Research Article

Generalized Carleson Measure Spaces and Their Applications

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We introduce the generalized Carleson measure spaces $\text{CMO}^{p,q}_\alpha$ that extend BMO. Using Frazier and Jawerth’s $\varphi$-transform and sequence spaces, we show that, for $\alpha \in \mathbb{R}$ and $0 < p \leq 1$, the duals of homogeneous Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_p$ for $1 < q < \infty$ and $0 < q \leq 1$ are $\text{CMO}^{-\alpha,q}_d$ and $\text{CMO}^{-\alpha+(n/p)-n,\infty}_d$ (for any $r \in \mathbb{R}$), respectively. As applications, we give the necessary and sufficient conditions for the boundedness of wavelet multipliers and paraproduct operators acting on homogeneous Triebel-Lizorkin spaces.

1. Introduction

In 1972, Fefferman and Stein [1] proved that the dual of $H^1$ is the BMO space. In 1990, Frazier and Jawerth [2, Theorem 5.13] generalized the above duality to homogeneous Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_p$. More precisely, they showed that the dual of $F^{\alpha,q}_1$ is $\dot{F}^{-\alpha,q}_\infty$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, where $q'$ is the conjugate index of $q$. Throughout the paper, $q'$ is interpreted as $q' = \infty$ whenever $0 < q \leq 1$, and $q' = q/(q - 1)$ for $1 < q \leq \infty$. Note that $F^{0,2}_1 = H^1$ and $\text{BMO} = \dot{F}^{0,2}_\infty$. For $\alpha \in \mathbb{R}$, $0 < p < 1$, and $0 < q < \infty$, it is known (cf. [2–4]) that the dual of $\dot{F}^{\alpha,q}_p$ is $\dot{F}^{-\alpha+(n/p)-n,\infty}_\infty$. Here, we will give another characterization for the duals of $\dot{F}^{\alpha,q}_p$ in terms of the generalized Carleson measure spaces for $\alpha \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$.

We say that a cube $Q \subseteq \mathbb{R}^n$ is dyadic if $Q = Q_{j,k} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 2^{-j}k_i \leq x_i < 2^{-j}(k_i + 1), i = 1, 2, \ldots, n\}$ for some $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Denote by $\ell(Q) = 2^{-j}$ the side length of $Q$ and by $x_Q = 2^{-j}k$ the “left lower corner” of $Q$ when $Q = Q_{j,k}$. We use $\sup_P$ and $\sum_P$ to express the supremum and summation taken over all dyadic cubes $P$, respectively. Also, denote the summation taken over all dyadic cubes $Q$ contained in $P$ by $\sum_{Q \subseteq P}$. For any dyadic cubes $P$ and $Q$, either $P$ and $Q$ are nonoverlapping or one contains the other. For any
function $f$ defined on $\mathbb{R}^n$, $j \in \mathbb{Z}$, and dyadic cube $Q = Q_{j,k}$, set

$$
\begin{align*}
  f_Q(x) &= |Q|^{-1/2} f \left( \frac{x - X_Q}{\ell(Q)} \right) = 2^{jn/2} f \left( 2^j x - k \right), \\
  f_j(x) &= 2^{jn} f \left( 2^j x \right), \\
  \tilde{f}(x) &= f(-x).
\end{align*}
$$

(1.1)

It is clear that $\tilde{g} \ast f(x_Q) = |Q|^{-1/2} (g, f_Q)$, where $(f, g)$ denotes the paring in the usual sense for $g$ in a Fréchet space $X$ and $f$ in the dual of $X$.

Choose a fixed function $\varphi$ in Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, the collection of rapidly decreasing $C^\infty$ functions on $\mathbb{R}^n$, satisfying

$$
\begin{align*}
  \text{supp}(\hat{\varphi}) &\subseteq \left\{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \right\}, \\
  |\hat{\varphi}(\xi)| &\geq c > 0 \quad \text{if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}.
\end{align*}
$$

(1.2)

For $\alpha \in \mathbb{R}$ and $0 < p, q \leq +\infty$, we say that $f$ belongs to the homogeneous Triebel-Lizorkin space $F^p_{\alpha,q}$ if $f \in \mathcal{S}'/\mathcal{D}$, the tempered distributions modulo polynomials, satisfies

$$
\begin{align*}
  \left\| f \right\|_{F^p_{\alpha,q}} := 
  \begin{cases}
  \left\{ \left( \sum_{k \in \mathbb{Z}} \left| 2^{ka} \varphi_k \ast f \right|^q \right)^{1/q} \right\}^{1/p} < \infty & \text{for } 0 < p < \infty, \\
  \sup_{P} \left\{ |P|^{-1} \int_{P_{k=-\log_2 P}} \sum_{k \in \mathbb{Z}} \left| 2^{ka} \varphi_k \ast f(x) \right|^q \langle x \rangle dx \right\}^{1/q} < \infty & \text{for } p = \infty.
  \end{cases}
\end{align*}
$$

(1.3)

When $0 < p < \infty$ and $q = \infty$, the above $\ell^1$-norm is modified to be the supremum norm as usual, and $F^p_{\alpha,\infty}$ is defined to be $B^p_{\alpha,\infty}$, which is

$$
\left\| f \right\|_{F^p_{\alpha,\infty}} := \sup_{k \in \mathbb{Z}} \sup_{x \in Q_{j,k} \cap \Omega} 2^{ka} |\varphi_k \ast f(x)| \approx \sup_Q |Q|^{-\alpha/n - (1/2)} \langle f, \varphi_Q \rangle < \infty.
$$

(1.4)

We now introduce a new space $CMO^{\alpha,q}_{r}$ as follows.

**Definition 1.1.** Let $\varphi \in \mathcal{S}$ satisfy (1.2). For $\alpha, r \in \mathbb{R}$ and $0 < q \leq \infty$, the generalized Carleson measure spaces $CMO^{\alpha,q}_{r}$ is the collection of all $f \in \mathcal{S}'/\mathcal{D}$ satisfying $\left\| f \right\|_{CMO^{\alpha,q}_{r}} < \infty$, where

$$
\begin{align*}
  \left\| f \right\|_{CMO^{\alpha,q}_{r}} := 
  \begin{cases}
  \sup_{P} \left\{ \left| P \right|^{-\alpha} \int_{P} \left| \langle Q \rangle^{-\alpha/n - (1/2)} \langle f, \varphi_Q \rangle \chi_Q(x) \right|^q \langle x \rangle dx \right\}^{1/q} & \text{for } 0 < q < \infty, \\
  \sup_{P} \left( \sup_{Q \subseteq P} |Q|^{-\alpha/n - (1/2)} \langle f, \varphi_Q \rangle \right) = \sup_Q |Q|^{-\alpha/n - (1/2)} \langle f, \varphi_Q \rangle, & q = \infty,
  \end{cases}
\end{align*}
$$

(1.5)

and $\chi_Q$ denotes the characteristic function of $Q$. 
Remark 1.2. By definition, we immediately have $\text{CMO}^{p,\infty}_r = F^{p,\infty}_r$ for $\alpha, r \in \mathbb{R}$, and it is easy to check $\text{CMO}^{p,q}_r = \{0\}$ for $r < 0$ and $0 < q < \infty$. Note that the zero element in $\text{CMO}^{p,q}_r$ means the class of polynomials. Also note that $\text{CMO}^{p,q}_r = \dot{F}^{p,q}_r$ with equivalent norms for $\alpha \in \mathbb{R}$ and $0 < q < \infty$. It follows from Proposition 3.3 that $\text{CMO}^{p,q}_1 = F^{p,q}_1$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$. In particular, $\text{CMO}^{0,2}_1 = \text{BMO}$, and hence the spaces $\text{CMO}^{p,q}_r$ generalize BMO.

Remark 1.3. For a dyadic cube $P$, denote by $k_P = -\log_2 \ell(P)$; that is, $k_P$ is the integer so that $\ell(P) = 2^{-k_P}$. In [5, 6], Yang and Yuan introduced the so-called “unified and generalized” Triebel-Lizorkin-type spaces $\dot{F}^{p,q}_{r,q}$ with four parameters by

$$\|f\|_{\dot{F}^{p,q}_{r,q}} := \sup_P |P|^{-r} \left\{ \int_P \left[ \sum_{k \geq k_P} \left( 2^{k\alpha} |\varphi_k * f(x)| \right)^q \right]^{p/q} dx \right\}^{1/p} < \infty,$$  

for $\alpha, r \in \mathbb{R}, p \in (0, \infty), q \in (0, \infty),$ and $f \in \mathcal{S}'/\mathcal{P}$. Note that in [5] the space $\dot{F}^{p,q}_{r,q}$ was defined for $r \in (0, \infty), p \in (1, \infty),$ and $q \in (1, \infty]$. It follows from [6, Theorem 3.1] that

$$\|f\|_{\dot{F}^{p,q}_{r,q}} \approx \sup_P |P|^{-r} \left\{ \int_P \left[ \sum_{Q \subset P} \left( |Q|^{(-\alpha/q)-(1/2)} |\langle f, \varphi_Q \rangle| \chi_Q(x) \right)^q \right]^{p/q} dx \right\}^{1/p}. $$

It is clear that $\text{CMO}^{p,q}_r = \dot{F}^{p,q/q}_{r,q}$ for $0 < q < \infty$, and hence $\text{CMO}^{p,q}_r$ “looks like” a special case of $\dot{F}^{p,q}_{r,q}$. In fact, it was proved in [7, 8] that the space $\dot{F}^{p,q}_{p,q}$ is the “same” as the space $\text{CMO}^{p,q}_r$.

The definition of $\text{CMO}^{p,q}_r$ is independent of the choice of $\varphi \in \mathcal{S}$ satisfying (1.2). To show that, we need the following Plancherel-Pólya inequalities.

**Theorem 1.4** (Plancherel-Pólya inequality for $0 < q < \infty$). Let $\varphi, \hat{\varphi} \in \mathcal{S}$ satisfy (1.2). For $\alpha, r \in \mathbb{R}$ and $0 < q < \infty$, if $f \in \mathcal{S}'/\mathcal{P}$ satisfies

$$\sup_P \left\{ |P|^{-r} \sum_{k = -\log_2 \ell(P)}^\infty \sum_{Q \subset P, \ell(Q) = 2^{-k}} \left( 2^{k\alpha} \sup_{u \in Q} |\hat{\varphi}_k * f(u)| \right)^q |Q| \right\}^{1/q} < \infty,$$

then

$$\sup_P \left\{ |P|^{-r} \sum_{k = -\log_2 \ell(P)}^\infty \sum_{Q \subset P, \ell(Q) = 2^{-k}} \left( 2^{k\alpha} \sup_{u \in Q} |\hat{\varphi}_k * f(u)| \right)^q |Q| \right\}^{1/q} \approx \sup_P \left\{ |P|^{-r} \sum_{k = -\log_2 \ell(P)}^\infty \sum_{Q \subset P, \ell(Q) = 2^{-k}} \left( 2^{k\alpha} \inf_{u \in Q} |\hat{\varphi}_k * f(u)| \right)^q |Q| \right\}^{1/q}. $$

Theorem 1.5 (Plancherel-Pólya inequality for \( q = \infty \)). Let \( \varphi, \psi \in S \) satisfy (1.2). For \( \alpha, r \in \mathbb{R} \), if \( f \in S'/\mathcal{D} \) satisfies

\[
\sup_Q \left( |Q|^{-r} \sup_{u \in Q} |\hat{\varphi}_k * f(u)| \right) < \infty,
\]

then

\[
\sup_Q \left( |Q|^{-r} \sup_{u \in Q} |\hat{\varphi}_k * f(u)| \right) \approx \sup_Q \left( |Q|^{-r} \inf_{u \in Q} |\hat{\varphi}_k * f(u)| \right).
\]

Remark 1.6. Let \( \varphi, \psi \in S \) satisfy (1.2). Denote by \( \text{CMO}^{\alpha,q}_r \) the collection of all \( f \in S'/\mathcal{D} \) satisfying \( \|f\|_{\text{CMO}^{\alpha,q}_r} < \infty \) defined in Definition 1.1 with respect to \( \varphi \). Then, by Theorem 1.4,

\[
\|f\|_{\text{CMO}^{\alpha,q}_r(\varphi)} \leq \sup_P \left\{ |P|^{-r} \sum_{k=\log_2(P)}^{\infty} \sum_{Q \in P, i(Q) \geq 2^{-k}} \left( 2^{k\alpha} \sup_{u \in Q} |\hat{\varphi}_k * f(u)| \right)^q |Q| \right\}^{1/q}
\]

\[
\leq C \sup_P \left\{ |P|^{-r} \sum_{k=\log_2(P)}^{\infty} \sum_{Q \in P, i(Q) \geq 2^{-k}} \left( 2^{k\alpha} \inf_{u \in Q} |\hat{\varphi}_k * f(u)| \right)^q |Q| \right\}^{1/q}
\]

\[
\leq C \|f\|_{\text{CMO}^{\alpha,q}_r(\psi)} \quad \text{for} \ 0 < q < \infty.
\]

Similarly, \( \|f\|_{\text{CMO}^{\alpha,q}_r(\psi)} \leq C \|f\|_{\text{CMO}^{\alpha,q}_r(\varphi)} \) by interchanging the roles of \( \varphi \) and \( \psi \). Hence, the definition of \( \text{CMO}^{\alpha,q}_r(\varphi) \) is independent of the choice of \( \varphi \) and, for short, denoted by \( \text{CMO}^{\alpha,q}_r \). Also, Theorem 1.5 shows that \( \text{CMO}^{\alpha,\infty}_r \) is independent of the choice of \( \varphi \) satisfying (1.2) in the same argument.

Remark 1.7. The classical Plancherel-Pólya inequality \([9]\) concludes that if \( \{x_k\} \) is an appropriate set of points in \( \mathbb{R}^n \), for example, lattice points, where the length of the mesh is sufficiently small, then

\[
\left( \sum_{k=1}^{\infty} |f(x_k)|^p \right)^{1/p} \approx \|f\|_p
\]

for all \( 0 < p \leq \infty \) with a modification if \( p = \infty \).

Using the Calderón reproducing formula (either continuous or discrete version), several authors obtain the variant Plancherel-Pólya inequalities \([10–13]\). These inequalities give characterizations of the Besov spaces and the Triebel-Lizorkin spaces. Moreover, using these inequalities, one can show that the Littlewood-Paley \( g \)-function and Lusin area \( S \)-function are equivalent in \( L^p \)-norm.
Define a linear map $S_{q}$ from $\mathcal{S}/\mathcal{D}$ into the family of complex sequences by

$$S_{q}(f) = \{\langle f, \varphi_{Q}\rangle\}_{Q}. \quad (1.14)$$

Let $S_{0}$ denote the family of $f \in \mathcal{S}$ satisfying $\int x^{k}f(x)dx = 0$ for all $k \in (\mathbb{N} \cup \{0\})^{n}$. For $g \in \text{CMO}_{p}^{-a,d}$, define a linear functional $L_{g}$ by

$$L_{g}(f) = \langle S_{q}(g), S_{q}(f)\rangle = \sum_{Q} \langle g, \varphi_{Q}\rangle \langle f, \varphi_{Q}\rangle \quad \text{for } f \in S_{0}. \quad (1.15)$$

We now state our first main result as follows.

**Theorem 1.8** (duality for $F_{p}^{a,d}$). Suppose that $a \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$.

(a) For $1 < q < \infty$, the dual of $F_{p}^{a,d}$ is $\text{CMO}_{(q/p)-(q/q)}^{-a,d}$ in the following sense.

(i) For $g \in \text{CMO}_{(q/p)-(q/q)}^{-a,d}$, the linear functional $L_{g}$ given by (1.15), defined initially on $S_{0}$, extends to a continuous linear functional on $F_{p}^{a,d}$ with $\|L_{g}\| \leq C\|g\|_{\text{CMO}_{(q/p)-(q/q)}^{-a,d}}$. 

(ii) Conversely, every continuous linear functional $L$ on $F_{p}^{a,d}$ satisfies $L = L_{g}$ for some $g \in \text{CMO}_{(q/p)-(q/q)}^{-a,d}$ with $\|g\|_{\text{CMO}_{(q/p)-(q/q)}^{-a,d}} \leq C\|L\|.

(b) For $0 < q \leq 1$, the dual of $F_{p}^{a,d}$ is $\text{CMO}_{r}^{a+(n/p)-n,\infty}$ (any $r \in \mathbb{R}$) in the following sense.

(i) For $g \in \text{CMO}_{r}^{a+(n/p)-n,\infty}$, the linear functional $L_{g}$ given by (1.15), defined initially on $S_{0}$, extends to a continuous linear functional on $F_{p}^{a,d}$ with $\|L_{g}\| \leq C\|g\|_{\text{CMO}_{r}^{a+(n/p)-n,\infty}}$.

(ii) Conversely, every continuous linear functional $L$ on $F_{p}^{a,d}$ satisfies $L = L_{g}$ for some $g \in \text{CMO}_{r}^{a+(n/p)-n,\infty}$ with $\|g\|_{\text{CMO}_{r}^{a+(n/p)-n,\infty}} \leq C\|L\|.

**Remark 1.9.** For $0 < p < 1$ and $0 < q \leq 1$, it follows immediately from [2, 3] (Verbitsky [4] corrected a gap of the proof) and definition that $(F_{p}^{a,d})' = F_{\infty}^{a+(n/p)-n,\infty} = \text{CMO}_{r}^{a+(n/p)-n,\infty}$ (any $r \in \mathbb{R}$). Theorem 1.8 (b) shows a different approach to the duality and includes the case of $p = 1$.

For $p = 1 < q < \infty$, we have $\text{CMO}_{q-(q/q)}^{a,d} = (F_{1}^{a,d})' = F_{\infty}^{a,d}$. For $0 < p < 1 < q < \infty$, $\text{CMO}_{(q/p)-(q/q)}^{a,d} = (F_{p}^{a,d})' = F_{\infty}^{a+(n/p)-n,\infty}$, and hence $\text{CMO}_{(q/p)-(q/q)}^{a,d} = \text{CMO}_{r}^{a+(n/p)-n,\infty}$. That is, each $\text{CMO}_{(q/p)-(q/q)}^{a,d}$ coincides with $\text{CMO}_{r}^{a+(n/p)-n,\infty}$ for $a, r \in \mathbb{R}$ and $0 < p < 1 < q < \infty$.

**Remark 1.10.** In Remark 1.2 we are aware that $\text{CMO}_{a,d}^{a,q}$ generalize BMO by the viewpoint of spaces directly. Choosing $a = 0$ and $q = 2$ in Theorem 1.8, we immediately have $(H^{p})' = (F_{p}^{0,2})' = \text{CMO}_{1}^{0,2}$ for $0 < p \leq 1$. In particular, BMO = $\text{CMO}_{1}^{0,2}$. Once again, we obtain that $\text{CMO}_{a,d}^{a,q}$ generalize BMO by the viewpoint of duality. It was also proved in [14] that the dual of the multiparameter product Hardy space is the generalized multiparameter Carleson measure space (cf. [14] for more details).
Remark 1.11. For \( \alpha, \beta \in \mathbb{R} \), in order to make each index works, we defined CMO\(^{\alpha,\infty}\) to be \( \sup_P |P|^\alpha \sup_{Q \subseteq P} |Q|^{-(\alpha/n) - (1/2)} |\langle f, \varphi_Q \rangle| \) in our earlier version and in [7]. In such a situation, for \( 0 < p, q \leq 1 \), the dual of \( F_p^{\alpha,q} \) would be CMO\(^{-\alpha,\infty}\). In this paper, however, we follow the referee’s suggestion and adopt a more “natural” definition of CMO\(^{\alpha,\infty}\) in Definition 1.1, that is, the limit of CMO\(^{\alpha,q}\) as \( q \to \infty \). The sequence space \( c_0^{\alpha,\infty} \) given in Definition 2.1 has a similar story as well.

As applications, we first recall the Haar multipliers introduced in [15, 16]. Given a sequence \( t = \{t_I\}_I \), where the \( I \)'s are dyadic intervals in \( \mathbb{R} \), a Haar multiplier on \( L^2(\mathbb{R}) \) is a linear operator of the form

\[
H_t f(x) := \sum_I t_I \langle f, \varphi_I \rangle \varphi_I(x), \quad f \in L^2(\mathbb{R}),
\]

(1.16)

where \( \varphi_I \) are the Haar functions corresponding to \( I \).

Using Meyer’s wavelets, we may generalize the above Haar multiplier to \( \mathbb{R}^n \) and obtain a necessary and sufficient condition for the boundedness on Triebel-Lizorkin spaces. Let \( \{\varphi_i\} \) for \( i \in E := \{1, 2, \ldots, 2^n - 1\} \) be Meyer’s wavelets (cf. [17], [18, pages 71–109]). Then, \( \{\varphi_{Q}^{i}\} \), where \( i \in E \) and \( Q^i \)'s are dyadic cubes in \( \mathbb{R}^n \), is a frame for \( \dot{F}_p^{\alpha,q} \) for \( \alpha \in \mathbb{R} \) and \( 0 < p, q \leq \infty \); that is, \( \|f\|_{\dot{F}_p^{\alpha,q}} \approx \sum_{i \in E} \|\{\langle f, \varphi_{Q}^{i}\rangle\}_{Q^i}\|_{L^p(\mathbb{R}^n)} \) for \( f \in \dot{F}_p^{\alpha,q} \). For \( t = \{t_Q\}_Q \), define a \textit{wavelet multiplier} \( \tilde{T}_t \) on \( \mathbb{R}^n \) by

\[
\tilde{T}_t(f) = \sum_{i \in E} \sum_{Q} |Q|^{-1/2} t_Q \langle f, \varphi_{Q}^{i}\rangle \varphi_{Q}^{i},
\]

(1.17)

for \( f \in \mathcal{S}'/\mathcal{D} \) such that the above summation is well defined.

Theorem 1.12. Suppose that \( \alpha, \beta \in \mathbb{R} \), \( 0 < p \leq 1 \) and \( 0 < q \leq \infty \). Then,

(a) for \( 1 < q < \infty \), \( \tilde{T}_t \) is bounded from \( \dot{F}_p^{\alpha,q} \) into \( \dot{F}_1^{\alpha+\beta,1} \) if and only if \( t \in c_0^{\alpha,q} \);  

(b) for \( 0 < q \leq 1 \) and \( r \in \mathbb{R} \), \( \tilde{T}_t \) is bounded from \( \dot{F}_p^{\alpha,q} \) into \( \dot{F}_1^{\alpha+\beta,1} \) if and only if \( t \in c_0^{\alpha+\beta+(n/p) - n,\infty} \), where \( c_0^{\alpha,q} \) is given in Definition 2.1.

We consider another application. Let \( \varphi \) and \( \varphi \) in \( \mathcal{S} \) satisfy (1.2) and (3.1). Choose a function \( \Phi \in \mathcal{S} \) supported on \( [0,1]^n \) and \( \int \Phi = 1 \). For \( \alpha \in \mathbb{R} \) and \( g \in \dot{F}_\infty^{\alpha,\infty} \), define the \textit{paraproduct operator} \( \Pi_g^\alpha \) by

\[
\Pi_g^\alpha(f) = \sum_Q \langle g, \varphi_Q \rangle |Q|^{-1/2} \langle f, \Phi_Q \rangle \varphi_Q.
\]

(1.18)

Thus, the adjoint operator \( \Pi_g^\alpha \) is

\[
\Pi_g^\alpha(f) = \sum_Q \langle g, \varphi_Q \rangle |Q|^{-1/2} \langle f, \varphi_Q \rangle \Phi_Q.
\]

(1.19)

Then, \( \Pi_g^1 = g \) and \( \Pi_g^\alpha 1 = 0 \) since \( \langle 1, \Phi_Q \rangle = |Q|^{1/2} \) and \( \langle 1, \varphi_Q \rangle = 0 \). Also, if \( g \in \dot{F}_\infty^{0,\infty} \), then
both $\Pi_g$ and $\Pi_g^*$ are singular integral operators satisfying the weak boundedness property. Moreover, $\Pi_g$ is a Calderón-Zygmund operator (i.e., $\Pi_g$ is bounded on $L^2(\mathbb{R}^n)$) if and only if $g \in F^{0,2}_\infty$ by David-Journé’s T1 theorem [19] (also see [12, Theorems 5.4 and 5.8]). The authors showed a more general type of paraproduct operators in [12, page 688], which were derived from the discrete Calderón reproducing formula.

**Theorem 1.13.** Suppose that $\beta \in \mathbb{R}$, $0 < r \leq 1$ and $0 < p \leq r < q < r/(1 - r)$.

(i) For $\alpha < 0$, $\Pi_g^*$ is bounded from $F^p_\alpha$ into $F^r_\alpha$ if and only if $g \in \text{CMO}^{\beta,q}_{\bar{r}(q-p)/(q-r)}$.

(ii) If $\alpha \in \mathbb{R}$ with $\alpha + \beta > 0$ and $g \in \text{CMO}^{\beta,q}_{r(q-p)/(q-r)}$, then $\Pi_g^*$ is bounded from $F^p_\alpha$ into $F^r_\alpha$.

**Remark 1.14.** When $r = 1$, $0 < p \leq 1 < q < \infty$, and $\beta \in \mathbb{R}$, Theorem 1.13 says that $\Pi_g$ is bounded from $F^p_\alpha$ into $F^p_\alpha$ if and only if $g \in \text{CMO}^{\beta,q}_{(q/p)-(q/q)}$ for $\alpha < 0$, and $\Pi_g^*$ is bounded from $F^p_\alpha$ into $F^p_\alpha$ for $\alpha > -\beta$ provided $g \in \text{CMO}^{\beta,q}_{(q/p)-(q/q)}$. In 1995, Youssfi [20] showed that, for $\beta \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq 2$, and $g \in F^{0,\infty}_{\bar{r}},$ $\Pi_g$ is bounded from $F^{0,\infty}_p$ into $F^{p,p}_\beta$ if and only if $g \in F^{0,\infty}_\beta$. The special case of Theorem 1.13(i), $p = r$, generalizes Youssfi’s result to $0 < p \leq 1$.

More precisely, for $\alpha < 0$, $\beta \in \mathbb{R}$, $0 < p \leq 1$, and $p < q < p/(1 - p)$, $\Pi_g$ is bounded from $F^p_\alpha$ to $F^p_\alpha$ if and only if $g \in \text{CMO}^{\beta,p/q}_{p/(q-p)} = F^{\infty}_{\alpha/q}$. The paper is organized as follows. In Section 2, we introduce the discrete version of the generalized Carleson measure spaces $c^{\alpha,q}_\hat{=}^{\hat{}}$ and show that the duals of sequence Triebel-Lizorkin spaces $f^{\alpha,q}_p$ for $1 < q < \infty$ and $0 < q \leq 1$ are $c^{q/(q/p)-(q/q)}_\hat{=}^{\hat{}}$ and $c^{q/(q/p)-(q/q)}_\hat{=}^{\hat{}}$ (for any $r \in \mathbb{R}$), respectively. In Section 3, we prove the duals of homogeneous Triebel-Lizorkin spaces $f^{\alpha,q}_p$ for $1 < q < \infty$ and $0 < q \leq 1$ to be the generalized Carleson measure spaces $\text{CMO}^{\alpha,q}_{p/(q-p)}$ and $\text{CMO}^{\alpha,q}_{p/(q-p)}$ (for any $r \in \mathbb{R}$), respectively. In Section 4, we prove the Plancherel-Pólya inequalities that give us the independence of the choice of $q$ for the definition of the generalized Carleson measure spaces. In the last section, we show the boundedness of wavelet multipliers and paraproduct operators. Throughout, we use $C$ to denote a universal constant that does not depend on the main variables but may differ from line to line. Also, $Q$ and $P$ always mean the dyadic cubes in $\mathbb{R}^n$, and, for $r > 0$, we denote by $rQ$ the cube concentric with $Q$ whose each edge is $r$ times as long.

## 2. Sequence Spaces

In this section, we introduce sequence spaces $c^{\alpha,q}_\hat{=}^{\hat{}}$ and then characterize the duals of $f^{\alpha,q}_p$ by means of $c^{\alpha,q}_\hat{=}^{\hat{}}$. Let us recall the definition of these sequence spaces $f^{\alpha,q}_p$ defined in [2]. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$, the space $f^{\alpha,q}_p$ consists all such sequences $s = \{s_Q\}_Q$ satisfying

\[
\|s\|_{f^{\alpha,q}_p} := \left\{ \left[ \sum_Q \left( \frac{|Q|^{\alpha/(n-1)}}{(1/2)} |s_Q| \chi_Q \right)^q \right]^{1/q} \right\}_{L^p} < \infty \quad \text{if} \quad 0 < p < \infty,
\]

\[
\sup_P \left\{ \int_P \left[ \sum_Q \left( \frac{|Q|^{\alpha/(n-1)}}{(1/2)} |s_Q| \chi_Q(x) \right)^q \right]^{1/q} \right\}_{L^p} < \infty \quad \text{if} \quad p = \infty.
\]
As before, the previous $\ell^q$-norm is modified to the supremum norm for $0 < p < \infty$ and $q = \infty$.
For $p = q = \infty$, we adopt the norm

$$
\|s\|_{J_{\infty}^p} := \sup_Q |Q|^{-\alpha(n)/(2)} |s_Q|.
$$

(2.2)

Note that $\|s\|_{J_{\infty}^p}^q$ is equivalent to the Carleson norm of the measure

$$
\sum_Q \left( |Q|^{-\alpha(n)/(2)} |s_Q| \right)^q |Q| \delta_{(x,t)}(Q),
$$

(2.3)

where $\delta_{(x,t)}$ is the point mass at $(x,t) \in \mathbb{R}^{n+1}$. See [2] for the details.

To study the duals of $J_{\infty}^{\alpha,q}$, we introduce a discrete version of the generalized Carleson measure spaces $c_r^{\alpha,q}$.

**Definition 2.1.** For $\alpha, r \in \mathbb{R}$ and $0 < q \leq \infty$, the space $c_r^{\alpha,q}$ is the collection of all sequences $t = \{t_Q\}_Q$ satisfying $\|t\|_{c_r^{\alpha,q}} < \infty$, where

$$
\|t\|_{c_r^{\alpha,q}} := \begin{cases} 
sup_P |P|^r \sum_Q \left( |Q|^{-\alpha(n)/(2)} |t_Q| \chi_Q(x) \right)^q dx & \text{for } 0 < q < \infty, \\
\sup_P \sup_Q |Q|^{-\alpha(n)/(2)} |t_Q| = \sup_Q |Q|^{-\alpha(n)/(2)} |t_Q| & \text{for } q = \infty.
\end{cases}
$$

(2.4)

It is obvious that

$$
\|t\|_{c_r^{\alpha,q}} = \|t\|_{J_{\infty}^p}^{\alpha,q} \quad \text{for } a, r \in \mathbb{R}.\]

Using embedding theorem, Frazier and Jawerth [2, equation (5.14) and Theorem 5.9] obtained that, for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, the dual of $J_{\infty}^{\alpha,q}$ is $J_{\infty}^{\alpha+n/p-n,\infty}$ where $0 < p < 1$, and the dual of $J_1^{\alpha,q}$ is $J_{\infty}^{\alpha,q}$. Note that $c_r^{\alpha,q} = \{0\}$ for $r < 0$ and $0 < q < \infty$. Here we give the dual relationship between sequence spaces $J_p^{\alpha,q}$ and $c_r^{\alpha,q}$.

**Theorem 2.2** (duality for $J_p^{\alpha,q}$). Suppose that $\alpha \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$.

(a) For $1 < q < \infty$, the dual of $J_p^{\alpha,q}$ is $c_0^{\alpha-n/q}$ in the following sense.

(i) For $t = \{t_Q\}_Q \in c_0^{\alpha-n/q}$, the linear functional $\ell_t$ on $J_p^{\alpha,q}$ given by $\ell_t(s) = \sum_Q s_Q t_Q$ is continuous with $\|\ell_t\| \leq C \|t\|_{c_0^{\alpha-n/q}}$ for $s = \{s_Q\}_Q \in J_p^{\alpha,q}$.

(ii) Conversely, every continuous linear functional $\ell$ on $J_p^{\alpha,q}$ satisfies $\ell = \ell_t$ for some $t \in c_0^{\alpha-n/q}$ with $\|\ell\|_{c_0^{\alpha-n/q}} \leq C \|\ell\|$.

(b) For $0 < q \leq 1$, the dual of $J_p^{\alpha,q}$ is $c_r^{\alpha+n/p-n,\infty}$ (any $r \in \mathbb{R}$) in the following sense.
Remark 2.3. For \( \alpha \in \mathbb{R} \) and \( 0 < q < \infty \), sequence spaces \( \ell_{r}^{\alpha,q} \) and \( \ell_{r}^{\alpha,\infty} \) (for any \( r \in \mathbb{R} \)) by definitions. Theorem 2.2 shows that \( (\ell_{1}^{\alpha,q})' = \ell_{\infty}^{\alpha,q} \), which gives a different but simpler proof of Frazier-Jawerth's result for the duality of \( \ell_{1}^{\alpha,q} \) (cf. [2, Theorem 5.9]).

Proof of Theorem 2.2. For \( s = \{ s_{Q} \} \in \ell_{p}^{\alpha,q} \) and \( t = \{ t_{Q} \} \in \ell_{r}^{\alpha,q} \), set \( \tilde{s} = \{ \tilde{s}_{Q} \} \) and \( \tilde{t} = \{ \tilde{t}_{Q} \} \)

\[
\tilde{s}_{Q} = |Q|^{-\alpha/n} s_{Q}, \quad \tilde{t}_{Q} = |Q|^\alpha/n t_{Q}.
\]

Then, \( \ell_t(\tilde{s}) = \ell_t(s) \). Also,

\[
\|\tilde{s}\|_{\ell_{p}^{\alpha,q}} = \|s\|_{\ell_{p}^{\alpha,q}}, \quad \|\tilde{t}\|_{\ell_{r}^{\alpha,q}} = \|t\|_{\ell_{r}^{\alpha,q}}.
\]

Without loss of generality, we may assume that \( \alpha = 0 \).

We first consider the case \( 1 < q < \infty \). Let \( t \in \ell_{1}^{0,q} \) and define a linear functional \( \ell_t \) on \( \ell_{p}^{0,q} \) by

\[
\ell_t(s) = \sum_{Q} s_{Q} t_{Q} \quad \text{for} \ s \in \ell_{p}^{0,q}.
\]

For \( s = \{ s_{Q} \} \in \ell_{p}^{0,q} \), let

\[
V_{q}(x) := \left( \sum_{Q} \left( |Q|^{-1/2} |s_{Q}| x_{Q}(x) \right) \right)^{1/q}.
\]

For \( k \in \mathbb{Z} \), let

\[
\Omega_{k} := \{ x \in \mathbb{R}^{n} : 2^{k} < V_{q}(x) \leq 2^{k+1} \},
\]

\[
\tilde{\Omega}_{k} := \{ x \in \mathbb{R}^{n} : M \chi_{\Omega_{k}}(x) > \frac{1}{2} \},
\]

\[
B_{k} := \{ \text{dyadic } Q : |Q \cap \Omega_{j}| > |Q|/2, \ |Q \cap \Omega_{j+1}| \leq |Q|/2 \text{ for some } j \geq k \},
\]

where \( M \) is the Hardy-Littlewood maximal function. Then, for each dyadic cube \( Q \), there exists exactly a \( k \in \mathbb{Z} \) such that \( Q \in B_{k} \). For every \( Q \in B_{k} \), let \( \tilde{Q} \) denote the maximal
dyadic cube in $B_k$ containing $Q$. Then all of such $\tilde{Q}$’s are pairwise disjoint. Thus, by Hölder’s inequality for $q$ and the inequality $(a + b)^p \leq a^p + b^p$ for $0 < p \leq 1$,

$$\left| \sum_Q s_Q t_Q \right| \leq \sum_{k \in \mathbb{Z}} \sum_{Q \in B_k} \sum_{Q \subseteq Q_k} \left( |Q|^{-(1/2) + (1/q)} |s_Q| \right) \left( |Q|^{1/2} - (1/q) |t_Q| \right)$$

$$\leq \left\{ \sum_{k \in \mathbb{Z}} \sum_{Q \in B_k} \left( \sum_{Q \subseteq Q_k} \left( |Q|^{-(1/2) + (1/q)} |s_Q| \right)^q \right)^{p/q} \left( \sum_{Q \subseteq Q_k} \left( |Q|^{1/2} - (1/q) |t_Q| \right)^q \right)^{p/q} \right\}^{1/p}$$

$$\leq \| t \|_{L^{q/2} \cap (p,q)} \left\{ \sum_{k \in \mathbb{Z}} \left| \tilde{Q}_k \right|^{-1-(p/q)} \left( \sum_{Q \in B_k} \left( |Q|^{-(1/2) + (1/q)} |s_Q| \right)^q \right)^{p/q} \right\}^{1/p}.$$  (2.11)

Since $\tilde{Q} \in B_k$ implies $\tilde{Q} \subseteq \tilde{\Omega}_k$, the disjointness of $\tilde{Q}$’s and Hölder’s inequality yield

$$\left| \sum_Q s_Q t_Q \right| \leq \| t \|_{L^{q/2} \cap (p,q)} \left\{ \sum_{k \in \mathbb{Z}} \left| \tilde{Q}_k \right|^{-1-(p/q)} \left( \sum_{Q \in B_k} \left( |Q|^{-(1/2) + (1/q)} |s_Q| \right)^q \right)^{p/q} \right\}^{1/p}.$$  (2.12)

We claim that $\sum_{Q \in B_k} \left( |Q|^{-(1/2) + (1/q)} |s_Q| \right)^q \leq C 2^{kq} |\tilde{\Omega}_k|$ for $k \in \mathbb{Z}$ and $0 < q < \infty$. Assume the claim for the moment. The weak $(1,1)$ boundedness of $M$ gives $|\tilde{\Omega}_k| \leq C |\Omega_k|$, and hence

$$\left| \sum_Q s_Q t_Q \right| \leq C \| t \|_{L^{q/2} \cap (p,q)} \left( \sum_{k \in \mathbb{Z}} \left| \tilde{Q}_k \right|^{-1-(p/q)} \left( \sum_{Q \in B_k} \left( |Q|^{-(1/2) + (1/q)} |s_Q| \right)^q \right)^{p/q} \right)^{1/p}$$

$$\leq C \| t \|_{L^{q/2} \cap (p,q)} \left( \sum_{k \in \mathbb{Z}} 2^{kp} |\Omega_k| \right)^{1/p}$$

$$\leq C \| t \|_{L^{q/2} \cap (p,q)} \left( \sum_{k \in \mathbb{Z}} 2^{kp} |\Omega_k| \right)^{1/p} \| V_t \|_{L^p}$$

$$= C \| t \|_{L^{q/2} \cap (p,q)} \| s \|_{L^p}.$$  (2.13)

To prove the claim, we note that, for $k \in \mathbb{Z}$ and $0 < q < \infty$,

$$2^{q(k+1)} |\tilde{\Omega}_k| \geq \int_{\tilde{\Omega}_k \setminus \bigcup_{j=k+1}^{\infty} \Omega_j} (V_{q}(x))^q dx$$

$$= \int_{\tilde{\Omega}_k \setminus \bigcup_{j=k+1}^{\infty} \Omega_j} \sum_Q \left( |Q|^{-1/2} |s_Q| \chi_Q(x) \right)^q dx$$

$$\geq \sum_{Q \in B_k} \left( |Q|^{-1/2} |s_Q| \right)^q \left( |\tilde{\Omega}_k \setminus \Omega_j| \cap Q \right) \text{ for some } j \geq k+1.$$  (2.14)
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which implies

\[
2^{q(k+1)} |\tilde{Q}_k| \geq \frac{1}{2} \sum_{Q \subseteq B_k} \left( |Q|^{-(1/2)+(1/d)} |s_Q| \right)^q.
\]

(2.15)

For \(0 < q \leq 1\), with a modification, we have

\[
\left| \sum_Q s_Q t_Q \right| \leq \sum_{k \in Z} \sum_{Q \subseteq B_k} \left( \sum_Q \left| Q \right|^{1/2} |s_Q| \right) \left( \left| Q \right|^{-1/p} |t_Q| \right) \left( \left| \frac{Q}{Q} \right| \right)^{(1/p)-1} |Q|^{(1/p)-1}
\]

\[
\leq \|t\|_{c_p}\left( \sum_{k \in Z} \sum_{Q \subseteq B_k} \left( \sum_Q \left| Q \right|^{1/2} |s_Q| \right) \left| \frac{Q}{Q} \right|^{1-p} \right)^{1/p}
\]

\[
\leq C \|t\|_{c_p} \left( \sum_{k \in Z} |\tilde{Q}_k|^{1-p} \left( 2^k |\tilde{Q}_k| \right)^p \right)^{1/p}
\]

\[
\leq C \|t\|_{c_p} \|V\|_{L^p}
\]

\[
\leq C \|t\|_{c_p} \|s\|_{\ell^{p,q}}.
\]

(2.16)

On the other hand, suppose that \(\ell\) is a continuous linear functional on \(f^0_{q,q}\). For each dyadic cube \(P\), write \(e^p = \{ (e^p)_Q \}_{Q}\) to be the sequence defined by

\[
(e^p)_Q = \begin{cases} 
1 & \text{if } Q = P, \\
0 & \text{if } Q \neq P.
\end{cases}
\]

(2.17)

Let \(t_p = \ell(e^p)\) and \(t = \{ t_p \}_p\). Then, for \(s = \{ s_Q \}_Q \in f^0_{q,q}\),

\[
\ell(s) = \sum_Q s_Q t_Q = \ell_t(s).
\]

(2.18)

Fix a dyadic cube \(P\). For \(1 < q < \infty\), let \(X\) be the sequence space consisting of \(s = \{ s_Q \}_{Q \subseteq P}\), and define a counting measure on dyadic cubes \(Q \subseteq P\) by \(d\sigma(Q) = |Q|/|P|^{(q'/p)-(q'/q)}\). Then,

\[
\left( \frac{1}{|P|^{(q'/p)-(q'/q)}} \sum_{Q \subseteq P} \left( |Q|^{-(1/2)+(1/d')} |t_Q| \right)^q \right)^{1/q'}
\]

\[
= \sup_{\|s\|_{\ell^p(X)} \leq 1} \left| \frac{1}{|P|^{(q'/p)-(q'/q)}} \sum_{Q \subseteq P} |Q|^{-1/2} |s_Q| |Q|^{-1/2} |t_Q| \right|
\]

\[
\leq \|\ell\| \sup_{\|s\|_{\ell^p(X)} \leq 1} \left( \left\{ \frac{s_Q |Q|^{1/2}}{|P|^{(q'/p)-(q'/q)}} \right\}_{Q \subseteq P} \right)^{\ell_p}.
\]

(2.19)
Note that
\[
\left\| \left\{ \frac{s_Q|Q|^{1/2}}{|P|^{(q'/p)-(q'/q)}} \right\} \right\|_{L_p^q} \leq \frac{1}{|P|^{(q'/p)-(q'/q)}} \left\{ \left( \sum_{Q \in P} |Q| s_Q \right)^{p/q} \cdot |P|^{[1-(p/q)} \right\}^{1/p}
\leq C \|s\|_{\ell^p(X,d\sigma)}.
\]

Thus,
\[
\left( \frac{1}{|P|^{(q'/p)-(q'/q)}} \sum_{Q \in P} \left| Q \right|^{-1/2+1/q'} \left| t_Q \right| \right)^{1/q'} \leq C \|t\|,
\]
and hence \( t \in C_{(q'/p)-(q'/q)}^{0,q} \). For \( 0 < q \leq 1 \), consider \( e^p \) defined before. Then, \( \|e^p\|_{L_p^{p,q}} = |P|^{-1/2+(1/p)} \) and
\[
\left( \left| P \right|^{(1/2)-(1/p)} |t_P| \right) \left\| e^p \right\|_{L_p^{p,q}} = |t_P| = \left| e^p \right| \leq \left\| e^p \right\|_{L_p^{p,q}} \leq \left\| e^p \right\|_{L_p^{p,q}}.
\]
Hence, \( \|t\|_{L_p^{p,q}} = \sup_P |P|^{(1/2)-(1/p)} |t_P| \leq \left| e^p \right| \). This completes the proof. \( \square \)

3. Proof of the Main Theorem

Let us recall the \( \varphi \)-transform identity given by Frazier and Jawerth [2]. Choose a function \( \varphi \in S \) satisfying (1.2). Then there exists a function \( \varphi \in S \) satisfying the same conditions as \( \varphi \) such that \( \sum_{\xi \in \mathbb{Z}} \hat{\varphi}(2^{-j} \xi) \hat{\varphi}(2^{-j} \xi) = 1 \) for \( \xi \neq 0 \). The \( \varphi \)-transform identity is given by
\[
f = \sum_Q \left( f, \varphi_Q \right) \varphi_Q,
\]
where the identity holds in the sense of \( S' / \mathcal{D} \), \( S_0 \), and \( \tilde{f}^{\alpha,q}_{L_p} \)-norm.

Define a linear map \( S_{\varphi} \) from \( S' / \mathcal{D} \) into the family of complex sequences by
\[
S_{\varphi}(f) = \left\{ \left( f, \varphi_Q \right) \right\}_{Q'}
\]
and another linear map \( T_{\varphi} \) from the family of complex sequences into \( S' / \mathcal{D} \) by
\[
T_{\varphi} \left( \left\{ s_Q \right\} \right) = \sum_Q s_Q \varphi_Q.
\]

Then, \( T_{\varphi} \circ S_{\varphi} |_{\ell^p_{\alpha,q}} \) is the identity on \( \tilde{f}^{\alpha,q}_{L_p} \) by [2, Theorem 2.2].

**Proposition 3.1.** Suppose that \( \alpha \in \mathbb{R} \) and, \( 0 < p, q \leq +\infty \), and \( \varphi, \psi \in S \) satisfy (1.2) and (3.1). The linear operators \( S_{\varphi} : \tilde{f}^{\alpha,q}_{L_p} \rightarrow \tilde{f}^{\alpha,q}_{L_p} \) and \( T_{\varphi} : f^{\alpha,q}_{L_p} \rightarrow f^{\alpha,q}_{L_p} \) defined by (3.2) and (3.3), respectively, are
bounded. Furthermore, $T_{\psi} \circ S_{\phi}$ is the identity on $S_{\alpha,q}$. In particular, $\|f\|_{F_{\alpha,q}} \approx \|S_{\phi}(f)\|_{F_{\alpha,q}}$ and $F_{\alpha,q}$ can be identified with a complemented subspace of $S_{\alpha,q}$.

Figures 1 and 2 illustrate the relationship among $F_{\alpha,q}$, $S_{\alpha,q}$, $CMO_{\alpha,q}$, and $C_{\alpha,q}$.

One recalls the almost diagonality given by Frazier and Jawerth [2]. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$, let $J = n/\min\{1, p, q\}$. One says that a matrix $A = \{a_{QP}\}_{Q,P}$ is $(\alpha, p, q)$-almost diagonal if there exists $\varepsilon > 0$ such that

$$\sup_{Q,P} \frac{|a_{QP}|}{w_{QP}(\varepsilon)} < +\infty,$$  \hspace{1cm} (3.4)$$

where

$$w_{QP}(\varepsilon) = \left( \frac{\ell(Q)}{\ell(P)} \right)^{\alpha} \left(1 + \frac{|x_Q - x_P|}{\max(\ell(P), \ell(Q))} \right)^{-J-\varepsilon} \cdot \min \left\{ \left( \frac{\ell(Q)}{\ell(P)} \right)^{\frac{(n+\varepsilon)/2}{\max(\ell(P), \ell(Q))}}, \left( \frac{\ell(P)}{\ell(Q)} \right)^{(\frac{(n+\varepsilon)/2}{\max(\ell(P), \ell(Q))}+J-\varepsilon)} \right\}.$$  \hspace{1cm} (3.5)$$

**Lemma 3.2.** For $\alpha, r \in \mathbb{R}$ and $0 < q < \infty$, an $(\alpha + nr, q, q)$-almost diagonal matrix is bounded on $c_{\alpha,q}$. Furthermore, when $r \geq 0$, an $(\alpha + nr, \infty, \infty)$-almost diagonal matrix is bounded on $c_{\alpha,\infty}$.\[ \]

We postpone the proof of Lemma 3.2 until the end of Section 4.
Let $\alpha, r \in \mathbb{R}$. For $q = \infty$, we have $c^\infty,\infty = f^\infty,\infty$ and $\text{CMO}^\infty,\infty = \dot{f}^\infty,\infty$. Thus, $S_\psi : \text{CMO}^\infty,\infty \mapsto c^\infty,\infty$ and $T_\psi : c^\infty,\infty \mapsto \text{CMO}^\infty,\infty$ are bounded by Proposition 3.1. For $0 < q < \infty$ and $f \in \text{CMO}^\alpha,q$, let $s = \{s_0\}_Q = S_\psi(f)$. Then, the $\psi$-transform identity (3.1) shows that $f = \sum Q S_\psi f_Q$ and $\|f\|_{\text{CMO}^\alpha,q} = \|S_\psi(f)\|_{c^\alpha,q} = \|s\|_{c^\alpha,q}$. In particular, $\|f\|_{\text{CMO}^\alpha,q} = \|S_\psi(f)\|_{\dot{c}^\alpha,q} \approx \|f\|_{\dot{f}^\alpha,q}$. Furthermore, for $s \in c^\alpha,q$,

$$\|T_\psi(s)\|_{\text{CMO}^\alpha,q} = \left\| \sum_P s_P q_P \right\|_{\text{CMO}^\alpha,q} = \left\{ \left\| \sum_P s_P q_P \right\| \right\}_{Q,\psi} = \|A\| \|c\|_{c^\alpha,q},$$

where $A := \{(q_P, q_Q)\}_{Q,P} = (a + nr, q, q)$-almost diagonal (cf. [2, Lemma 3.6]) and hence $A$ is bounded on $c^\alpha,q$ by Lemma 3.2. Therefore, $S_\psi$ is bounded from $\text{CMO}^\alpha,q$ to $c^\alpha,q$ and $T_\psi$ is bounded from $c^\alpha,q$ to $\text{CMO}^\alpha,q$.

We summarize that $T_\psi \circ S_\psi |_{\text{CMO}^\alpha,q}$ is also the identity on $\text{CMO}^\alpha,q$.

**Proposition 3.3.** For $(\alpha, r, q) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$ or $(\alpha, r, q) \in \mathbb{R} \times \mathbb{R} \times \{\infty\}$, the linear operators $S_\psi : \text{CMO}^\alpha,q \mapsto c^\alpha,q$ and $T_\psi : c^\alpha,q \mapsto \text{CMO}^\alpha,q$ are bounded. Furthermore, $T_\psi \circ S_\psi$ is the identity on $\text{CMO}^\alpha,q$ and $\|f\|_{\text{CMO}^\alpha,q} = \|S_\psi(f)\|_{c^\alpha,q}$. In particular, $\|f\|_{\text{CMO}^\alpha,q} = \|S_\psi(f)\|_{c^\alpha,q} \approx \|f\|_{\dot{f}^\alpha,q}$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, and $\|f\|_{\text{CMO}^\alpha,q} = \|S_\psi(f)\|_{c^\alpha,q} \approx \|f\|_{\dot{f}^\alpha,q}$ for $\alpha, r \in \mathbb{R}$.

Theorem 1.8 can be proved as a consequence of Propositions 3.1–3.3 and a duality result between two sequence spaces.

**Proof of Theorem 1.8.** First let us consider the case for $1 < q < \infty$. Let $g \in \text{CMO}^\alpha,q \cap F_p^\alpha,q$. Then, by Proposition 3.3, $\|g\|_{\text{CMO}^\alpha,q} = \|S_\psi(g)\|_{c^\alpha,q}$. It follows from Theorem 2.2 that $\ell(S_\psi(g))$ is a continuous linear functional on $F_p^\alpha,q$ and $\|\ell(S_\psi(g))\| \approx \|S_\psi(g)\|_{c^\alpha,q}$. Hence, for $f \in \mathcal{S}_0$,

$$|L_\psi(f)| \leq C \|S_\psi(g)\|_{c^\alpha,q} \|S_\psi(f)\|_{\dot{f}^\alpha,q} \leq C \|g\|_{\text{CMO}^\alpha,q} \|f\|_{\dot{f}^\alpha,q}. \quad (3.7)$$

Since $\mathcal{S}_0$ is dense in $F_p^\alpha,q$, the functional $L_\psi$ can be extended to a continuous linear functional on $F_p^\alpha,q$ satisfying $\|L_\psi\| \leq C \|g\|_{\text{CMO}^\alpha,q}$.

Conversely, let $L \in (F_p^\alpha,q)'$, and set $\ell = L \circ T_\psi$ on $f^\alpha,q$. By Proposition 3.1, $\ell \in (F_p^\alpha,q)'$. Thus, by Theorem 2.2, there exists $t = \{t_Q\}_Q \in c^{\alpha,q} \cap F_p^\alpha,q$ such that

$$\ell\left(\{s_0\}_Q\right) = \sum_Q s_0 t_Q \quad \text{for} \quad \{s_0\}_Q \in f^\alpha,q, \quad (3.8)$$

and $\|t\|_{\text{CMO}^\alpha,q} = \|\ell\| \leq C \|L\|$. For $f \in F_p^\alpha,q$, we have

$$\ell \circ S_\psi(f) = L \circ T_\psi \circ S_\psi(f) = L(f). \quad (3.9)$$
So, for \( f \in S_0 \) and letting \( g = T_p(t) = \sum_Q t_Q \psi_Q \),
\[
L(f) = \ell \circ S_p(f) = \sum_Q \langle f, \varphi_Q \rangle t_Q = \langle t, S_p(f) \rangle.
\]

It follows from [2, equations (2.7)-(2.8)] that \( \langle g, f \rangle = \langle S_p(g), S_p(f) \rangle \) and \( \langle t, S_p(f) \rangle = \langle T_p(t), f \rangle \) for \( f \in S_0 \) and \( g \in S'/\mathcal{D} \). This shows that \( L(f) = \langle T_p(t), f \rangle = L_g(f) \) for \( f \in S_0 \).

Proposition 3.3 and Theorem 2.2 give
\[
\|g\|_{CMO_{a,p}^{q,p-q}} \leq C\|t\|_{CMO_{a,p}^{q,p-q}} \leq C\|L\|.
\]

A similar argument gives the desired result for \( 0 < q \leq 1 \) with a slight modification, and hence the proof is finished.

Remark 3.4. As pointed out by one of the referees, Yang and Yuan [8, Theorem 1] show that if \( \tau > 1/p \) and \( 0 < p,q < \infty \), then \( F_p^{a,q} = F_{\infty}^{a+(n/p)\tau,n,\infty} \), where the definition of \( F_p^{a,q} \) is given in Remark 1.3. Thus, for \( 0 < p < 1 \) and \( 1 < q < \infty \),
\[
(\hat{F}_p^{a,q})' = F_{\infty}^{-a+(n/p)\tau-n,\infty} = F_q^{-a,(1/p)-(1/q)} = CMO_{(q/p)-(q/q)}
\]
which demonstrates a different approach to the duality.

### 4. Proofs of the Plancherel-Pólya Inequalities

In this section we demonstrate the Plancherel-Pólya inequalities.

**Proof of Theorem 1.4.** Without loss of generality, we may assume that \( \alpha = 0 \). By (3.1), we rewrite \( \tilde{\phi}_j \ast f(u) \) as
\[
\tilde{\phi}_j \ast f(u) = \sum_Q \langle f, \varphi_Q \rangle \int \tilde{\phi}_j(u-x)\varphi_Q(x)dx
= \sum_{k \in \mathbb{Z}} \sum_{Q:\ell(Q)=2^k} |Q| \langle f, \varphi_k(\cdot-x_Q) \rangle \int \tilde{\phi}_j(u-x)\varphi_k(x-x_Q)dx.
\]

Using the inequality [2, page 151, equation (B.5)]
\[
\left| \int \tilde{\phi}_j(u-x)\varphi_k(x-x_Q)dx \right| \leq C2^{-k|j-k|} \frac{2^{-|j|}}{(2^{-|j|} + |u-x_Q|)^{n+1}},
\]
where \( j \wedge k = \min\{j, k\} \) and \( K > 1 + nr \), we obtain

\[
\left| \tilde{\phi}_j \ast f(u) \right| \leq C \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q| \frac{2^{-(j\wedge k)}}{(2^{-j\wedge k} + |x_Q - x_Q'|)^{n+1}} |\tilde{\phi}_k \ast f(x_Q)|.
\]  

(4.3)

Thus, for \( \ell'(Q') = 2^{-j} \),

\[
\left( \sup_{u \in Q} \left| \tilde{\phi}_j \ast f(u) \right| \right)^q \leq C \left( \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q| \frac{2^{-(j\wedge k)}}{(2^{-j\wedge k} + |x_Q - x_Q'|)^{n+1}} |\tilde{\phi}_k \ast f(x_Q)| \right)^q \leq C \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q| \frac{2^{-(j\wedge k)}}{(2^{-j\wedge k} + |x_Q - x_Q'|)^{n+1}} |\tilde{\phi}_k \ast f(x_Q)|^q,
\]

where the last inequality is followed by Hölder’s inequality and

\[
\sum_{Q \in \mathcal{Q}} |Q| \frac{2^{-(j\wedge k)}}{(2^{-j\wedge k} + |x_Q - x_Q'|)^{n+1}} \leq C.
\]  

(4.5)

Denote \( T_Q \) by

\[
T_Q := \inf_{u \in Q} \left| \tilde{\phi}_k \ast f(u) \right|^q.
\]  

(4.6)

Since \( x_Q \) can be replaced by any point in \( Q \) in the last inequality,

\[
\left( \sup_{u \in Q'} \left| \tilde{\phi}_j \ast f(u) \right| \right)^q \leq C \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q| \frac{2^{-(j\wedge k)}}{(2^{-j\wedge k} + |x_Q - x_Q'|)^{n+1}} T_Q.
\]  

(4.7)

Given a dyadic cube \( P \) with \( \ell(P) = 2^{-k_0} \), the above estimates yield

\[
\sum_{j=k_0}^{\infty} \sum_{Q \subseteq P} \left( \sup_{u \in Q'} \left| \tilde{\phi}_j \ast f(u) \right| \right)^q |Q'| \leq C \sum_{j=k_0}^{\infty} \sum_{Q \subseteq P} \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j\wedge k)}}{(2^{-j\wedge k} + |x_Q - x_Q'|)^{n+1}} T_Q |Q|
\]

\[
:= CA_1 + CA_2,
\]

(4.8)
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where

\[
A_1 = \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{P}} \sum_{k < k_0} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_Q - x_0|)^{n+1}} T_Q |Q|,
\]

\[
A_2 = \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{P}} \sum_{k < k_0} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_Q - x_0|)^{n+1}} T_Q |Q|.
\]

Thus, \( A_1 \) can be further decomposed as

\[
A_1 = \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{P}} \sum_{k < k_0} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_Q - x_0|)^{n+1}} T_Q |Q|
\]

\[
+ \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{P}} \sum_{k < k_0} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_Q - x_0|)^{n+1}} T_Q |Q|.
\]

\[\tag{4.9}\]

Then, \( A_1 \) can be further decomposed as

\[
A_1 = \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{P}} \sum_{k < k_0} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_Q - x_0|)^{n+1}} T_Q |Q|
\]

\[
+ \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{P}} \sum_{k < k_0} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_Q - x_0|)^{n+1}} T_Q |Q|.
\]

\[\tag{4.10}\]

\[\sum_{Q \in \mathcal{Q}} T_Q |Q| \leq 3^n \sup_{P' \in \mathcal{P}} \sum_{Q \in \mathcal{Q} \cap 3P} T_Q |Q|. \]

\[\tag{4.11}\]

Thus,

\[
|P|^{-r} A_{11} \leq C |P|^{-r} \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{P}} \sum_{k < k_0} \sum_{Q \in \mathcal{Q}} 2^{-K|j-k|} |Q'| \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_Q - x_0|)^{n+1}} T_Q |Q|
\]

\[
\leq C \sup_{P'} |P'|^{-r} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q \in \mathcal{Q} \cap 3P} \inf_{u \in Q} |\hat{f}_k * f(u)|^q |Q|.
\]

\[\tag{4.12}\]

Next we decompose the set of dyadic cubes \( \{Q : Q \cap 3P = \emptyset, \ell(Q) = \ell(P)\} \) into \( \{B_i\}_{i \in \mathbb{N}} \) according to the distance between each \( Q \) and \( P \). Namely, for each \( i \in \mathbb{N} \),

\[
B_i := \left\{ P' : P' \cap 3P = \emptyset, \ell(P') = \ell(P), 2^{i-k_0} \leq |y_{P'} - y_P| < 2^{i-k_0+1} \right\}, \]

\[\tag{4.13}\]
where \( y_Q \) denotes the center of \( Q \). Then, we obtain

\[
|P|^{-r} A_{12} \leq C \sum_{i=1}^{\infty} \sum_{P \in B_i} |P'|^{-r} \sum_{j=k_0}^{\infty} \sum_{Q \subseteq P \atop j \in \mathbb{Z}} \sum_{k \geq k_0} \sum_{Q \subseteq P \atop k \in \mathbb{Z}} 2^{-K(j-k)} |Q| \\
\times \frac{2^{-(j/k)}}{(2^{-(j/k)} + |x_{P'} - x_P|)^{n+1}} T_Q|Q|.
\]  

(4.14)

Since \( \sum_{Q \subseteq P \atop j \in \mathbb{Z}} |Q'| = |P| \) for each \( j \geq k_0 \) and \( |x_{P'} - x_P| \approx 2^{i-k_0} \) for \( P' \in B_i \), the right-hand side of (4.14) is dominated by

\[
C \sum_{i=1}^{\infty} |P| \frac{2^{-(k_0)}}{2^{(i-k_0)(n+1)}} \sum_{k \geq k_0} \sum_{j=k_0}^{\infty} 2^{2(k_0-(j/k)-|k|)} \left( |P'|^{-r} \sum_{Q \subseteq P \atop k \in \mathbb{Z}} T_Q|Q| \right).
\]  

(4.15)

There are at most \( 2^{(i+2)n} \) cubes in \( B_i \), and hence

\[
|P|^{-r} A_{12} \leq C \left\{ \sup_{P'} |P'|^{-r} \sum_{k \geq k_0} \sum_{Q \subseteq P \atop k \in \mathbb{Z}} T_Q|Q| \right\} \sum_{i=1}^{\infty} |P| \frac{2^{-(k_0)}}{2^{(i-k_0)(n+1)}} 2^{2in} \\
= C \sup_{P'} |P'|^{-r} \sum_{k=-\log_2 e(P')}^{\infty} \sum_{Q \subseteq P \atop k \in \mathbb{Z}} \inf_{u \in Q} |\tilde{g}_k * f(u)|^r |Q|.
\]  

(4.16)

To estimate \( A_2 \), for \( i \in \mathbb{N} \) and \( k < k_0 \), set

\[
E_{i,k} := \left\{ Q : \ell(Q) = 2^{-k}, \ x_Q \in 2^i P \setminus 2^{i-1} P \right\}.
\]  

(4.17)

Then, \( |x_Q - x_P| \approx 2^{i-k_0} \) for \( Q \in E_{i,k} \) and

\[
A_2 = \sum_{j=k_0}^{\infty} \sum_{k < k_0} \sum_{i=1}^{\infty} |P| \sum_{Q \in E_{i,k}} 2^{-K(j-k)} \left\{ \left( 2^{-K(j-k)} + |x_Q - x_P| \right)^{n+1} |Q|^{-r} T_Q|Q| \right\}.
\]  

(4.18)

Since, for \( Q \in E_{i,k} \),

\[
|Q|^{-r} T_Q|Q| \leq \sup_{P'} |P'|^{-r} \sum_{m=-\log_2 e(P')}^{\infty} \sum_{Q \subseteq P \atop m \in \mathbb{Z}} T_Q|Q|.
\]  

(4.19)
and the number of dyadic cubes contained in $E_{i,k}$ is at most $2^{(i+k-k_0)n}$,

$$
|P|^{-r} A_2 \leq C \left\{ \sup_{P'} |P'|^{-r} \sum_{m=-\log_2 \ell(P')}^{\infty} \sum_{Q \subset P' \atop \ell(Q) \geq 2^{-m}} T_Q |Q'| \right\} \\
\times \sum_{j=k_0}^{\infty} \sum_{k \geq k_0} \sum_{i=1}^{2^{(k-j)(n+\ell(P)/2)}} 2^{-(i-k_0)n} 2^{K(k-j)} \frac{1}{2^{(i-k_0)(n+1)}} 2^{(i+k-k_0)n} \tag{4.20}
$$

where the condition $K > 1 + nr$ is used in the last equality. Combining the estimates of $A_1$ and $A_2$, we prove Theorem 1.4.

By modifying the proof above, we may easily show Theorem 1.5. Detailed verifications are left to the reader.

We now return to show Lemma 3.2.

**Proof of Lemma 3.2.** For $r < 0$, $c_{r}^{a,q} = [0]$, and hence the result holds. For $r = 0$, $c_{0}^{a,q} = f_{q}^{a,q}$, and so the matrix is bounded by $[2, \text{Theorem 3.3}]$. To complete the proof, it suffices to show the boundedness of $(a + nr, q, q)$-almost diagonal matrices for the case $r > 0$.

We may assume that $\alpha = 0$ since the case implies the general case. The proof is similar to the proof of Theorem 1.4. Here, we only outline the proof. First let us consider the case for $q > 1$. Let $A = \{a_{Q,P}\}_{Q,P}$ be an $(nr, q, q)$-almost diagonal matrix. Then, for $\ell'(Q) = 2^{-k}$,

$$
\left| (A_{s}Q) \right| \leq C \sum_{j \in \mathbb{Z}} \sum_{\ell(P) = 2^{-j}} 2^{(j-k)(nr+((n+\epsilon)/2))} \left( 1 + 2^j |x_Q - x_P| \right)^{-n-\epsilon} |s_P|, \\
\left( |Q|^{-1/2} (A_{s}Q) \right)^q \leq C \sum_{j \in \mathbb{Z}} \sum_{\ell(P) = 2^{-j}} 2^{(j-k)(nr+\ell(P)/2)} \left( 1 + 2^j |x_Q - x_P| \right)^{-n-\epsilon} \left( |Q|^{1/2} |s_P| \right)^q \tag{4.21}
$$

due to Hölder’s inequality. Given a dyadic cube $R$ with $\ell(R) = 2^{-\delta}$,

$$
\sum_{k \geq \delta} \sum_{Q \subset R \atop \ell(Q) = 2^{-k}} \left( \left| Q \right|^{-1/2} \left| (A_{s}Q) \right| \right)^q \left| Q \right| \leq CI + CII \tag{4.22}
$$

where

$$
I = \sum_{k \geq \delta} \sum_{Q \subset R \atop \ell(Q) = 2^{-k}} \sum_{j \geq \delta} \sum_{\ell(P) = 2^{-j}} 2^{(j-k)(nr+n+\ell(P)/2)} \left( 1 + 2^j |x_Q - x_P| \right)^{-n-\epsilon} \left( |Q|^{1/2} |s_P| \right)^q |P|, \\
II = \sum_{k \geq \delta} \sum_{Q \subset R \atop \ell(Q) = 2^{-k}} \sum_{j \leq \delta} \sum_{\ell(P) = 2^{-j}} 2^{(j-k)(nr+n+\ell(P)/2)} \left( 1 + 2^j |x_Q - x_P| \right)^{-n-\epsilon} \left( |Q|^{1/2} |s_P| \right)^q |P| \tag{4.23}
$$
Then, $I$ can be further decomposed as

$$
I = \sum_{k \geq 0} \sum_{Q \in \mathcal{R}_{\ell}^{0} \cap \{Q: |Q| = 2^{-k}\}} \sum_{j \geq 0} \sum_{P \in \mathcal{R}_{\ell}^{1} \cap \{P: |P| < 2^{-j}\}} 2^{(j-k)(n+\epsilon)/2} \left(1 + 2^j |x_Q - x_P|\right)^{-n-\epsilon} \left(|P|^{-1/2}|s_P|\right)^q |P|
$$

$$
+ \sum_{k \geq 0} \sum_{Q \in \mathcal{R}_{\ell}^{0} \cap \{Q: |Q| = 2^{-k}\}} \sum_{j \geq 0} \sum_{P \in \mathcal{R}_{\ell}^{1} \cap \{P: |P| < 2^{-j}\}} 2^{(j-k)(nr+\epsilon)/2} \left(1 + 2^j |x_Q - x_P|\right)^{-n-\epsilon} \left(|P|^{-1/2}|s_P|\right)^q |P|
$$

$$
:= I_{11} + I_{12}.
$$

The same argument showed in the proof of Theorem 1.4 for the term $A_1$ gives us

$$
|R|^{-\epsilon} I \leq C \|s\|_{c_{\ell}^q}^q.
$$

To estimate $II$, for $i \in \mathbb{N}$ and $j < \delta$, let

$$
E_{i,j} := \left\{ Q: \ell(Q) = 2^{-i}, \ x_Q \in 2^i R \setminus 2^{i-1} R \right\}.
$$

Then, using the same argument as Theorem 1.4 for $A_2$, we have

$$
|R|^{-\epsilon} II \leq C \|s\|_{c_{\ell}^q}^q.
$$

Both estimates for $I$ and $II$ show the desired result for $q > 1$.

When $q \leq 1$, we modify the previous proof by replacing Hölder’s inequality with $q$-triangle inequality to get the result.

When $q = \infty$ and $r > 0$, the space $c_{\ell}^{\alpha, \infty} = f_{\infty}^{\alpha, \infty}$, and hence an $(\alpha + nr, \infty, \infty)$-almost diagonal matrix is bounded on $c_{\ell}^{\alpha, \infty}$ by Proposition 5.3.

**Remark 4.1.** Note that $c_{\ell}^{\alpha,d} = f_{\infty}^{\alpha,d}$. By a duality argument and [2, Theorem 3.3 and page 81], one can show that the $(\alpha + n, q, q)$-almost diagonal matrix is bounded on $f_{\infty}^{\alpha,d}$. When $q > 1$ and $r > 1$, we can prove Lemma 3.2 by duality in Theorem 2.2. Let $A = \{a_{QP}\}_{Q,P}$ be an $(nr, q, q)$-almost diagonal matrix. Also define the transpose of $A$ by $A' = \{a_{QP}\}_{Q,P}$. For $q > 1$ and $r > 1$, let $p = (q + q')/(q' r + q)$. Then, $p < 1$. Since $A$ is $(nr, q, q)$-almost diagonal, $A'$ is $(0, p, q')$-almost diagonal by a calculation for a different value of $\epsilon$. Thus, by Theorem 2.2 (a) and Proposition 5.3, $A'$ is bounded on $c_{0}^{\alpha,d}$.

## 5. Applications

We define another wavelet multiplier on $\mathbb{R}^n$ by using $\varphi$-transform identity as follows. Let $\varphi$ and $\varphi$ in $\mathcal{S}$ satisfy (1.2) and (3.1). For a sequence $t = \{t_Q\}_{Q}$, where the $Q'$s are dyadic cubes in $\mathbb{R}^n$, define the wavelet multiplier $T_t$ by

$$
T_t(f) = \sum_Q |Q|^{-1/2} t_Q(f, \varphi_Q) \varphi_Q
$$

(5.1)
for $f \in S'/\mathcal{B}$ such that the above summation is well defined. Thus, we have the following characterization.

**Theorem 5.1.** Suppose that $\alpha, \beta \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$. Then,

(a) for $1 < q < \infty$, $T_i$ is bounded from $F_{i_1}^{a,q}$ into $F_{i_1}^{a+\beta,1}$ if $t \in c_{(q'/p)-(q/q)'}^{(p/q)}$

(b) for $0 < q \leq 1$ and $r \in \mathbb{R}$, $T_i$ is bounded from $F_{i_1}^{a,q}$ into $F_{i_1}^{a+\beta,1}$ if $t \in c_r^{(n/p)-n,\infty}$.

**Proof.** We show the case $\alpha = 0$ only, which implies the general case by (2.7). For $\beta \in \mathbb{R}$, $0 < p \leq 1$, and $1 < q < \infty$, let $f \in F_{p}^{0,q}$ and $t \in c_{(q'/p)-(q/q)'}^{(p/q)}$. It follows from Theorem 2.2 and Proposition 3.1 that

$$
\|T_i(f)\|_{\dot{F}_{i_1}^{\beta,1}} \leq C \|\{\|Q\|^{-1/2}t_Q\langle f, \varphi_Q \rangle\}_{Q} \|_{\dot{F}_{i_1}^{\beta,1}}
$$

$$
= C \sum_Q \{\|Q\|^{-\beta/n} |t_Q| \} |\langle f, \varphi_Q \rangle|\nn
\leq C \|\{\langle f, \varphi_Q \rangle\}_{Q} \|_{\dot{F}_{i}^{\beta,q}} \{\|Q\|^{-\beta/n} |t_Q|\}_{Q} \|_{\alpha,q}_{(q'/p)-(q/q)'}
$$

$$
\leq C \|\|f\|_{\dot{F}_{i}^{a,q}}\|_{\beta,q}_{(q'/p)-(q/q)'}.
$$

This shows that $T_i$ is bounded from $F_{p}^{0,q}$ into $F_{i_1}^{\beta,1}$ and $\|T_i\| \leq C \|t\|_{\beta,q}_{(q'/p)-(q/q)'}$. A similar argument yields the boundedness of $T_i$ for the case $0 < q \leq 1$. \hfill \Box

In order to prove Theorem 1.12, we demonstrate a similar result in sequence spaces first. For a sequence $t = \{t_Q\}_Q$, define $D_i$ by

$$
D_i(s) = \{\|Q\|^{-1/2}t_Q s_Q\}_Q \quad \text{for } s = \{s_Q\}_Q \text{ with finitely many nonzero terms.} \quad \text{(5.3)}
$$

**Theorem 5.2.** Suppose that $\alpha, \beta \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$. Then,

(a) for $1 < q < \infty$, $D_i$ is extendible to be bounded from $F_{p}^{a,q}$ into $F_{i_1}^{a+\beta,1}$ if and only if $t \in c_{(q'/p)-(q/q)'}^{(p/q)}$

(b) for $0 < q \leq 1$ and $r \in \mathbb{R}$, $D_i$ is extendible to be bounded from $F_{p}^{a,q}$ into $F_{i_1}^{a+\beta,1}$ if and only if $t \in c_r^{(n/p)-n,\infty}$.
Proof. We still assume that $\alpha = 0$. For $\beta \in \mathbb{R}$, $0 < p \leq 1$, and $1 < q < \infty$, let $s = \{s_Q\}_Q \in \mathcal{F}^{\alpha,p,q}_{\beta,q'}$. It follows from Theorem 2.2 that

$$
\|D_t(s)\|_{\beta,1} = \sum_Q \left\{ |Q|^{-\beta/n} |t_Q| \right\} |s_Q|
$$

$$
\leq C\|s\|_{\beta,1} \left\{ |Q|^{-\beta/n} |t_Q| \right\} \|s_Q\|_{\beta,q'}
$$

$$
= C\|s\|_{\beta,1} \|t\|_{\beta,q'}.
$$

Conversely, suppose that $D_t$ maps from $\mathcal{F}^{\alpha,q}_{\beta,q'}$ into $\mathcal{F}^{\alpha,1,q}_{\beta,q'}$ boundedly. For $t = \{t_Q\}_Q$, let $\tilde{t} = \{ |Q|^{-\beta/n} t_Q \}_Q$. Define a linear functional $\hat{\ell}_t$ by

$$
\hat{\ell}_t(s) = \sum_Q s_Q \tilde{t}_Q
$$

for $s = \{s_Q\}_Q$ with finitely many nonzero terms.

Then,

$$
|\hat{\ell}_t(s)| \leq \sum_Q \left( |Q|^{-\beta/n} |t_Q| \right) |s_Q| = \|D_t(s)\|_{\beta,1}.
$$

The assumption shows that $\hat{\ell}_t$ is a continuous linear functional on $\mathcal{F}^{\alpha,q}_{\beta,q'}$. Using Theorem 2.2, we have $\tilde{t} \in \mathcal{F}^{\alpha,q}_{\beta,q'}$ and hence $t \in \mathcal{F}^{\alpha,q}_{\beta,q'}$.

For $0 < q \leq 1$, a similar argument gives the desired result of (b). \hfill \Box

Proof of Theorem 1.12. The “if” part follows from Theorem 5.1. To show the “only if” part, define $\bar{T}_t$ by

$$
\bar{T}_t(f) = \sum_Q |Q|^{-1/2} t_Q \langle f, \varphi_Q \rangle \varphi_Q.
$$

The boundedness of $\bar{T}_t$ says that $\bar{T}_t$ is bounded from $\mathcal{F}^{\alpha,q}_{\beta,q'}$ into $\mathcal{F}^{\alpha,1,q}_{\beta,q'}$. Clearly,

$$
S_{\varphi'} \circ \bar{T}_t \circ T_{\varphi}(s) = D_t(s) \quad \text{for} \quad s \in \mathcal{F}^{\alpha,q}_{\beta,q'}.
$$

It follows from Proposition 3.1 that $D_t$ is bounded from $\mathcal{F}^{\alpha,q}_{\beta,q'}$ into $\mathcal{F}^{\alpha,1,q}_{\beta,q'}$, and hence $t \in \mathcal{C}_{(q/p)^{-n}, q}^{\beta,q}$ for $1 < q < \infty$ and $t \in \mathcal{C}_{r}^{\beta,(n/p)-n,\infty}$ for $0 < q \leq 1$ and $r \in \mathbb{R}$ by Theorem 5.2. \hfill \Box

In order to study the boundedness of the paraproduct operators acting on Triebel-Lizorkin spaces, we need more results described as follows.

Proposition 5.3 ([2, pages 54 and 81]). For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$, an $(\alpha, p, q)$-almost diagonal matrix is bounded on $\mathcal{F}^{\alpha,q}_{p}$. 

Lemma 5.4. Define a matrix by $G = \{(\psi_p, \Phi_Q)\}_{Q,p}$. Then, for $\alpha < 0$ and $0 < p, q \leq +\infty$, $G$ is $(\alpha, p, q)$-almost diagonal and hence is bounded on $f_p^{\alpha,q}$.

Proof. For $\ell(P) \leq \ell(Q)$, since $\int x^\gamma \psi_p(x)dx = 0$ for all $\gamma$, by [2, page 150, Lemma B.1], we have

$$|\langle \psi_p, \Phi_Q \rangle| \leq C \left( \frac{\ell(Q)}{\ell(P)} \right)^{\alpha} \left( 1 + \frac{|x_Q - x_P|}{\ell(P)} \right)^{-J-\varepsilon} \left( \frac{\ell(Q)}{\ell(P)} \right)^{(n+\varepsilon)/2+J-n}$$

(5.9)

for $\varepsilon > 0$ and $\alpha < J - n + (\varepsilon/2)$, where $J = n/\min\{1, p, q\}$ and $C$ is independent of $P$ and $Q$.

For $\ell(Q) < \ell(P)$, by [2, page 152, Lemma B.2], we obtain

$$|\langle \psi_p, \Phi_Q \rangle| \leq C \left( 1 + \frac{|x_Q - x_P|}{\ell(P)} \right)^{-J-\varepsilon} \left( \frac{\ell(Q)}{\ell(P)} \right)^{n/2} \left( \frac{\ell(Q)}{\ell(P)} \right)^{(n-2\alpha)/2}.$$  

(5.10)

Choosing $\varepsilon = -2\alpha$, we obtain the result. $\Box$

We now can prove Theorem 1.13.

Proof of Theorem 1.13. To simplify notations, let $q_0 = qr / (q - r)$ and $(1/p_0) = (1/p) - (1/q) + (1/q_0)$. The requirement $p \leq r < q < r / (1 - r)$ guarantees that $p_0 \leq 1 \leq q_0$. Now assume that $g \in \text{CMO}_{(q_0/p_0) - (q_0/q)}^{\beta, \beta_0}$ and $f \in f_p^{\alpha,q}$. To prove part (i), by (3.1) we rewrite $\Pi_g(f)$ as

$$\Pi_g(f) = \sum_Q \langle g, \varphi_Q \rangle |Q|^{-1/2} \left( \sum_P \langle f, \varphi_P \rangle \psi_P, \Phi_Q \right) \varphi_Q$$

(5.11)

$$= \sum_Q \langle g, \varphi_Q \rangle |Q|^{-1/2} (G_s)_Q \varphi_Q,$$

where $s = \{(f, \varphi_p)\}_{p}$. Proposition 3.1 and Theorem 2.2 give

$$\|\Pi_g(f)\|_{f_p^{\beta,\beta_0}} \leq C \left\| \left( |Q|^{-1/2} \langle g, \varphi_Q \rangle (G_s)_Q \right) \right\|_{f_p^{\beta,\beta_0}}$$

$$= C \sum_Q \left( |Q|^{-(\beta/n) - (1/2) + (1/2r)} \right) \left( \langle g, \varphi_Q \rangle \right)^{\tau} \left( |Q|^{-(\alpha/n) - (1/2)} \right) \| (G_s)_Q \|^r$$

(5.12)

$$\leq C \left\| \left( |Q|^{-(\beta/n) - (1/2) + (1/2r)} \right) \langle g, \varphi_Q \rangle \right\|_{f_p^{\beta,\beta_0}} \left\| (G_s)_Q \right\|_{f_p^{\beta,\beta_0}}.$$
It is clear that

\[ \left\| \left\{ \left| Q \right|^{-\beta/n} \left| \langle g, \varphi_Q \rangle \right| \right\} \right\|_{L^{1/q}(r/q', r/q)} = \sup_{p'} \left\{ \left| P \right|^{-r/q'} \left( \left| Q \right|^{(1/p) - (1/q)} \right) \right\|_{L^{1/q}(r/q', r/q)} \right\} \left\{ \left| Q \right|^{-\beta/n} \left| \langle g, \varphi_Q \rangle \right| \chi_Q(x) \right\|_{L^{1/q}(r/q', r/q)} \right\}^{1/(q/r)} \]

and

\[ \left\| \left\{ \left| Q \right|^{-(\alpha/n) - \beta/n} \left| (G_s)_Q \right| \right\} \right\|_{L^{1/q}(r/q', r/q)} = \left\| \left( \sum_Q \left| Q \right|^{-\beta/n} \left| (G_s)_Q \right| \chi_Q(x) \right\|_{L^{1/q}(r/q', r/q)} \right\|^{1/q} \]

Hence, by Propositions 3.1 and 3.3, and Lemma 5.4,

\[ \left\| \Pi_f(f) \right\|_{F^{\alpha,q}_p} \leq C \left\| \left\| \left\{ \langle g, \varphi_Q \rangle \right\} \right\|_{L^{1/q}(r/q', r/q)} \right\|_{F^{\alpha,q}_p} \leq C \left\| g \right\|_{\text{CMO}^{\alpha_0,q_0}_{(a_0/q_0) - (a_0/q_0)}} \left\| s \right\|_{F^{\alpha,q}_p} \leq C \left\| g \right\|_{\text{CMO}^{\alpha_0,q_0}_{(a_0/q_0) - (a_0/q_0)}} \left\| f \right\|_{F^{\alpha,q}_p}. \]

Next suppose that \( \Pi_f \) is bounded from \( F^{\alpha,q}_p \) into \( F^{\alpha+\beta,q}_p \). Without lost of generality, we may assume that \( \alpha = 0 \). A computation yields

\[ \left( \left| P \right|^{-\beta/q_1(q_1)} \left( \int \sum_{Q \subseteq P} \left| Q \right|^{-\beta/n} \left| \langle g, \varphi_Q \rangle \chi_Q(x) \right|^{q_0} \right)^{1/q_0} \right)^{1/q_0} \]

\[ = \left| P \right|^{\beta/q_0} \left( \int \sum_{Q \subseteq P} \left| Q \right|^{-\beta/n} \left| \langle g, \varphi_Q \rangle \chi_Q(x) \right|^{(q_0/q_0')^{-1/q_0}} \right) \]

\[ \leq C \left| P \right|^{\beta/q_0} \left( \int \sum_{Q \subseteq P} \left| Q \right|^{-\beta/n} \left| \langle g, \varphi_Q \rangle \chi_Q(x) \right|^{q_0/q_0'} \right)^{1/q_0'}. \]
Fix an integer \( N > (n/p) - n \). Choose a function \( \theta \in S(\mathbb{R}^n) \) satisfying \( \theta(x) = 1 \) on \([0,1]^n\), \( \theta(x) = 0 \) if \( x \notin [0,1]^n \) and \( \int x^y \theta(x) \, dx = 0 \) for all multi-indices \( y \) with \( |y| \leq N \). By the molecular theory [2, page 56], it follows that \( \theta \in F^{0,q}_P \). For each dyadic cube \( P \), define \( \theta^P \) by

\[
\theta^P(x) = \theta\left(\frac{x - x_P}{\ell(P)}\right). \tag{5.17}
\]

Then, \( \langle \theta^P, \Phi_Q \rangle = \int \Phi_Q(x) \, dx = |Q|^{1/2} \) for all dyadic cubes \( Q \subseteq P \) and \( \|\theta^P\|_{F^0_q} = C|P|^{1/r} \) by the translation invariance and the dilation properties of \( F^0_q \). By Proposition 3.1,

\[
\left\| \Pi_g(\theta^P) \right\|_{f^r_q} \approx \left\| \left\{ \langle g, \varphi_Q \rangle |Q|^{-1/2} \langle \theta^P, \Phi_Q \rangle \right\}_Q \right\|_{f^r_q}^r \geq \left( \int_P \sum_{Q \subseteq P} \left( |Q|^{-(\beta/n)-(1/2)} \right) \left( \langle g, \varphi_Q \rangle |X_Q(x)\rangle \right)^r \, dx \right)^{1/r}, \tag{5.18}
\]

and hence, by the boundedness of \( \Pi_g \),

\[
\left( |P|^{-q_0(1/p_0)+(1/q_0)-1} \right) \int_P \sum_{Q \subseteq P} \left( |Q|^{-(\beta/n)-(1/2)} \right) \left( \langle g, \varphi_Q \rangle |X_Q(x)\rangle \right)^{q_0} \, dx \leq C. \tag{5.19}
\]

Taking the supremum on \( P \), we show that \( g \in \text{CMO}^{\beta,q_0}_{(q_0/p_0) - (q_0/q_0)} \).

To prove part (ii), assume that \( g \in \text{CMO}^{\beta,q_0}_{(q_0/p_0) - (q_0/q_0)} \) and \( f \in F^{\alpha,q}_{P} \). Let \( t = \{ \langle g, \varphi_Q \rangle \}_Q \) and \( s = \{ \langle f, \varphi_Q \rangle \}_Q \). By Proposition 3.1,

\[
\left\| \Pi_g(t) \right\|_{f^r_{q'}} \approx \left\| \sum_P \left( |P|^{-1/2} \langle g, \varphi_P \rangle \Phi_P, \varphi_Q \rangle \langle f, \varphi_P \rangle \right)_Q \right\|_{f^r_{q'}} = \| \tilde{G} \|_{f^r_{q'}} \tag{5.20}
\]

where \( \tilde{G} := \{ \langle \Phi_P, \varphi_Q \rangle \}_Q \) is the transpose of \( \{ \langle \varphi_P, \Phi_Q \rangle \}_Q \). Since \( \alpha + \beta > 0 \), by Lemma 5.4, \( \tilde{G} \) is \((\alpha + \beta, r, r)\)-almost diagonal and hence is bounded on \( F^r_{\alpha + \beta,r} \). Following the same argument as the proof of part (i), we get

\[
\left\| \Pi_g(t) \right\|_{f^r_{q'}} \leq C \|Dts\|_{f^r_{\alpha r}},
\]

\[
= C \sum_{Q} \left( |Q|^{-(\beta/n)-(1/2)+(1/2r)} \right) \| g \|_{\text{CMO}^{\beta,q_0}_{(\alpha/n) - (\alpha/q_0)}} \| f \|_{f^r_{\alpha,q}} \tag{5.21}
\]

which completes the proof.
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