

*Research Article*

# Some Chaotic Properties of Discrete Fuzzy Dynamical Systems

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Received 16 September 2012; Revised 21 November 2012; Accepted 21 November 2012

Academic Editor: Gani Stamov

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Letting  $(X, d)$  be a metric space,  $f : X \rightarrow X$  a continuous map, and  $(\mathcal{F}(X), D)$  the space of nonempty fuzzy compact subsets of  $X$  with the Hausdorff metric, one may study the dynamical properties of the Zadeh's extension  $\hat{f} : \mathcal{F}(X) \rightarrow \mathcal{F}(X) : u \mapsto \hat{f}u$ . In this paper, we present, as a response to the question proposed by Román-Flores and Chalco-Cano 2008, some chaotic relations between  $f$  and  $\hat{f}$ . More specifically, we study the transitivity, weakly mixing, periodic density in system  $(X, f)$ , and its connections with the same ones in its fuzzified system.

## 1. Introduction

Throughout this paper, let  $(X, d)$  be a compact metric space with metric  $d$  and let  $f : X \rightarrow X$  be continuous. A discrete dynamical system is a pair  $(X, f)$ . For other notions and notations mentioned in this section, we refer to Section 2. The main goal of the theory of discrete dynamical system is to understand the dynamics of individuals (points) in the state space  $X$ . However, in many cases such as biological species and migration phenomenon, it is not enough to know how the points move, but it is necessary to know how the subsets of  $X$  move, which is so-called collective dynamics. When studying the chaotic dynamics of individual members in a certain ecosystem, the natural question that arise is what the relationship between individual chaos and collective chaos is.

Motivated by this question, the study of set-valued discrete systems has recently become active [1–6]. Moreover, when the available datas are uncertain, the fuzzy system should be considered:

$$u_{n+1} = \widehat{f}(u_n), \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where  $f : X \rightarrow X$  is a continuous map on a metric space  $(X, d)$  and  $\widehat{f} : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$  is the Zadeh's extension of  $f$  to  $\mathcal{F}(X)$ , the class of all nonempty compact fuzzy subsets of  $X$ . Consequently, the fundamental question here is to analyze relations between  $f$  and  $\widehat{f}$ : When does the chaoticity of  $f$  imply the chaoticity of  $\widehat{f}$ ? And conversely?

As a partial response to this question, in the case of Devaney chaos, Román-Flores and Chalco-Cano [7] investigate the discrete fuzzy dynamical system associated to  $f$  given by  $(\mathcal{F}(X), \widehat{f})$ , and then obtain the following results:

$$\begin{aligned} \widehat{f} \text{ transitive} &\Rightarrow f \text{ transitive,} \\ \widehat{f} \text{ transitive} &\not\Leftarrow f \text{ transitive,} \\ \widehat{f} \text{ sensitive} &\Rightarrow f \text{ sensitive,} \\ f \text{ has periodic density} &\Rightarrow \widehat{f} \text{ has periodic density.} \end{aligned}$$

In addition, by analyzing connections between the fuzzified dynamical system related to the original one, the authors have pointed out that this kind of investigation should be useful in many real problems, such as in ecological modelling and demographic sciences. Some recent works along these lines appear, for example, we refer to [8, 9], where different dynamical properties were demonstrated. In [8], Kupka shows that the dynamical behavior of the set-valued and fuzzy extension of the original system mutually inherits some global characteristics. In particular, the author proves that there exists a transitive fuzzification on the space of normal fuzzy sets, which contains the solution of the problem that was partially solved in [7]. Specifically, the author considers a symbolic dynamical system as the original system and then shows that Zadeh's extension of the shift map is transitive. As regards periodic density, a concept of piecewise constant fuzzy set is introduced in [8], and then period density equivalence of  $f$  and  $\widehat{f}$  is proposed. Together, Kupka presents a complete solution of the open problem that has been established in [7].

In this paper, unless otherwise stated a chaotic map is always Devaney chaotic. We investigate relations between  $f$  and  $\widehat{f}$  and other dynamical concepts that describe chaos. Although the problem has been completely solved in [8], we propose some different methods to solve the problem. Following [8, 10] and references therein, the space of fuzzy sets does not admit a transitive fuzzification, thus it cannot be mixing, weakly mixing, and so forth. Hence, it makes sense to consider the space of normal fuzzy set on  $X$ . We show that in case where  $f$  is weakly mixing,  $\widehat{f}$  is chaotic provided that  $f$  is chaotic. In the converse direction of the question mentioned above, we prove that, in certain conditions,  $\widehat{f}$  chaotic implies  $f$  chaotic. Moreover, we also prove that a totally transitive map with dense small period set is weakly mixing.

Below, Section 2 gives some known definitions and notations. In Section 3 and Section 4 the main results are presented. We conclude this paper with some discussions in Section 5.

## 2. Preliminaries

In this section, we complete notations and recall some known definitions. Let  $\mathcal{K}(X)$  be the class of all nonempty and compact subset of  $X$ . If  $A \in \mathcal{K}(X)$  we define the  $\varepsilon$ -neighbourhood of  $A$  as the set

$$N(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}, \quad (2.1)$$

where  $d(x, A) = \inf_{a \in A} \|x - a\|$ .

The Hausdorff separation  $\rho(A, B)$  of  $A, B \in \mathcal{K}(X)$  is defined by

$$\rho(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N(B, \varepsilon)\}. \quad (2.2)$$

The Hausdorff metric on  $\mathcal{K}(X)$  is defined by letting

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}. \quad (2.3)$$

Define  $\mathcal{F}(X)$  as the class of all upper semicontinuous fuzzy sets  $u : X \rightarrow [0, 1]$  such that  $[u]_\alpha \in \mathcal{K}(X)$ , where  $\alpha$ -cuts and the *support* of  $u$  are defined by

$$\begin{aligned} [u]_\alpha &= \{x \in X \mid u(x) \geq \alpha\}, \quad \alpha \in [0, 1], \\ \text{supp}(u) &= \overline{\{x \in X \mid u(x) > 0\}}, \end{aligned} \quad (2.4)$$

respectively.

Moreover, let  $\mathcal{F}^1(X)$  denote the space of all *normal fuzzy sets* on  $X$  and  $\emptyset_X$  denote the *empty fuzzy set* ( $\emptyset_X(x) = 0$  for all  $x \in X$ ).

A *levelwise metric*  $D$  on  $\mathcal{F}(X)$  is defined by

$$D(u, v) = \sup_{\alpha \in [0, 1]} H([u]_\alpha, [v]_\alpha), \quad (2.5)$$

for all  $u, v \in \mathcal{F}(X)$ , where  $[u]_\alpha = \{x \in X \mid u(x) \geq \alpha\}$ . It is well known that if  $(X, d)$  is complete, then  $(\mathcal{F}(X), D)$  is also complete but is not compact and is not separable (see [9, 11, 12]).

We say that  $f$  is *transitive* if for any pair of nonempty open sets  $U$  and  $V$  there exists  $n \geq 1$  such that  $f^n(U) \cap V \neq \emptyset$ ;  $f$  is *totally transitive* if all its iterates  $f^n$  is transitive;  $f$  is *weakly mixing* if for all nonempty open sets  $U_1, U_2, V_1$ , and  $V_2$  there exists  $k \geq 1$  such that  $f^k(U_1) \cap V_1 \neq \emptyset$  and  $f^k(U_2) \cap V_2 \neq \emptyset$ ;  $f$  is *mixing* if for any pair of nonempty open sets  $U$  and  $V$  there exists  $N \geq 1$  such that for all  $k \geq N$  one has  $f^k(U) \cap V \neq \emptyset$ . We say that  $f$  has *dense small period sets* if for every nonempty open set  $U \subset X$  there is a closed subset  $A \subset U$  and  $n > 0$  such that  $f^n(A) = A$ .

A point  $x$  is *periodic* if  $f^n(x) = x$  for some  $n \geq 1$ . We say that  $f$  has *sensitive dependence on initial conditions* if there is a constant  $\delta > 0$  such that for every point  $x$  and every neighborhood  $U$  about  $x$ , there is a  $y \in U$  and a  $k \geq 1$  such that  $d(f^k(x), f^k(y)) \geq \delta$ . A map that is transitive, has a dense set of periodic points, and has sensitive dependence on initial conditions is called *Devaney chaotic* [13]. However, sensitive dependence on initial conditions is a consequence of

transitivity together with a dense set of periodic points [14, 15]. More precisely, sensitivity is redundant in the definition if the state space  $X$  is infinite. This fact reveals the topological, rather than metric, nature of chaos. In this paper, we say that a map  $f$  is *Devaney chaotic* (*chaotic* for short), if it is transitive and has dense set of periodic points.

**Proposition 2.1** (see [7]). *The family  $\{[u]_\alpha : \alpha \in [0, 1]\}$  satisfies the following properties:*

- (1)  $[u]_0 \supseteq [u]_\alpha \supseteq [u]_\beta$ , for all  $0 \leq \alpha \leq \beta \leq 1$ ,
- (2)  $u = v$  if and only if  $[u]_\alpha = [v]_\alpha$ , for all  $\alpha \in [0, 1]$ ,
- (3)  $[\widehat{f}(u)]_\alpha = f([u]_\alpha)$ , for all  $\alpha \in [0, 1]$ ,
- (4)  $\widehat{f}^n = \widehat{f^n}$ .

**Proposition 2.2** (see [7, 8]). *Let  $A$  be an open subset of  $X$ . Define  $e(A) = \{u \in \mathcal{F}(X) : [u]_0 \subseteq A\}$ , then  $e(A)$  is an open subset of  $\mathcal{F}(X)$ .*

**Proposition 2.3** (see [7, 8]). *If  $f$  has periodic density on  $X$ , then  $\widehat{f}$  has periodic density on  $\mathcal{F}(X)$ .*

**Proposition 2.4** (see [4]). *For any continuous map  $f$ ,  $f$  is weakly mixing if and only if for any nonempty open sets  $A$  and  $B$  there is a  $k \geq 1$  such that  $f^k(A) \cap B \neq \emptyset$  and  $f^k(B) \cap A \neq \emptyset$ .*

### 3. $\widehat{f}$ Chaotic Implies $f$ Chaotic

In this section, some conditions are discussed, under which  $\widehat{f}$  chaotic implies  $f$  chaotic. Let  $\mathcal{M}$  be a subspace of  $\mathcal{F}(X)$ . Notice that  $\widehat{f}_\mathcal{M}(u) = \widehat{f}(u)$  for all  $u \in \mathcal{M}$ . We say that a topological space  $X$  has the fixed point property (in short, *f.p.p.*) if every continuous map  $f : X \rightarrow X$  has a fixed point. We will denote the family of all nonempty compact subsets of  $X$  which have the *f.p.p.* by  $\mathcal{K}_p(X)$ . Define  $\mathcal{F}_p(X) = \{u \in \mathcal{F}(X) : [u]_\alpha \in \mathcal{K}_p(X)\}$ . Theorem 3.2 below shows that when  $\widehat{f}$  chaotic will imply  $f$  chaotic. Note that in [7, 8],  $\widehat{f}$  transitive implies  $f$  transitive, to prove  $f$  is chaotic, it suffices to prove that  $f$  has periodic density.

*Remark 3.1.* Let  $A$  be a subset of  $X$  and let  $e_\mathcal{M}(A) = \{u \in \mathcal{M} : [u]_0 \subseteq A\}$ . We can conclude that if  $A$  is an open subset of  $X$ , then  $e_\mathcal{M}(A)$  is an open subset of  $\mathcal{F}(X)$ . Consequently, we obtain  $\widehat{f}_\mathcal{M}$  transitive implies  $f$  transitive, the proof is similar to the Theorem 3 in [7].

**Theorem 3.2.** *Let  $f : X \rightarrow X$  be continuous,  $\widehat{f}$  the Zadeh's extension of  $f$ , and  $\mathcal{M}$  a subspace of  $\mathcal{F}(X)$  such that  $\chi_{\{x\}} \in \mathcal{M}$  for all  $x \in X$ . If  $\mathcal{M} \subseteq \mathcal{F}_p(X)$ , then  $\widehat{f}_\mathcal{M}$  chaotic implies  $f$  chaotic.*

*Proof.* Since  $\widehat{f}_\mathcal{M}$  is transitive, it follows, by Remark 3.1, that  $f$  is transitive. Therefore, it suffices to show that  $f$  is periodically dense.

If  $x \in X$  and  $\varepsilon > 0$ , then  $\chi_{\{x\}} \in \mathcal{M}$  and, by periodic density of  $\widehat{f}_\mathcal{M}$ , there exist  $v \in \mathcal{M}$  and  $n \in \mathbb{N}$  such that

- (a)  $D(\chi_{\{x\}}, v) < \varepsilon$ ,
- (b)  $\widehat{f}_\mathcal{M}^n(v) = \widehat{f}^n(v) = v$ .

On one hand, by Proposition 2.1 (3) and (4), we have  $f^n([\nu]_\alpha) = [\nu]_\alpha$ . Thus, combing (a) and (b), we have

$$d(x, f^n(y)) < \varepsilon, \tag{3.1}$$

for all  $y \in [\nu]_\alpha$ .

On the other hand, the map  $g : [\nu]_\alpha \rightarrow [\nu]_\alpha$  given by  $g(y) = f^n(y)$  for every  $y \in [\nu]_\alpha$  is a continuous map. Since  $L_\alpha \nu$  has the *f.p.p.* (recall that  $\mathcal{M} \subseteq \mathcal{F}_p(X)$ ), it follows that  $g$  has a fixed point  $y_p$  such that  $g(y_p) = f^n(y_p) = y_p$ , that is to say,  $y_p$  is a periodic point of  $f$  contained in  $[\nu]_\alpha$ . Thus, due to (3.1), we obtain  $d(x, y_p) < \varepsilon$  for all  $x \in X$ . Consequently,  $f$  has periodic density on  $X$ . The theorem is proved.  $\square$

*Remark 3.3.* The conditions on  $\mathcal{M}$  in Theorem 3.2 are restrictive. In fact, we could consider  $\mathcal{M}$  with different conditions, for instance  $\mathcal{M}$  being the subspace of compact convex fuzzy set. More specifically, let  $\mathcal{F}_c$  be the family of all fuzzy sets  $u : \mathbb{R}^n \rightarrow [0, 1]$  that satisfy

- (i)  $u \in \mathcal{F}(\mathbb{R}^n)$ , which means that  $u$  is upper semicontinuous and  $[u]_\alpha$  is compact,
- (ii)  $u$  is fuzzy convex, that is,  $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ ,
- (iii)  $[u]_1$  is nonempty.

If  $\mathcal{F}_c$  is endowed with the topology generated by the *levelwise metric*  $D$ , then each compact convex subset of  $(\mathcal{F}_c, D)$  has the *f.p.p.* [12]. Hence, the conditions in Theorem 3.2 could be slightly changed.

**Proposition 3.4.** *If  $\widehat{f}$  is transitive, then  $f$  is weakly mixing.*

*Proof.* Suppose  $\widehat{f}$  is transitive. By Proposition 2.4, it suffices to show that for any nonempty open sets  $A$  and  $B$ , there is a  $k \geq 1$  such that

$$f^k(A) \cap B \neq \emptyset, \quad f^k(B) \cap A \neq \emptyset. \tag{3.2}$$

Due to Proposition 2.2,  $e(A)$  and  $e(B)$  are open subsets of  $X$  and so,  $e(A) \cap e(B)$  is open. Thus, by transitivity of  $\widehat{f}$ , there is a  $k \geq 1$  such that

$$\emptyset \neq \widehat{f}^k(e(A) \cap e(B)) \cap e(B) \subset \widehat{f}^k(e(A)) \cap \widehat{f}^k(e(B)) \cap e(B). \tag{3.3}$$

Therefore, we have

$$\widehat{f}^k(e(A)) \cap e(B) \neq \emptyset, \quad \widehat{f}^k(e(B)) \cap e(A) \neq \emptyset. \tag{3.4}$$

On one hand, since  $\widehat{f}^k(e(A)) \cap e(B) \neq \emptyset$ , there exist  $u \in e(A)$  and  $v \in e(B)$  such that  $\widehat{f}^k(u) = v$ . By Proposition 2.1(3) and (4), we have

$$f^k([u]_0) = [v]_0 \subseteq B. \tag{3.5}$$

For any  $x \in [u]_0 \subseteq A$ , we have  $f^k(A) \cap B \neq \emptyset$ . On the other hand, we can choose  $\omega_1, \omega_2 \in e(B)$  such that  $\hat{f}^k(\omega_1) = \omega_2$ , it would then follow by induction that  $f^k(B) \cap B \neq \emptyset$ .  $\square$

It is known that a totally transitive system having dense period points is weakly mixing. The following proposition shows that a totally transitive map with dense small period sets is also weakly mixing.

**Proposition 3.5.** *If  $\hat{f}$  is a totally transitive map with dense small period sets, then  $\hat{f}$  is weakly mixing.*

*Proof.* Let  $U, V, Y$ , and  $Z$  be any nonempty open subsets of  $\mathcal{F}(X)$ . It suffices to show that  $(\hat{f} \times \hat{f})^n(U \times V) \cap (Y \times Z) \neq \emptyset$  for some  $n \geq 1$ . Since  $f$  is transitive, there is a  $k \neq 1$  such that  $W = U \cap f^{-k}(Y)$  is open and nonempty. Hence  $W$  contains a closed subset  $G \subset W$  of period  $m$ . For  $j \geq 1$ , we have  $f^{mj+k}(G) = f^k(G) \subset Y$ , whence  $f^{mj+k}(U) \cap Y \neq \emptyset$ . Since  $f^{-k}(Z)$  is nonempty and open and  $\hat{f}^m$  is transitive, there is a  $j_0 \geq 1$  such that  $f^{mj_0+k}(V) \cap Z \neq \emptyset$  whence  $f^{mj_0+k}(V) \cap Z \supset f^k(f^{mj_0}(V) \cap f^{-k}(Z)) \neq \emptyset$ . Setting  $n = mj_0 + k$  implies that  $\hat{f}$  is weakly mixing.  $\square$

#### 4. $f$ Weakly Chaotic Implies $\hat{f}$ Chaotic

Concerning the transitivity of fuzzy dynamical systems, the authors in [7] has proved that  $\hat{f}$  transitive implies  $f$  transitive, but the converse is not true. In [8, 10] the author presents that no fuzzification can be transitive on the whole  $\mathcal{F}(X)$ , but there exists a transitive fuzzification on the space of normal fuzzy sets. In this section, we propose another method to prove that  $f$  is weakly mixing implies  $\hat{f}$  is weakly mixing and thus transitive. It should be mentioned that our approach was inspired by the idea presented in [8, 10].

We say that a map  $f$  is *weakly Devaney chaotic* (*weakly chaotic* for short) if it is weakly mixing and periodically dense. Let  $U$  be a subset of  $\mathcal{F}^1(X)$ . Set

$$r(U) = \{A \in \mathcal{K}(X) \mid \exists u \in U \text{ s.t. } A \subseteq [u]_0\}. \quad (4.1)$$

**Proposition 4.1.** *Let  $U, V$ , and  $W$  be subsets of  $\mathcal{F}^1(X)$ ,*

- (1)  $r(U) \neq \emptyset$  if and only if  $U \neq \emptyset_X$ , where  $\emptyset_X$  is the empty fuzzy set ( $\emptyset_X = 0$  for each  $x \in X$ ),
- (2) suppose that  $u \neq v$  implies  $[u]_0 \cap [v]_0 = \emptyset$ , then  $r(U \cap V) = r(U) \cap r(V)$ ,
- (3)  $f(r(U)) \subseteq r(\hat{f}(U))$ ,
- (4) if  $U$  is a nonempty open subset of  $\mathcal{F}^1(X)$ , then  $r(U)$  is a nonempty open subset of  $X$ .

*Proof.* (1) Follows directly from the definitions.

(2) If  $A \in r(U \cap V)$ , then there exists  $\omega \in U \cap V$  such that  $A \in [\omega]_0$ . Then  $A \in r(U)$  and  $A \in r(V)$ . Therefore, the inclusion  $r(U \cap V) \subseteq r(U) \cap r(V)$  follows. Conversely, let  $A \in r(U) \cap r(V)$ . Then there exist  $u \in U$  and  $v \in V$  such that  $A \subseteq [u]_0$  and  $A \subseteq [v]_0$ , respectively. Hence, by hypothesis,  $A \subseteq [u]_0 \cap [v]_0$  which means that  $[u]_0 \cap [v]_0 \neq \emptyset$  and so,  $u = v$ . Consequently, there is  $u \in U \cap V$  such that  $A \in r(U \cap V)$  and the inclusion  $r(U \cap V) \supseteq r(U) \cap r(V)$  is true.

(3) If  $y \in f(r(U))$ , then there exists  $x \in A \subseteq [u]_0$  such that  $y = f(x)$ . Thus, by Proposition 2.1 (3), we have  $y = f(x) \in f([u]_0) = [\hat{f}(u)]_0$ , consequently,  $y \in r(\hat{f}(U))$ .

(4) Suppose that  $r(U)$  is not open. For any  $A \in r(U) \setminus \text{int}(r(U))$  and  $\varepsilon > 0$ , there exists open  $\varepsilon$ -neighborhood  $N$  of  $A$  such that  $N \cap r(U) \neq \emptyset$  and  $N \not\subseteq r(U)$ . Consider a fuzzy set  $\chi_{\{A\}}$ . Since  $\chi_{\{A\}} \in U$  and

$$D(\chi_N, \chi_{\{A\}}) = \sup_{\alpha \in [0,1]} H([\chi_N]_\alpha, [\chi_{\{A\}}]_\alpha) \leq \varepsilon, \quad (4.2)$$

we obtain  $\chi_N \in B(\chi_{\{A\}}, \varepsilon)$ , where  $B(\chi_{\{A\}}, \varepsilon)$  is an open ball in  $\mathcal{F}^1(X)$ . However,  $\chi_N \notin U$ , and consequently,  $B(\chi_{\{A\}}, \varepsilon) \not\subseteq U$ . That is to say, no  $\varepsilon$ -neighborhood of  $\chi_{\{A\}}$  contains in  $U$ , this contradicts the fact that  $U$  is open in  $\mathcal{F}^1(X)$ .  $\square$

**Proposition 4.2.** *If  $f$  is weakly mixing then  $\widehat{f}$  is weakly mixing.*

*Proof.* Suppose  $f$  is weakly mixing. By Proposition 2.4, it suffices to show that for any nonempty open sets  $U$  and  $V$  of  $\mathcal{F}^1(X)$ , there is a  $k \geq 1$  such that

$$\widehat{f}^k(U) \cap V \neq \emptyset, \quad \widehat{f}^k(V) \cap V \neq \emptyset. \quad (4.3)$$

Since  $U$  and  $V$  are open, by Proposition 4.1(4),  $r(U)$  and  $r(V)$  are also open sets. Due to  $f$  is weakly mixing, there is a  $k \geq 1$  such that

$$f^k(r(U)) \cap r(V) \neq \emptyset, \quad f^k(r(V)) \cap r(V) \neq \emptyset. \quad (4.4)$$

By Propositions 4.1(3) and 2.1(4), we have

$$r(\widehat{f}^k(U)) \cap r(V) \neq \emptyset, \quad r(\widehat{f}^k(V)) \cap r(V) \neq \emptyset. \quad (4.5)$$

Thus, using Proposition 4.1 (2), it follows that

$$r(\widehat{f}^k(U) \cap V) \neq \emptyset, \quad r(\widehat{f}^k(V) \cap V) \neq \emptyset. \quad (4.6)$$

Therefore,

$$\widehat{f}^k(U) \cap V \neq \emptyset, \quad \widehat{f}^k(V) \cap V \neq \emptyset. \quad (4.7)$$

$\square$

Theorem 4.3 below is an immediate consequence, it shows that in case when  $f$  is weakly mixing,  $\widehat{f}$  will be chaotic provided  $f$  is chaotic.

**Theorem 4.3.** *Let  $f : X \rightarrow X$  be continuous and  $\widehat{f}$  the Zadeh's extension of  $f$ . If  $f$  is weakly chaotic, then  $\widehat{f}$  is chaotic.*

*Proof.* By Proposition 4.2,  $\widehat{f}$  is weakly mixing and hence transitive, combine this assertion and Proposition 2.3, we can conclude that  $\widehat{f}$  is chaotic.  $\square$

## 5. Conclusions and Discussions

In this present investigation, we discuss relations between dynamical properties of the original and fuzzified dynamical systems. More specifically, we study transitivity, periodic density, and weakly mixing and so forth. And we show that the dynamical properties of the original system and its fuzzy extension mutually inherits some global characteristics. More precisely, the following implications are obtained

- (a)  $\widehat{f}_M$  chaotic implies  $f$  chaotic (Theorem 3.2),
- (b)  $\widehat{f}$  is transitive  $\Rightarrow f$  is weakly mixing (Proposition 3.4),
- (c)  $f$  is weakly mixing  $\Rightarrow \widehat{f}$  is weakly mixing (Proposition 4.2),
- (d)  $f$  is weakly chaotic  $\Rightarrow \widehat{f}$  is chaotic (Theorem 4.3).

Actually the open question raised in [7] has been completely solved in [9], here we propose another approaches to answer the question. It is worth noting that to complete the generalization of the system  $(\mathcal{F}(X), \widehat{f})$ , we need to endow  $\mathcal{F}(X)$  with a metric. When considering the practical interpretation and computer realization, a question can be addressed here: Is there any other metric available? In [9], the author discusses some other metrics on this subject.

On the other hand, it is well known that any given discrete dynamical system uniquely induces its fuzzified counterpart, that is, a discrete fuzzy dynamical system. There have been various attempt to “fuzzify” the discrete dynamical systems. One of these methods appeared in [16] where either a  $t$ -norm or  $t$ -conorm  $\Gamma$  was used to fuzzify discrete systems and then elaborated and specified in [9] in a more general way, namely,  $g$ -fuzzification. Therefore, on the basis of the idea presented in [9], it would be interesting to study the relations between dynamical properties of the original and  $g$ -fuzzified dynamical systems. And this will be one aspect of our future works.

## Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (no. 11226268, no. 11071061, no. 61202349), the National Basic Research Program of China (no. 2011CB311808), NNSF of Hunan Province (no. 10JJ2001), and NSF of CUAS (no. Z2010ST14).

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