad u = 0, \quad x \in \partial \Omega,$$

where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded domain, $p(x)$ is Lipschitz continuous, radially symmetric on $\Omega$, and $1 < p^- \leq p(x) \leq p^+ < N$. $s(x)$ is Lipschitz continuous, radially symmetric on $\Omega$ and $0 \leq s(x) \ll p(x)$. $p_*(x) = ((N-s(x))/(N-p(x)))p(x)$ is the critical Sobolev-Hardy exponent, and $p_0^*(x) = Np(x)/(N-p(x)) = p^*(x)$ is the critical Sobolev exponent. Throughout this paper we assume the following:

(F-1) $f(x, t)$ satisfies the Carathéodory condition.

(F-2) $|f(x, t)| \leq c_1 + c_2|t|^{p(x)-1}$, $q : \Omega \rightarrow \mathbb{R}$ is measurable and satisfies $p(x) \ll q(x) \ll p_0^*(x)$ or $1 < q^- \leq q(x) \ll p(x)$, for any $x \in \Omega$. 

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Let \( p(x) \in \mathbb{P}(\Omega) \) and denote by \( \|u\|_p = \inf \left\{ \lambda > 0 : \int_\Omega \frac{|u|^p}{\lambda^p} \, dx \leq 1 \right\} \).

The variable exponent Lebesgue space \( L^{p(x)}(\Omega) \) is the class of functions \( u \) such that \( \int_\Omega |u(x)|^{p(x)} \, dx < \infty \). \( L^{p(x)}(\Omega) \) is a Banach space endowed with the norm (2.1).

For a given \( p(x) \in \mathbb{P}(\Omega) \), we define the conjugate function \( p'(x) \) as:

\[
p'(x) = \frac{p(x)}{p(x) - 1}.
\]

**Theorem 2.1.** Let \( p(x) \in \mathbb{P}(\Omega) \). Then the inequality

\[
\int_\Omega |f(x) \cdot g(x)| \, dx \leq 2\|f\|_p \|g\|_{p'}
\]

holds for every \( f \in L^{p(x)}(\Omega) \) and \( g \in L^{p'(x)}(\Omega) \).
Theorem 2.2. Suppose that \( p(x) \) satisfies\(^1\)
\[
1 < p^- \leq p^+ < \infty. \tag{2.4}
\]

Let \( \Omega < \infty \), \( p_1(x), p_2(x) \in \mathcal{P}(\Omega) \), then the necessary and sufficient condition for \( L_{p(x)(\Omega)} \subset L_{p(x)(\Omega)} \) is that for almost all \( x \in \Omega \) we have \( p_1(x) \leq p_2(x) \), and in this case, the imbedding is continuous.

Theorem 2.3. Suppose that \( p(x) \) satisfies (2.4). Let \( \rho(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx \). If \( u, u_k \in L_{p(x)}(\Omega) \), then

1. \( \|u\|_p < 1(=1; >1) \) if and only if \( \rho(u) < 1(=1; >1) \).
2. If \( \|u\|_p > 1 \), then \( \|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+} \).
3. If \( \|u\|_p < 1 \), then \( \|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+} \).
4. \( \lim_{k \to \infty} \|u_k\|_p = 0 \) if and only if \( \lim_{k \to \infty} \rho(u_k) = 0 \).
5. \( \|u_k\|_p \to \infty \) if and only if \( \rho(u_k) \to \infty \).

We assume that \( k \) is a given positive integer.

Given a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_n \), we set \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( D_\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \), where \( D_i = \partial / \partial x_i \) is the generalized derivative operator.

The generalized Sobolev space \( W_{p(x)}^{k}(\Omega) \) is the class of functions \( f \) on \( \Omega \) such that \( D_\alpha f \in L_{p(x)} \) for every multi-index \( \alpha \) with \( |\alpha| \leq k \). \( W_{p(x)}^{k}(\Omega) \) is endowed with the norm
\[
\|f\|_{k,p} = \sum_{|\alpha| \leq k} \|D_\alpha f\|_p. \tag{2.5}
\]

By \( W_{0}^{k,p(x)}(\Omega) \) we denote the subspace of \( W_{p(x)}^{k}(\Omega) \) which is the closure of \( C_{0}^{\infty}(\Omega) \) with respect to the norm (2.5).

In this paper we use the following equivalent norm of \( W_{1,p(x)}^{1}(\Omega) \):
\[
\|u\|_{1,p} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{\|\nabla u\|_p}{\lambda} \leq \frac{1}{\lambda} \|u\|_p^{p(x)} \, dx \leq 1 \right\}. \tag{2.6}
\]

Then we have the inequality \( (1/2)(\|\nabla u\|_p + \|u\|_p) \leq \|u\|_{1,p} \leq 2(\|\nabla u\|_p + \|u\|_p) \).

Theorem 2.4. The spaces \( W_{p(x)}^{k}(\Omega) \) and \( W_{0}^{k,p(x)}(\Omega) \) are separable reflexive Banach spaces if \( p(x) \) satisfies (2.4).

Theorem 2.5. Suppose that \( p(x) \) satisfies (2.4). Let \( \varphi(u) = \int_{\Omega} |\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \, dx \). If \( u, u_k \in W_{1,p(x)}^{1}(\Omega) \), then

1. \( \|u\|_{1,p} < 1(=1; >1) \) if and only if \( \varphi(u) < 1(=1; >1) \).
2. If \( \|u\|_{1,p} > 1 \), then \( \|u\|_{1,p}^{p^-} \leq \varphi(u) \leq \|u\|_{1,p}^{p^+} \).
3. If \( \|u\|_{1,p} < 1 \), then \( \|u\|_{1,p}^{p^-} \leq \varphi(u) \leq \|u\|_{1,p}^{p^+} \).
Theorem 2.10. Assume that $\lim_{k \to \infty} ||u_k||_{1,p} = 0$ if and only if $\lim_{k \to \infty} \varphi(u_k) = 0$.

(5) $||u_k||_{1,p} \to \infty$ if and only if $\varphi(u_k) \to \infty$.

Theorem 2.6. Let $\Omega$ be a bounded in $\mathbb{R}^N$, $p \in C(\overline{\Omega})$ and satisfies (2.4). Then for any measurable function $q(x)$ with $1 \leq q(x) \leq p^*(x)$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Theorem 2.7. If $p : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous and satisfies (2.4), then for any measurable function $q(x)$ with $p(x) \leq q(x) \leq p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Next let us consider the weighted variable exponent Lebesgue space. Let $a(x) \in P(\Omega)$ and $a(x) > 0$ for $x \in \Omega$. Define

$$L^{p(x)}_{a(x)}(\Omega) = \left\{ u \in P(\Omega) : \int_{\Omega} a(x)|u(x)|^{p(x)} dx < \infty \right\}$$

(2.7)

with the norm

$$|u|_{L^{p(x)}_{a(x)}(\Omega)} = ||u||_{p,a} = \inf\left\{ \lambda > 0 : \int_{\Omega} a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}$$

(2.8)

then $L^{p(x)}_{a(x)}(\Omega)$ is a Banach space.

Theorem 2.8. Suppose that $p(x)$ satisfies (2.4). Let $\rho(u) = \int_{\Omega} a(x)|u(x)|^{p(x)} dx$. If $u, u_k \in L^{p(x)}_{a(x)}(\Omega)$, then

1. For $u \neq 0$, $||u||_{p,a} = \lambda$ if and only if $\rho(u/\lambda) = 1$.
2. $||u||_{p,a} < 1 (= 1; > 1)$ if and only if $\rho(u) < 1 (= 1; > 1)$.
3. If $||u||_{p,a} > 1$, then $||u||_{p,a}^p \leq \rho(u) \leq ||u||_{p,a}^p$.
4. If $||u||_{p,a} < 1$, then $||u||_{p,a}^p \leq \rho(u) \leq ||u||_{p,a}^p$.
5. $\lim_{k \to \infty} ||u_k||_{p,a} = 0$ if and only if $\lim_{k \to \infty} \rho(u_k) = 0$.
6. $||u_k||_{p,a} \to \infty$ if and only if $\rho(u_k) \to \infty$.

Theorem 2.9. Let $\Omega \subset \mathbb{R}^n$ be a measurable subset. Suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies

$$|g(x,u)| \leq a(x) + \beta |u|^{(p_1(x))/(p_2(x))}$$

for any $x \in \Omega$, $t \in \mathbb{R}$,

(2.9)

where $p_i(x) \geq 1$, $i = 1, 2$, $a(x) \in L^{p_i(x)}(\Omega)$, $a(x) \geq 0$, $\beta \geq 0$ is a constant, then the Nemitsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_{\alpha}u)(x) = g(x,u(x))$ is a continuous and bounded operator.

Theorem 2.10. Assume that $0 \in \overline{\Omega}$ and the boundary of $\Omega$ possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega})$, $0 \leq s(x) < N$ for $x \in \overline{\Omega}$. If $q(x)$ satisfies $1 \leq q(x) < p_1^*(x)$ for $x \in \overline{\Omega}$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{[x]=s(x)}(\Omega)$. 
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**Theorem 2.11.** Assume that $0 \in \mathbb{R}$ and the boundary of $\Omega$ possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega}), 0 \leq s(x) \ll p(x)$ for $x \in \overline{\Omega}$. There is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}_{|x|^{-\alpha}}(\Omega)$.

**Proof.** Let $u \in W^{1,p(x)}(\Omega)$. Note that

$$
\int_{\Omega} \frac{|u|^{p(x)}|x|^{s(x)}}{|x|^{p(x)}} \, dx = \int_{\Omega} \frac{|u|^{p(x)}|x|^{p(x)-s(x)}}{|x|^{p(x)}} \, dx \\
\leq C_1 \left( \left\| \frac{|u|}{x} \right\|_{p/s} \left\| |u|^{N(p(x)-s(x))/(N-p(x))} \right\|_{p/(p-s)} \right).
$$

By Theorems 2.7 and 2.10, we have $\|u\|_{p,s} \leq C_2 \|u\|_{1,p} < \infty$ and $\|u\|_{p} \leq C_3 \|u\|_{1,p} < \infty$. So we get

$$
\int_{\Omega} \left( \frac{|u|}{x} \right)^{p(x)/s(x)} \, dx = \int_{\Omega} \frac{|u|^{p(x)}}{x} \, dx < \infty,
$$

$$
\int_{\Omega} |u|^{N(p(x)-s(x))/(N-p(x))-(p(x)/(p-s(x)))} \, dx = \int_{\Omega} |u|^{p(x)} \, dx < \infty.
$$

Furthermore, we obtain $\int_{\Omega} |u|^{p(x)}/|x|^{s(x)} \, dx < \infty$. This shows $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, then by the closed graph theorem in Banach space, we get the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}_{|x|^{-\alpha}}(\Omega)$. \qed

**3. The Principle of Concentration Compactness**

In this section, we will establish the principle of concentration compactness in $W^{1,p(x)}_{0}(\Omega)$.

We denote by $\mathcal{M}(\overline{\Omega})$ the space of finite nonnegative Borel measures on $\overline{\Omega}$. A sequence $\mu_n \rightharpoonup \mu$ weakly-* in $\mathcal{M}(\overline{\Omega})$ is defined by $(\mu_n, u) \rightarrow (\mu, u)$, for any $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$.

We first give two lemmas. From [13] we can obtain the proof of the following lemmas. Assume that $p(x)$ is Lipschitz continuous satisfying (2.4) and $s(x)$ is continuous on $\overline{\Omega}$.

**Lemma 3.1.** Let $\{u_n\} \subset L^{p(x)}_{|x|^{-\alpha}}(\Omega)$ be bounded, and $u_n \rightarrow u \in L^{p(x)}_{|x|^{-\alpha}}(\Omega)$ a.e. on $\Omega$, then

$$
\lim_{n \to \infty} \int_{\Omega} \frac{|u_n|^{p(x)}}{|x|^{s(x)}} - \frac{|u_n - u|^{p(x)}}{|x|^{s(x)}} \, dx = \int_{\Omega} \frac{|u|^{p(x)}}{|x|^{s(x)}} \, dx.
$$

**Lemma 3.2.** Let $\delta > 0, 0 < r < R < 1$, and $r/R \leq k(\delta) = \min\{\exp(-\delta/(2\tilde{C}))^{n/p} (1-n), e^{-|s-1|/(n-1)}\}$, where $\tilde{C} = ((1/(1 + (\delta/2))^{1/(p-1)} - 1)) + 1)^{p-1} \max\{2C^n, 2Cr \}|s-1| p/n, |s-1|$. 

denotes the surface area of the unit sphere in \( \mathbb{R}^n \) and \( C \) satisfies the inequality \( \|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \).

Then for every \( u \in W_0^{1,p(x)}(\Omega) \),

\[
\int_{B_r(x_0)} \frac{|u|^{p(x)}}{|x|^{n(p(x))}} \, dx \leq C^* \max \left\{ \left( \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx + \delta \max \left\{ \|u\|_{L^p}^{p^*_r}, \|u\|_{L^p_1}^{p^*_r} \right\} \right)^{p^*_r/p^*}, \left( \int_{B_r(x_0)} |u|^{p(x)} \, dx + \delta \max \left\{ \|u\|_{L^p}^{p^*_r}, \|u\|_{L^p_1}^{p^*_r} \right\} \right)^{p^*_r/p^*} \right\},
\]

(3.2)

where \( C^* = \sup \{ \int_\Omega |u|^{p(x)} / |x|^{n(p(x))} \, dx : \|u\|_{L^p(\Omega)} \leq 1, \ u \in W_0^{1,p(x)}(\Omega) \} \).

Theorem 3.3. Let \( \{u_n\} \subset W_0^{1,p(x)}(\Omega) \) with \( \|u_n\|_{L^p(\Omega)} \leq 1 \) such that

\[
u_n \rightarrow u \quad \text{weakly in } W_0^{1,p(x)}(\Omega),
\]

\[
|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \rightarrow \mu \quad \text{weakly-\star in } M(\overline{\Omega}),
\]

\[
|u_n|^{p(x)} \rightarrow \nu \quad \text{weakly-\star in } M(\overline{\Omega}),
\]

(3.3)

as \( n \rightarrow \infty \). Then the limit measures are of the form

\[
\mu = |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0 + \tilde{\mu}, \quad \mu(\overline{\Omega}) \leq 1,
\]

\[
\nu = |u|^{p(x)} \frac{|x|^{n(p(x))}}{|x|^{n(p(x))}} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0, \quad \nu(\overline{\Omega}) \leq C^*,
\]

(3.4)

where \( J \) is a countable set, \( \{\mu_j\} \subset [0, \infty) \), \( \{\nu_j\} \subset [0, \infty) \), \( \mu_0 \geq 0, \nu_0 \geq 0 \), \( \{x_j\} \subset \overline{\Omega}, \tilde{\mu} \in M(\overline{\Omega}) \) is a nonatomic positive measure. \( \delta_{x_j} \) and \( \delta_0 \) are atomic measures which concentrate on \( x_j \) and 0, respectively. \( C^* \) is as defined in Lemma 3.2. The atoms and the regular part satisfy the generalized Sobolev inequalities

\[
\nu(\overline{\Omega}) \leq C^* \max \left\{ \mu(\overline{\Omega})^{p^*_r/p^*}, \mu(\overline{\Omega})^{p^*_r/p^*} \right\},
\]

\[
\nu_j \leq C^* \max \left\{ \mu_j^{p^*_r/p^*}, \mu_j^{p^*_r/p^*} \right\},
\]

\[
\nu_0 \leq C^* \max \left\{ \mu_0^{p^*_r/p^*}, \mu_0^{p^*_r/p^*} \right\}.
\]

(3.5)
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Proof. By Lemma 3.2, for every \( \delta > 0 \), there exists \( k(\delta) > 0 \) such that for \( 0 < r < R \) with \( r/R \leq k(\delta) \),

\[
\int_{B_r(0)} \frac{|u_n|^{p'(x)}_{\Omega}}{|x|^{p(x)}} \, dx
\]

\[
\leq C^* \max \left\{ \left( \int_{B_r(0)} \left| \nabla u_n \right|^{p(x)} + |u_n|^{p(x)} \, dx + \delta \max \left\{ \|u_n\|^{p'_r}_{1,p'}, \|u_n\|^{p_r}_{1,p'} \right\} \right)^{p'_r/p'} , \left( \int_{B_r(0)} \left| \nabla u_n \right|^{p(x)} + |u_n|^{p(x)} \, dx + \delta \max \left\{ \|u_n\|^{p'_r}_{1,p'}, \|u_n\|^{p_r}_{1,p'} \right\} \right)^{p'_r/p'} \right\}. \tag{3.6}
\]

Let \( \eta_1 \in C_0^\infty(B_r(0)) \) and \( \eta_2 \in C_0^\infty(B_{2R}(0)) \) such that \( 0 \leq \eta_1, \eta_2 \leq 1 \), \( \eta_1 \equiv 1 \) in \( B_{r/2}(0) \) and \( \eta_2 \equiv 1 \) in \( B_R(0) \). Then we have

\[
\int_{B_r(0)} \frac{|u_n|^{p'(x)}_{\Omega}}{|x|^{p(x)}} \eta_1 \, dx \rightarrow \int_{B_r(0)} \eta_1 \, d\nu,
\]

\[
\int_{B_{2R}(0)} \left( \left| \nabla u_n \right|^{p(x)} + |u_n|^{p(x)} \right) \eta_2 \, dx \rightarrow \int_{B_{2R}(0)} \eta_2 \, d\mu.
\tag{3.7}
\]

Thus,

\[
\int_{B_r(0)} \eta_1 \, d\nu \leq C^* \max \left\{ \left( \int_{B_{2R}(0)} \eta_2 \, d\mu + \delta \right)^{p'_r/p'} , \left( \int_{B_{2R}(0)} \eta_2 \, d\mu + \delta \right)^{p'_r/p'} \right\}. \tag{3.8}
\]

Furthermore,

\[
\nu\left( \{0\} \right) \leq \nu(B_{r/2}(0)) \leq C^* \max \left\{ (\mu(B_{2R}(0)) + \delta)^{p'_r/p'} , (\mu(B_{2R}(0)) + \delta)^{p'_r/p'} \right\}. \tag{3.9}
\]

Let \( \delta \rightarrow 0 \) and \( R \rightarrow 0 \), then we get

\[
\nu\left( \{0\} \right) \leq C^* \max \left\{ \mu(\{0\})^{p'_r/p'} , \mu(\{0\})^{p'_r/p'} \right\}, \tag{3.10}
\]

that is,

\[
\nu_0 \leq C^* \max \left\{ \mu_0^{p'_r/p'} , \mu_0^{p'_r/p'} \right\}. \tag{3.11}
\]

By Theorem 2.11 and the definition of \( C^* \), we have

\[
\int_{\Omega} \frac{|u|^{p'(x)}_{\Omega}}{|x|^{p(x)}} \, dx \leq C^* \max \left\{ \left( \int_{\Omega} \left| \nabla u \right|^{p(x)} + |u|^{p(x)} \, dx \right)^{p'_r/p'} , \left( \int_{\Omega} \left| \nabla u \right|^{p(x)} + |u|^{p(x)} \, dx \right)^{p'_r/p'} \right\}. \tag{3.12}
\]
Similar to the proof of Theorem 3.1 in [13], we get
\begin{equation}
\nu = \frac{|u|^{p'(x)}}{|x|^{p(x)}} + \sum_{j=1}^{n} \nu_j \delta_{x_j} + \nu_0 \delta_0
\end{equation}
(3.13)
and the other results.

\section{4. Existence of Solutions}

Let \(O(N)\) be the group of orthogonal linear transformations in \(\mathbb{R}^N\), and \(G\) is a subgroup of \(O(N)\). For \(x \neq 0\), we denote the cardinality of \(G_x = \{gx : g \in G\}\) by \(|G_x|\) and set \(|G| = \inf_{x \in \mathbb{R}^N, x \neq 0}|G_x|\). An open subset \(\Omega \subset \mathbb{R}^N\) is \(G\)-invariant if \(g\Omega = \Omega\) for any \(g \in G\).

\textbf{Definition 4.1.} Let \(\Omega\) be a \(G\)-invariant open subset of \(\mathbb{R}^N\). The action of \(G\) on \(W^{1,p(x)}_0(\Omega)\) is defined by \(gu(x) = u(g^{-1}x)\) for any \(u \in W^{1,p(x)}_0(\Omega)\). The subspace of invariant functions is defined by
\[
W^{1,p(x)}_{0,G}(\Omega) = \left\{ u \in W^{1,p(x)}_0(\Omega) : gu = u, \forall g \in G \right\}.
\] (4.1)

A functional \(I : W^{1,p(x)}_0(\Omega) \to \mathbb{R}^N\) is \(G\)-invariant if \(I \circ g = I\) for any \(g \in G\).

Set
\[
I(u) = \int_\Omega \frac{1}{p(x)} \left( \left| \nabla u \right|^{p(x)} + \left| u \right|^{p(x)} \right) - \frac{h(x)}{p'(x)} \frac{\left| u \right|^{p'(x)}}{\left| x \right|^{p(x)}} - F(x,u)dx
\]
(4.2)
\[
F(x,t) = \int_0^t f(x,s)ds.
\]

The critical points of \(I(u)\), that is,
\[
0 = I'(u)\varphi = \int_\Omega \left| \nabla u \right|^{p(x)-2} \nabla u \nabla \varphi + \left| u \right|^{p(x)-2}u\varphi - h(x)\frac{\left| u \right|^{p'(x)-2}u}{\left| x \right|^{p(x)}} \varphi - f(x,u)\varphi dx
\] (4.3)
for all \(\varphi \in W^{1,p(x)}_0(\Omega)\), are weak solutions of the problem (1.1). So next we need only to consider the existence of nontrivial critical points of \(I(u)\).

In this paper, assume that \(G = O(N)\) and \(\Omega\) is \(O(N)\)-invariant. By (F-3) and (F-5), we get that \(I\) is \(O(N)\)-invariant. By the principle of symmetric criticality of Krawcewicz and Marzantowicz [20], \(u\) is a critical point of \(I\) if and only if \(u\) is a critical point of \(\tilde{I} = I|_{W^{1,p(x)}_0(\Omega)}\).

So we only need to prove the existence of critical points of \(\tilde{I}\) on \(W^{1,p(x)}_{0,\Omega}(\Omega)\).

\textbf{Lemma 4.2.} Any \((PS)_c\) sequence \(\{u_n\} \subset W^{1,p(x)}_{0,\Omega}(\Omega)\) possesses a convergent subsequence.
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Proof. Suppose that \( I(u_n) \to c, c \in \mathbb{R} \), and \( \tilde{I}(u_n) \to 0 \) in \( (W^{1,p(x)}_0(\Omega))^* \). Let \( I(x) = (p(x) + p_*^r(x))/2 \) and \( |\nabla (1/I(x))| \leq C \). Denote \( a = \inf_{x \in \tilde{\Omega}} ((1/p(x)) - (1/I(x))) > 0 \) and \( b = \inf_{x \in \tilde{\Omega}} ((1/I(x)) - (1/p_*^r(x))) > 0 \). Then we have

\[
\tilde{I}(u_n) - \left< \tilde{I}(u_n), \frac{u_n}{I(x)} \right> \\
= \int_{\Omega} \left( \frac{1}{p(x)} - \frac{1}{I(x)} \right) \left( |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) + h(x) \left( \frac{1}{I(x)} - \frac{1}{p_*^r(x)} \right) \frac{|u_n|^{p^*_r(x)}}{|x|^{p^*_r(x)}} \\
+ \frac{1}{I(x)} f(x, u_n) u_n - F(x, u_n) dx - \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \left( \frac{1}{I(x)} \right) u_n dx \\ \\
\geq \int_{\Omega} a \left( |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) + b h(x) \frac{|u_n|^{p^*_r(x)}}{|x|^{p^*_r(x)}} + \frac{1}{I(x)} f(x, u_n) u_n - F(x, u_n) dx \\
- \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \left( \frac{1}{I(x)} \right) u_n dx.
\]

By Young's inequality, for \( \varepsilon_1 \in (0, 1) \), we get

\[
\left| |\nabla u_n|^{p(x)-2} \nabla u_n \cdot u_n \right| \leq \varepsilon_1 |\nabla u_n|^{p(x)} + \varepsilon_1 |u_n|^{p^*_r(x)} + C(\varepsilon_1). 
\]

By (F-2), \(|(1/I(x)) f(x, u_n) u_n - F(x, u_n)| \leq C(|u_n| + |u_n|^{q(x)})\), then we have for \( \varepsilon_2 \in (0, 1) \)

\[
|u_n| + |u_n|^{q(x)} \leq \varepsilon_2 |u_n|^{p^*_r(x)} + C(\varepsilon_2). 
\]

From \( h(x)/|x|^{p(x)} \to \infty \) as \( x \to 0 \), we get that there exists \( \overline{H} > 0 \) such that \( h(x)/|x|^{p(x)} > \overline{H} \) for any \( x \in \Omega \), so we have

\[
\tilde{I}(u_n) - \left< \tilde{I}(u_n), \frac{u_n}{I(x)} \right> \\
\geq \int_{\Omega} a \left( |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) dx + \int_{\Omega} b \overline{H} |u_n|^{p^*_r(x)} dx - C\varepsilon_1 \int_{\Omega} |\nabla u_n|^{p(x)} dx \\
- C(\varepsilon_1 + \varepsilon_2) \int_{\Omega} |u_n|^{p^*_r(x)} dx - C(\varepsilon_1) - C(\varepsilon_2).
\]

Take \( \varepsilon_1 \) and \( \varepsilon_2 \) sufficiently small such that \( C\varepsilon_1 < a/2 \) and \( C(\varepsilon_1 + \varepsilon_2) \leq \overline{b}\overline{H} \), thus,

\[
c + 1 > I(u_n) \geq \int_{\Omega} \frac{a}{2} \left( |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) dx - C,
\]

if \( n \) is sufficiently large. Furthermore, we obtain \( \|u_n\|_{1,p} < \infty \).
Note that
\[
\Omega \left( |\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{{p(x)-2}} \nabla u \right) (\nabla u_n - \nabla u) dx \\
\leq \left| \left\langle \tilde{T}(u_n), u_n - u \right\rangle + \int_\Omega \left| \left( |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right)(u_n - u) \right| dx \\
+ \int_\Omega \left| h(x) \left( \frac{|u_n|^{p'_n(x)-2} u_n - |u|^{p'_n(x)-2} u}{|x|^{q(x)}} \right)(u_n - u) \right| dx \\
+ \int_\Omega \left| (f(x,u_n) - f(x,u))(u_n - u) \right| dx \right) \\
\triangleq \sum_{i=1}^5 I_i.
\] (4.9)

Because \{u_n\} is bounded in \(W^{1,p(x)}_{0,\Omega(N)}(\Omega)\), there exists a subsequence (still denoted by \(u_n\)) such that \(u_n \rightharpoonup u\) weakly in \(W^{1,p(x)}_{0,\Omega(N)}(\Omega)\). Then we have \(u_n \rightarrow u\) in \(L^{q(x)}(\Omega)\). It is easy to get \(I_1 \rightarrow 0\), \(I_2 \rightarrow 0\), and \(I_3 \rightarrow 0\). By (F-2)

\[
\int_\Omega |f(x,u_n)|^{q(x)} dx \\
\leq \int_\Omega \left( c_1 + c_2 |u_n|^{q(x)-1} \right)^{q(x)} dx \\
\leq C \int_\Omega (1 + |u_n|)^{(q(x)-1)q(x)} dx \\
\leq C \left( |\Omega| + \int_\Omega |u_n|^{q(x)} dx \right).
\] (4.10)

Then we have that \(\|f(x,u_n)\|_q\) is bounded. By

\[
I_5 \leq 2 \|f(x,u_n)\|_q \|u_n - u\|_q + 2 \|f(x,u)\|_q \|u_n - u\|_{q'}
\] (4.11)

we get \(I_5 \rightarrow 0\).

Next we show that \(I_4 \rightarrow 0\). Note that

\[
I_4 \leq h^0 \left( \int_\Omega \frac{|u_n|^{p'_n(x)-1}}{|x|^{q(x)}} - |u_n - u| dx + \int_\Omega \frac{|u|^{p'_n(x)-1}}{|x|^{q(x)}} - |u_n - u| dx \right) \\
\leq 2h^0 \left( \left\| \frac{|u_n|^{p'_n(x)-1}}{|x|^{q(x)/p'_n(x)}} \right\|_{p'_n} \left\| u_n - u \right\|_{p'_n} + \left\| \frac{|u|^{p'_n(x)-1}}{|x|^{q(x)/p'_n(x)}} \right\|_{p'_n} \left\| u_n - u \right\|_{p'_n} \right)
\] (4.12)
where $h^0 = \max_{x \in \Omega} h(x)$. By Theorem 2.11, $\|u_n\|_{L^p(\Omega)}^{-1} / |x|^{\sigma(\cdot)} / \mu_{\ast}$ is bounded. If we show that there exists a subsequence (still denoted by $\{u_n\}$) such that $\int_{\Omega} |u_n - u| |x|^{\sigma(\cdot)} dx \to 0$ as $n \to \infty$, then $I_4 \to 0$.

As $u_n \rightharpoonup u$ weakly in $W_{0,0}^{1,p(x)}(\Omega)$, passing to a subsequence, still denoted by $\{u_n\}$, by Theorem 3.3 we assume that there exist $\mu, \nu \in \mathcal{M}(\overline{\Omega})$ and $\{x_j\}_{j \in J}$ in $\overline{\Omega}$ such that $|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \to \mu$ weakly in $\mathcal{M}(\overline{\Omega})$ and $|u_n|^{p(x)} / |x|^{\sigma(\cdot)} \to \nu$ weakly in $\mathcal{M}(\overline{\Omega})$, where

$$\nu = \frac{|u_n|^{p(x)}}{|x|^{\sigma(x)}} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0,$$ (4.13)

$J$ is a countable set, $\{\mu_j\} \subset [0, \infty)$, $\{\nu_j\} \subset [0, \infty)$, $\mu_0 \geq 0$, $\nu_0 \geq 0$, $\tilde{\mu} \in \mathcal{M}(\overline{\Omega})$ is a nonatomic positive measure. Take $\eta \equiv 1$, then

$$\lim_{n \to \infty} \int_{\Omega} \frac{|u_n|^{p(x)} - \eta}{|x|^{\sigma(x)}} dx = \int_{\Omega} \eta d\nu = \int_{\Omega} \frac{|u|^{|p(x)} - \eta}{|x|^{\sigma(x)}} dx + \sum_{j \in J} \nu_j + \nu_0.$$ (4.14)

We claim $\nu_0 = 0$ and $\nu_j = 0$ for any $j \in J$. First we consider $\nu_0$.

For any $\varepsilon > 0$, choose $\varphi_0 \in C_0^\infty(B_{2\varepsilon}(0))$ such that $0 \leq \varphi_0 \leq 1$, $\varphi_0 = 1$ on $B_{\varepsilon}(0)$ and $|\nabla \varphi_0| \leq 2 / \varepsilon$. Then

$$\langle \tilde{I}(u_n), u_n \varphi_0 \rangle = \int_{\Omega} \left( |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) \varphi_0 dx - \int_{\Omega} h(x) \frac{|u_n|^{p(x)} \varphi_0}{|x|^{\sigma(x)}} dx$$

$$- \int_{\Omega} f(x, u_n) u_n \varphi_0 dx + \int_{\Omega} |\nabla u_n|^{p(x) - 2} \nabla u_n \nabla \varphi_0 u_n dx.$$ (4.15)

We have

$$\lim_{n \to \infty} \int_{B_{2\varepsilon}(0)} \frac{|u_n|^{p(x)} \varphi_0}{|x|^{\sigma(x)}} dx = \int_{B_{2\varepsilon}(0)} \varphi_0 d\mu,$$ (4.16)

$$\lim_{n \to \infty} \int_{B_{2\varepsilon}(0)} |u_n^{p(x)} \varphi_0| dx = \int_{B_{2\varepsilon}(0)} \varphi_0 d\nu.$$ (4.17)

By Theorem 2.1,

$$\int_{B_{2\varepsilon}(0)} |\nabla u_n|^{p(x) - 2} \nabla u_n \nabla \varphi_0 u_n dx$$

$$\leq 2 \|u_n \varphi_0\|_{p, B_{2\varepsilon}(0)} \left\| |\nabla u_n|^{p(x) - 1} \right\|_{p', B_{2\varepsilon}(0)}$$ (4.17)

$$\leq C \|u_n \varphi_0\|_{p, B_{2\varepsilon}(0)}.$$
By Theorem 2.6, we have \( u_n \to u \) in \( L^{p(x)}(\Omega) \), then
\[
\lim_{n \to \infty} \int_{B_2(0)} |u_n \nabla \varphi_0|^{p(x)} \, dx = \int_{B_2(0)} |u \nabla \varphi_0|^{p(x)} \, dx.
\]
(4.18)

Furthermore,
\[
\int_{B_2(0)} |u \nabla \varphi_0|^{p(x)} \, dx \leq 2 \left\| \nabla \varphi_0 \right\|_{N/p, B_2(0)} \left\| u \right\|_{p^{(0)}, B_2(0)} \left\| u \right\|_{N/(N-p), B_2(0)}.
\]
(4.19)

where \( \omega_N \) is the volume of the unit ball. By \( \lim_{\epsilon \to 0} \int_{B_2(0)} |u|^p \, dx = 0 \), then we have
\[
\lim \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} \left| \nabla u_n \right|^{p(x)-2} \nabla u_n \nabla \varphi_0 u_n \, dx = 0.
\]
(4.20)

Since \( \| f(x,u_n) \|_q \) is bounded and by Theorem 2.9 we have
\[
\lim_{n \to \infty} \int_{B_2(0)} \left| f(x,u_n) - f(x,u) \right|^{q(x)} \, dx = 0.
\]
(4.21)

From
\[
\int_{B_2(0)} \left| f(x,u_n) u_n - f(x,u) u \right| \, dx
\]
(4.22)

\[
\leq 2 \| f(x,u_n) \|_q \| u_n - u \|_q + 2 \| f(x,u_n) - f(x,u) \|_q \| u \|_q,
\]
we have
\[
\lim_{n \to \infty} \int_{B_2(0)} f(x,u_n) u_n \varphi_0 \, dx = \int_{B_2(0)} f(x,u) u \varphi_0 \, dx.
\]
(4.23)

Therefore,
\[
\lim \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{B_2(0)} f(x,u_n) u_n \varphi_0 \, dx = \lim_{\epsilon \to 0} \int_{B_2(0)} f(x,u) u \varphi_0 \, dx = 0.
\]
(4.24)

Thus, we have
\[
0 = \lim_{n \to \infty} \left( \tilde{I}(u_n), u_n \varphi_0 \right) = \int_{B_2(0)} \varphi_0 \, d\mu - \int_{B_2(0)} h(x) \varphi_0 \, dv - \int_{B_2(0)} f(x,u) u \varphi_0 \, dx
\]
(4.25)

\[
+ \lim_{n \to \infty} \int_{B_2(0)} \left| \nabla u_n \right|^{p(x)-2} \nabla u_n \nabla \varphi_0 \cdot u_n \, dx.
\]
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Furthermore, we obtain

$$0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left< \tilde{T}(u_n), u_n \varphi_0 \right> = \mu_0 - h(0) \nu_0. \quad (4.26)$$

As \( h(0) = 0, \mu_0 = 0, \) thus, \( \nu_0 = 0. \)

Next we consider \( \nu_j \) for any \( j \in J. \) Suppose \( \exists j_0 \in J \) such that \( \nu_{j_0} > 0. \) Note that \( u_n \in W^{1,p(x)}_0(\Omega) \), then for any \( g \in O(N), \nu(gx_{j_0}) = \nu(x_{j_0}) > 0. \) By \( |O(N)| = \infty, \) we get \( \nu(\{gx_{j_0} : g \in O(N)\}) = \infty. \) As the measure \( \nu \) is finite, that is a contradiction. So we obtain that \( \nu_0 = 0 \) and \( \nu_j = 0 \) for any \( j \in J. \) Thus,

$$\lim_{n \to \infty} \int_{\Omega} \frac{|u_n|^{p_j(x)}}{|x|^{s_j(x)}} \, dx = \int_{\Omega} \frac{|u|^{p_j(x)}}{|x|^{s_j(x)}} \, dx. \quad (4.27)$$

By Lemma 3.1, we obtain \( \lim_{n \to \infty} \int_{\Omega} |u_n - u|^{p_j(x)} / |x|^{s_j(x)} \, dx = 0, \) that is, \( u_n \to u \) strongly in \( L^{p_j(x)}(\Omega). \)

We obtain that \( \{u_n\} \) possesses a subsequence (still denoted by \( \{u_n\} \)), such that \( I_i \to 0, \) \( i = 1, \ldots, 5, \) as \( n \to \infty. \) Thus, \( \int_{\Omega} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u)(\nabla u_n - \nabla u) \, dx \to 0, \) as \( n \to \infty. \) As in the proof of Theorem 3.1 in [5], we divide \( \Omega \) into two parts:

$$\Omega_1 = \{ x \in \Omega : p(x) \geq 2 \}, \quad \Omega_2 = \{ x \in \Omega : p(x) < 2 \}. \quad (4.28)$$

We have

$$\int_{\Omega_1} |\nabla u_n - \nabla u|^{p(x)} \, dx + \int_{\Omega_2} |\nabla u_n - \nabla u|^{p(x)} \, dx \to 0, \quad (4.29)$$

that is, \( \int_{\Omega} |\nabla u_n - \nabla u|^{p(x)} \, dx \to 0. \) Then \( u_n \to u \) in \( W^{1,p(x)}_0(\Omega) \).

Since \( W^{1,p(x)}_0(\Omega) \) is a separable and reflexive Banach space, \( W^{1,p(x)}_0(\Omega) \) is also a separable and reflexive Banach space. So there exist \( \{e_n\}_{n=1}^{\infty} \subset W^{1,p(x)}_0(\Omega) \) and \( \{e_n^*\}_{n=1}^{\infty} \subset (W^{1,p(x)}_0(\Omega))^* \) such that

$$\left\{ e^*_j, e_i \right\} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad (4.30)$$

$$W^{1,p(x)}_0(\Omega) = \overline{\text{span}\{e_n : n = 1, 2, \ldots\}},$$

$$\left(W^{1,p(x)}_0(\Omega)\right)^* = \overline{\text{span}\{e^*_n : n = 1, 2, \ldots\}}.$$

For \( k = 1, 2, \ldots, \) denote \( X_k = \text{span}\{e_k\}, \) \( Y_k = \overline{\bigoplus_{j=1}^k X_j}, \) \( Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \)
Theorem 4.3. Under assumptions (F-1)–(F-5), the problem (1.1) admits a sequence of solutions \(
abla u_n \subset W^{1,p(x)}_{0,\Omega(N)}(\Omega)\) such that \(I(u_n) \to \infty\).

Proof. Set \(\varphi(u) = \int_{\Omega} F(x,u) \, dx\). We first show that \(\varphi(u)\) is weakly strongly continuous. Let \(u_n \to u\) weakly in \(W^{1,p(x)}_{0,\Omega(N)}(\Omega)\). So we have \(u_n \to u\) in \(L^q(x)(\Omega)\). Note that

\[|F(x,u)| \leq C \left(|u| + |u|^{q(x)}\right) \leq C \left(1 + |u|^{q(x)}\right),\]

(4.31)

then by Theorem 2.9 we obtain \(F(x,u_n) \to F(x,u)\) in \(L^1(\Omega)\). By Proposition 3.5 in [18],

\[\beta_k = \beta_k(r) = \sup_{u \in Z_k, \|u\|_p \leq r} \int_{\Omega} |F(x,u)| \, dx \to 0,\]

(4.32)

as \(k \to \infty\) for \(r > 0\).

Set

\[\theta_k = \theta_k(r) = \sup_{u \in Z_k, \|u\|_p \leq r} \int_{\Omega} \frac{|u|^{p(x)}}{|x|^{s(x)}} \, dx.\]

(4.33)

Next we show \(\theta_k \to \sum_{j \in I} \nu_j + \nu_0\) as \(k \to \infty\). Note that \(0 \leq \theta_{k+1} \leq \theta_k\), then \(\theta_k \to \theta \geq 0\), as \(k \to \infty\). There exists \(u_k \in Z_k\) with \(\|u_k\|_1 \leq r\) such that \(0 \leq \theta_k - \int_{\Omega} (|u_k|^{p(x)}/|x|^{s(x)}) \, dx < 1/k\), for each \(k = 1, 2, \ldots\). As \(W^{1,p(x)}_{0,\Omega(N)}(\Omega)\) is reflexive, passing to a subsequence, still denoted by \(\{u_k\}\), we assume \(u_k \to u\) weakly in \(W^{1,p(x)}_{0,\Omega(N)}(\Omega)\). We claim \(u = 0\). In fact, for any \(e_m^*(u_k) = 0\), when \(k > m\), then \(e_m^*(u_k) \to 0\) as \(k \to \infty\). It is immediate to get \(e_m^*(u_k) = 0\) for any \(m \in \mathbb{N}\). Then we have \(u = 0\). By Theorem 3.3, there exist a finite measure \(\nu\) and a sequence \(\{x_j\} \subset \Omega\) such that

\[\frac{|u_k|^{p(x)}}{|x|^{s(x)}} \to \nu = \frac{|u|^{p(x)}}{|x|^{s(x)}} + \sum_{j \in I} \nu_j \delta_{x_j} + \nu_0 \delta_0,\]

(4.34)

where \(I\) is countable. Set \(\eta \equiv 1\), we obtain \(\int_{\Omega} (|u_k|^{p(x)}/|x|^{s(x)}) \, \eta dx \to \sum_{j \in I} \nu_j + \nu_0\). So we have

\[
\lim_{k \to \infty} \theta_k = \sum_{j \in I} \nu_j + \nu_0 \leq \nu(\Omega) < \infty.
\]

For any \(n \in \mathbb{N}\), there exists a positive integer \(k_n\) such that \(\beta_k(n) \leq 1\) and \(\theta_k(n) \leq \sum_{j \in I} \nu_j + \nu_0 + 1\) for all \(k \geq k_n\). Assume that \(k_n < k_{n+1}\) for each \(n\). Define \(\{r_k : k = 1, 2, \ldots\}\) in the following way:

\[r_k = \begin{cases} n, & k_n \leq k < k_n + 1, \\ 1, & 1 \leq k < k_1. \end{cases}\]

(4.35)
Then we get \( r_k \to \infty \) as \( k \to \infty \). Hence, for \( u \in Z_k \) with \( \|u\|_{1,p} = r_k \), we get

\[
\bar{T}(u) \geq \frac{1}{p^+} \|u\|_{1,p}^{p^+} - \frac{h^0}{p_g} \theta_k(r_k) - \beta_k(r_k)
\]

\[
\geq \frac{1}{p^+} \|u\|_{1,p}^{p^+} - \frac{h^0}{p_g} \left( \sum_{j=1} \nu_j + \nu_0 + 1 \right) - 1,
\]

(4.36)

where \( h^0 \) is as defined in Lemma 4.2. So

\[
\inf_{u \in Z_k, \|u\|_{1,p} = r_k} \bar{T}(u) \to \infty \quad \text{as } k \to \infty.
\]

(4.37)

Note that for \( \epsilon \in (0,1) \), \(|F(x,u)| \leq C \epsilon \|u\|^{p^*(x)} + C(\epsilon)\), then

\[
\int_{\Omega} F(x,u)dx \leq C \epsilon \int_{\Omega} |u|^{p^*(x)}dx + C(\epsilon)|\Omega|.
\]

(4.38)

We have

\[
\bar{T}(u) \leq \int_{\Omega} |\nabla u|^{p(x)} + |u|^{p(x)}dx - \int_{\Omega} \bar{H}|u|^{p^*(x)}dx + C \epsilon \int_{\Omega} |u|^{p^*(x)}dx + C(\epsilon)|\Omega|.
\]

(4.39)

Take \( \epsilon \) sufficiently small so that \( C \epsilon \leq \bar{H}/2p^*_s \), then

\[
\bar{T}(u) \leq \int_{\Omega} |\nabla u|^{p(x)} + |u|^{p(x)}dx - m \int_{\Omega} |u|^{p^*(x)}dx + C,
\]

(4.40)

where \( m = \bar{H}/2p^*_s \). Since the dimension of \( Y_k \) is finite, any two norms on \( Y_k \) are equivalent, then \( k_1 \|u\|_{1,p} \leq \|u\|_{p^*_s} \leq k_2 \|u\|_{1,p} \), \( k_1, k_2 > 0 \). As in the proof of Theorem 4.2 in [13], we can find hypercubes \( \{Q_i\}_{i=1}^Q \) which mutually have no common points such that \( \overline{Q} \subseteq \bigcup_{i=1}^Q Q_i \) and \( p^*_s = \sup_{y \in \overline{Q}} p(y) < \inf_{y \in \Omega} p_s(y) = p^*_s \), where \( \Omega_i = Q_i \cap \Omega \). Then we have

\[
\bar{T}(u) \leq \sum_{\|u\|_{1,p,Q_i} > 1} \left( \|u\|_{1,p,Q_i}^{p^*_s} - m k_{Q_i}^{p^*_s} \|u\|_{1,p,Q_i}^{p^*_s} \right) \\
+ \sum_{\|u\|_{1,p,Q_i} \leq 1} \left( \|u\|_{1,p,Q_i}^{p^*_s} - m k_{Q_i}^{p^*_s} \|u\|_{1,p,Q_i}^{p^*_s} \right) + C
\]

(4.41)

Let \( f_i(t) = t^{p^*_s} - m k_{Q_i}^{p^*_s} t^{p^*_s} \), for \( i = 1, \ldots, Q \). Take \( s_i > 0 \) such that \( f_i(s_i) = \max_{t \geq 0} f_i(t) \geq f_i(0) = 0 \).

Denote \( g_i(t) = t^{p^*_s} - m k_{Q_i}^{p^*_s} t^{p^*_s} + \sum_{j=1}^Q f_j(s_i) + C \), for \( i = 1, \ldots, Q \). By \( \lim_{t \to \infty} g_i(t) = -\infty \), there
exists $t_0 > 0$ such that $g_i(t) \leq 0$ for $t \in [t_0, +\infty)$, for all $i = 1, \ldots, Q$. For any $k = 1, 2, \ldots$, take $\|u\|_{1,p} = \rho_k = \max \{Qt_0, r_k + 1\}$. Note that $\exists i_0$ such that

$$\|u\|_{1,p,\Omega_0} \geq \frac{1}{Q} \sum_{i=1}^{Q} \|u\|_{1,p,\Omega_k} \geq \frac{\rho_k}{Q} \geq t_0. \quad (4.42)$$

Then we have $g_{\rho_k}(\|u\|_{1,p,\Omega_0}) \leq 0$. Thus,

$$\bar{I}(u) \leq g_0 \left( \|u\|_{1,p,\Omega_0} \right) = \sum_{i=1}^{Q} f_i(s_i) + f_i \left( \|u\|_{1,p,\Omega_0} \right) + Q + C \leq 0. \quad (4.43)$$

Therefore, $\bar{I}(u) \leq 0$ for $u \in Y_k \cap S_{\rho_k}$, where $S_{\rho_k} = \{ u : \|u\|_{1,p} = \rho_k \}$. From Lemma 4.2 we have that $\bar{I}(u)$ satisfies $(PS)_c$ condition. In view of (F-4), by Fountain Theorem [21], we conclude the result.

\section*{Acknowledgments}

This research is supported by the Mathematical Tianyuan Foundation of China (Grant No. 11126027), NPU Foundation for Fundamental Research (NPU-FFR-JC20100220, NPU-FFR-JC20110229).

\section*{References}


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