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Research Article

Generalized Chessboard Structures Whose Effective Conductivities Are Integer Valued

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We consider generalized chessboard structures where the local conductivity takes two values a and b. All integer combinations of a and b which make the components of effective conductivity matrix integer valued are found. Moreover, we discuss the problem of estimating the effective conductivity matrix by using the finite-element method.

1. Introduction

Let h be a large positive integer and consider a periodic composite material with period equal to $1/h$. The stationary heat conduction problem can then be formulated by the following minimum principle:

$$E_h = \min_u \left(\mathcal{F}_h(u) - \int_\Omega u(x) g(x) dx \right), \tag{1.1}$$

where

$$\mathcal{F}_h(u) = \int_\Omega \left(\frac{1}{2} \operatorname{grad} u(x) \cdot C_h(x) \operatorname{grad} u(x) \right) dx. \tag{1.2}$$

Here, u is the temperature; the conductivity matrix $C_h(x)$ is given by

$$C_h(x) = C(hx), \tag{1.3}$$

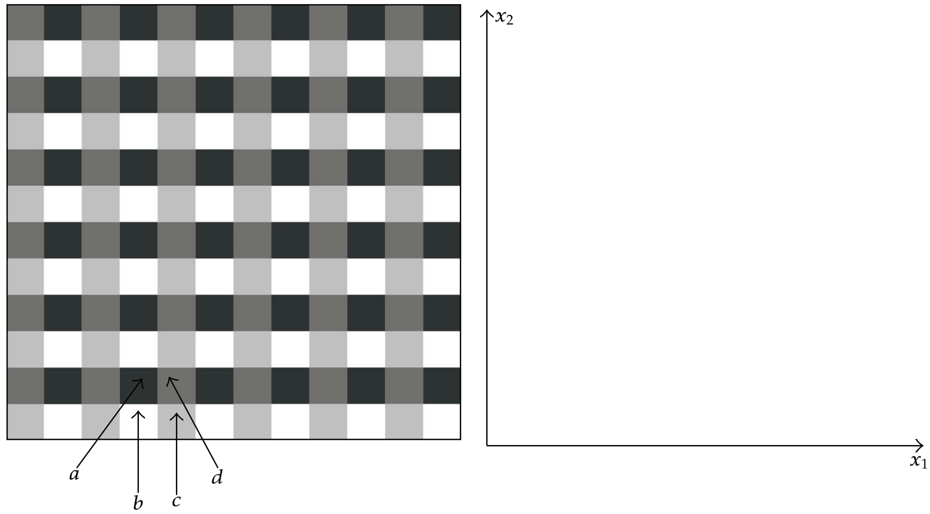


Figure 1: The four component chessboard structure.

where $C(\cdot)$ is periodic relative to the unit cube of $\mathbb{R}^m$, g is the source field, Ω is a bounded-open subset of $\mathbb{R}^m$, and the minimization is taken over some Sobolev space depending on the boundary conditions. It is possible to prove that the "energy" E_h converges to the so-called homogenized "energy" E_{hom} as $h \to \infty$, defined by

$$E_{\text{hom}} = \min_u \left(\mathcal{F}_{\text{hom}}(u) - \int_\Omega u(x) g(x) dx \right), \tag{1.4}$$

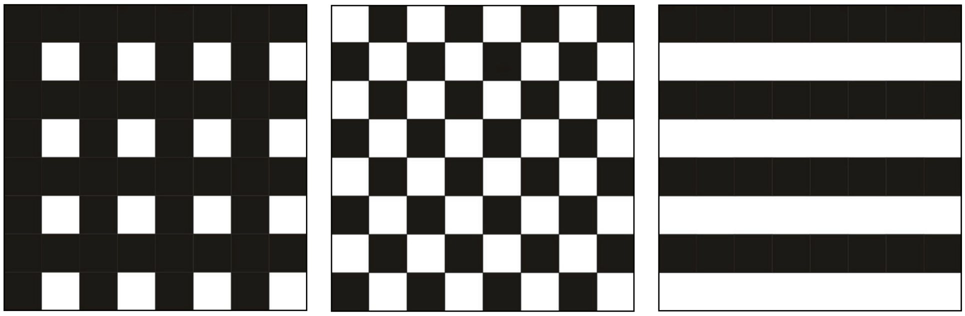
where

$$\mathcal{F}_{\text{hom}}(u) = \int_\Omega \left(\frac{1}{2} Du(x) \cdot \sigma^* Du(x) \right) dx. \tag{1.5}$$

The matrix σ^* is often called the homogenized or effective conductivity matrix and is found by solving a number of boundary value problems on the cell of periodicity. For an elementary introduction to the theory of homogenization, see, for example, the book Persson et al. [1].

There are very few microstructures where all elements of the effective conductivity matrix are known in terms of closed form explicit formulae. Laminates and chessboards are the most classical. Mortola and Steffé [2] studied in 1985 a chessboard structure consisting of four equally sized squares in each period with (isotropic) conductivities a, b, c, and d, respectively (see Figure 1). They conjectured that the corresponding effective conductivity matrix of this composite structure is given by

$$\sigma^* = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}, \tag{1.6}$$



Connected case (a) Chessboard case (b) Laminate case (c)

Figure 2: The three special cases when the local conductivity only takes two values.

where

$$\sigma_{11} = G(H(A(a,b), A(c,d)), A(H(a,d), H(b,c))),$$
$$\sigma_{22} = G(H(A(a,d), A(b,c)), A(H(a,b), H(c,d))). \quad (1.7)$$

Here, A, G, and H denote the arithmetic mean, the geometric mean and the harmonic mean, respectively, given by the formulae:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \qquad G(\alpha,\beta) = \sqrt{\alpha\beta}, \qquad H(\alpha,\beta) = \frac{2\alpha\beta}{\alpha+\beta}. \quad (1.8)$$

Mortola and Steffés conjecture was ultimately proved by Craster and Obnosov [3] and Milton [4] in 2001 (see also [5]).

In this paper, we consider the special cases when the local conductivity only takes two values (see Figure 2). One such case is when $a = c = d$, which will be referred to as the *connected case*. The remaining cases are the *chessboard case* and the *laminate case*. The first of these is characterized by the property $a = c$ and $b = d$, and the second is characterized by the property $a = d$ and $b = c$. For these three cases, we find all integer combinations of a and b which make the components of σ^* integer valued. We also discuss the problem of estimating σ^* by using the finite-element method.

2. The Connected Case

For the connected case, it is easily verified that $\sigma_{11} = \sigma_{22} = \sigma$, where σ is given by the formula:

$$\sigma = \sqrt{a^2\frac{3b+a}{3a+b}}. \quad (2.1)$$

Assume that a and b are positive integers. In order to obtain an integer value of σ, there must exist integers r_1, r_2, k, and l such that

$$3b + a = kr_1^2, \quad 3a + b = kr_2^2, \quad a = lr_2, \tag{2.2}$$

where

$$\gcd(r_1, r_2) = 1. \tag{2.3}$$

We note that (2.2) is equivalent with

$$a = k\frac{3(r_2)^2 - (r_1)^2}{8}, \qquad b = k\frac{3(r_1)^2 - (r_2)^2}{8}, \qquad a = lr_2. \tag{2.4}$$

Hence,

$$k\left(3(r_2)^2 - (r_1)^2\right) = 8lr_2. \tag{2.5}$$

It is clear that $\gcd(3(r_2)^2 - (r_1)^2, r_2) = 1$ (otherwise, $3(r_2)^2 - (r_1)^2 = ds$ and $r_2 = dt$ for some integers ts and $d > 1$, which would mean that $r_1^2 = d(3dt^2 - s)$, and since $r_2^2 = d^2t^2$, we obtain that $\gcd(r_1^2, r_2^2) \geq d$, which contradicts (2.3)). According to (2.5), this gives that

$$k = s_2 r_2 \tag{2.6}$$

for some integer s_2. By checking the four combinations when r_1 and r_2 are odd/even, we can easily verify that

$$\gcd\left(3(r_2)^2 - (r_1)^2, 8\right) = \gcd\left(3(r_1)^2 - (r_2)^2, 8\right). \tag{2.7}$$

Hence, using (2.2), we obtain that

$$a = sr_2\frac{3(r_2)^2 - (r_1)^2}{d}, \qquad b = sr_2\frac{3(r_1)^2 - (r_2)^2}{d}, \tag{2.8}$$

where

$$d = \gcd\left(r_2\left(3(r_2)^2 - (r_1)^2\right), 8\right), \tag{2.9}$$

and s is some positive integer. The corresponding integer value of σ is then given by

$$\sigma = a\sqrt{\frac{3b + a}{3a + b}} = sr_2\frac{3(r_1)^2 - (r_2)^2}{d}\sqrt{\frac{kr_1^2}{kr_2^2}} = sr_1\frac{3(r_1)^2 - (r_2)^2}{d}. \tag{2.10}$$

In order to have positive values of a and b, (2.8) shows that we only can choose values of r_1 and r_2 such that

$$\frac{1}{3}(r_2)^2 \leq (r_1)^2 \leq 3(r_2)^2. \tag{2.11}$$

Summing up, all integers a and b making σ integer valued are of the form (2.8) for some positive integers s, r_1 and r_2, satisfying (2.11), where d is given by (2.9). The corresponding value of σ is

$$\sigma = sr_1 \frac{3(r_2)^2 - (r_1)^2}{d}. \tag{2.12}$$

Conversely, all a and b of the form (2.8), satisfying (2.11), are positive integers and make σ integer valued.

For $a \neq b$, the smallest integer value of σ is obtained when $r_1 = 5$, $r_2 = 3$ (making $d = 2$) and $s = 1$, corresponding to the values $a = 3$, $b = 99$, and $\sigma = 5$.

3. The Chessboard-and Laminate Case

For the chessboard case, $\sigma_{11} = \sigma_{22} = \sigma$, where

$$\sigma = G(a, b) = \sqrt{ab}. \tag{3.1}$$

This case is very simple. We just use the representation:

$$a = kr_1^2, \qquad b = kr_2^2, \tag{3.2}$$

where r_1, r_2, and k are integers (giving $\sigma = kr_1r_2$).

For the laminate case, we find that $\sigma_{11} = A(a, b)$ and $\sigma_{22} = H(a, b)$. It is possible to show that the integers a and b making the harmonic mean integer valued are precisely those on the form:

$$a = tp(p + q), \qquad b = tq(p + q), \tag{3.3}$$

(in this case, $H(a, b) = 2tqp$) and the form:

$$a = t(2p + 1)(p + q + 1), \qquad b = t(2q + 1)(p + q + 1), \tag{3.4}$$

(in this case, $H(a, b) = t(2q + 1)(2p + 1)$) where p, q, and t are positive integers. For a proof of this fact, see [6]. Therefore, if $\sigma_{11} = A(a, b)$ and $\sigma_{22} = H(a, b)$ are integer valued, (a, b) must belong to the class (3.3) or (3.4). The latter is directly seen to generate integer values also for A. However, (3.3) gives integer values of A only if both p and q are odd (for which (3.3) may be written on the form (3.4) with t replaced by $2t$) or both even, or t is even. In any case, if both H and A are integers, we end up with the form:

$$a = 2tp(p+q), \qquad b = 2tq(p+q), \tag{3.5}$$

(in this case $\sigma_{11} = t(p+\mathrm{q})^2$ and $\sigma_{22} = 4tqp$) and the form:

$$a = t(2p+1)(p+q+1), \qquad b = t(2q+1)(p+q+1) \tag{3.6}$$

(in this case $\sigma_{11} = t(p+q+1)^2$ and $\sigma_{22} = t(2q+1)(2p+1)$).

4. Calculating σ^* by Numerical Methods

As mentioned in the introduction, the effective conductivity matrix is found by solving a number of boundary value problems on the cell of periodicity. For the connected case, the effective conductivity σ can be found by solving the following boundary value problem:

$$\begin{gathered} \operatorname{div}(\lambda(x)\operatorname{grad} u) = 0 \quad \text{on } Y, \\ u(0,x_2) = 0, \qquad u(1,x_2) = 1, \\ \frac{\partial u}{\partial x_2(x_1,0)} = \frac{\partial u}{\partial x_2(x_1,1)} = 0. \end{gathered} \tag{4.1}$$

Here, Y is the unit cell $Y = [0,1]^2$ and the conductivity λ is defined by

$$\lambda(x) = \begin{cases} a & \text{if } x \in Y \setminus [0.25, 0.75]^2, \\ b & \text{if } x \in [0.25, 0.75]^2. \end{cases} \tag{4.2}$$

In addition, we must assume continuity of normal component of $\lambda(x)\operatorname{grad} u$ through the four surfaces where $\lambda(x)$ changes its value from a to b. The effective conductivity σ is then found by calculating the integral:

$$\sigma = \int_Y \lambda(x)\left|\operatorname{grad} u\right|^2 dx. \tag{4.3}$$

The above boundary value problem can be solved numerically by using the finite-element method with relatively good accuracy. The hardest case is assumed to be the one when $b = 0$, but even in this case, we obtain a numerical value close to the exact one. Using a couple of thousand elements (built up by second-order polynomials), the numerical value of σ for the case $a = 1$ and $b = 0$ turns out to be 0.5773, which is close to the exact value:

$$\sigma = \frac{1}{\sqrt{3}} \approx 0.5773502692. \tag{4.4}$$

Using the class of integer values for a and b giving integer valued effective conductivity, we may relatively easily make internet-based student projects where the students themselves are supposed to train their skills in using FEM programs, simply by randomly generating a and b from (2.8) and ask the students to find the integer which is the closest to their numerical estimate. Without knowing the exact formula, there are no way that they can guess the correct value without doing the numerical FEM calculation. The actual evaluation can be done by a simple Java-script or other programs which can register the students progress.

For the chessboard structure, we can use the same method with the only difference that $\lambda(x)$ is defined by

$$\lambda(x) = \begin{cases} a, & \text{if } x \in [0, 0.5]^2 \cup [0.5, 1]^2, \\ b, & \text{otherwise.} \end{cases} \tag{4.5}$$

However, the numerical estimation of the effective conductivity turns out to be significantly more difficult. In order to illustrate, we have made a numerical estimation for the case when $a = 1$ and $b = 10000$. Even with about 10000 quadrilateral 8-node elements (with increasing number of elements close to the midpoint of the unit-cell), our numerical value turned out to be as high as 557, which is more than 5 times higher than the actual value which is $\sigma = 100$. There exists a numerical method which is much more efficient than the finite-element method in such problems. Concerning this, we refer to the paper [7].

The numerical calculation of σ^* for the laminate case turns out to be trivial. In fact, we only need two elements to obtain a numerical value which is exactly equal to σ_{ii}.

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