

*Research Article*

# **Unsteady Unidirectional MHD Flow of Voigt Fluids Moving between Two Parallel Surfaces for Variable Volume Flow Rates**

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The velocity profile and pressure gradient of an unsteady state unidirectional MHD flow of Voigt fluids moving between two parallel surfaces under magnetic field effects are solved by the Laplace transform method. The flow motion between parallel surfaces is induced by a prescribed inlet volume flow rate that varies with time. Four cases of different inlet volume flow rates are considered in this study including (1) constant acceleration piston motion, (2) suddenly started flow, (3) linear acceleration piston motion, and (4) oscillatory piston motion. The solution for each case is elaborately derived, and the results of associated velocity profile and pressure gradients are presented in analytical forms.

## **1. Introduction**

Magnetohydrodynamics (MHD) is an academic discipline, which studies the dynamic behaviors of the interaction between magnetic fields and electrically conducting fluids. Examples of such fluids are numerous including plasmas, liquid metals, and salt water or electrolytes. The MHD flow is encountered in a variety of applications such as MHD power generators, MHD pumps, MHD accelerators, and MHD flowmeters, and it can also be expanded into various industrial uses.

During the past decades, a great deal of papers in literatures used a combination of Navier-Stokes equations and Maxwell's equations to describe the MHD flow of the Newtonian and electrically conducting fluid. Sayed-Ahmed and Attia [1] examined the effect of the Hall term and the variable viscosity on the velocity and temperature fields of the MHD flow. Attia [2] studied the unsteady Couette flow and heat transfer of a dusty conducting

fluid between two parallel plates with variable viscosity and electrical conductivity. Osalusi et al. [3] solved unsteady MHD and slip flow over a porous rotating disk in the presence of Hall and ion-slip currents by using a shooting method.

However, the Newtonian fluid is the simplest to be solved and its application is very limited. In practice, many complex fluids such as blood, suspension fluids, certain oils, greases and polymer solution, elastomers, and many emulsions have been treated as non-Newtonian fluids.

From the literature, the non-Newtonian fluids are principally classified on the basis of their behavior in shear. A fluid with a linear relationship between the shear stress and the shear rate, giving rise to a constant viscosity, is always characterized to be a Newtonian fluid. Based on the knowledge of solutions to Newtonian fluid, the different fluids can be extended, such as Maxwell fluids, Voigt fluids, Oldroyd-B fluids, Rivlin-Ericksen fluids, and power-law fluids. In this study, we investigate the flow characteristics of the MHD flow of Voigt fluids.

The exact solutions for laminar flow in a duct with a given pressure gradient varying with time are well known. For example, Szymanski [4] gave solutions for impulsively imposed pressure gradient. For more complex cases of the Maxwell fluid, Rahaman and Ramkissoon [5] provided solutions for a pressure gradient varying exponentially with time, sinusoidal pulsating pressure gradient and constant pressure gradient. Hayat et al. [6] solved for exact solution to some simple flows of an Oldroyd-B fluid between two parallel surfaces with and without pressure gradient. Some researchers studied the flow motion caused by the movement of a boundary. Bandelli and Rajagopal [7] solved for various start-up flows of second grade fluids in domains with one finite dimension by an integral transform method. Huang and Liu [8] analytically investigated the character of viscoelastic fluids in a double-gap concentric cylinder rheometer.

In general for more realistic applications, the volume flow rates are given as the inlet condition instead of the pressure gradient. For a power law fluid, J. P. Pascal and H. Pascal [9] solved this problem by similarity transformation method. Das and Arakeri [10] gave an analytical solution for various transient volume flow rates for a Newtonian fluid, which complemented with earlier experimental work [11]. Chen et al. [12–16] extended Das and Arakeri's work by considering various non-Newtonian fluids. Hayat et al. considered the unsteady flow of an incompressible second-grade fluid in a circular duct with a given volume flow rate variation [17]. And further, Hayat et al. presented a lot of researches about the MHD flows of non-Newtonian fluids [18–36].

Based upon previous studies, we, therefore, further investigate in this paper the flow characteristics of Voigt fluids under magnetic field effects.

## 2. Mathematical Formulations

The unidirectional rheological equation of state for a Voigt fluid in  $x$ -direction is given by Lee and Tsai [37]

$$\vec{T}_{ij} = -p\delta_{ij} + \tau_{ij}, \quad i = x, y, z, \quad j = x, \quad (2.1)$$

$$\tau_{yx} = G\nu + \mu\dot{\nu}, \quad (2.2)$$

where  $T_{ij}$  is the total stress, subscript  $i$  denotes the normal direction of  $i$ -plane, subscript  $j$  denotes the stress acting direction,  $p$  is the static fluid pressure ( $p = p(x, y, z)$ ),  $\delta_{ij}$  is the Kronecker delta,  $\tau_{ij}$  is the shear stress,  $G$  is the rigidity modulus,  $\nu$  is the shear strain,  $\dot{\nu}$  is the

rate of shear strain, and  $\mu$  is the viscosity coefficient. Here  $G$ ,  $\mu$  are the material properties and are assumed to be constant. When  $G = 0$ , (2.2) reduces to that of Newtonian fluid.

The problem of the unsteady flow of incompressible Voigt fluid between the parallel surfaces is considered. The dynamic equation is

$$\nabla \cdot \vec{T} + \rho \vec{b} = \rho \frac{d\vec{V}}{dt}. \quad (2.3)$$

In the above equation,  $\vec{T}$  denotes the total stress tensor,  $\rho$  the fluid density,  $\vec{V}$  the velocity vector,  $\vec{b}$  the body force field, and  $\nabla$  the divergence operator.

The continuity equation is

$$\nabla \cdot \vec{V} = 0. \quad (2.4)$$

Using the Cartesian coordinate system  $(x, y, z)$ , the  $x$ -axis is taken as the centerline direction between these two parallel surfaces,  $y$  is the coordinate normal to the plate,  $z$  is the coordinate normal to  $x$  and  $y$ , respectively, and the velocity field is assumed in the form

$$\vec{V} = u(y, t)\vec{i}, \quad (2.5)$$

where  $u$  is the velocity in the  $x$ -coordinate direction and  $\vec{i}$  is the unit vector in the  $x$ -coordinate direction. This effectively assumes that the flow is fully developed at all points in time.

Substituting of (2.5) into (2.4) shows that the continuity equation automatically satisfied the result of substituting in (2.1) and (2.3). So we have the following scalar equation:

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} - \rho \frac{\partial u}{\partial t} - \sigma B_0^2 u, \quad (2.6)$$

where  $B_0$  is the electromagnetic field, subscript  $y$  denotes the plane normal to  $y$  direction,  $x$  the direction along the shear stress, and

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \quad (2.7)$$

where the body force is incorporated into the term of pressure gradient.

Equations (2.6) and (2.7) imply that the pressure gradient is a function of time only.

Solving (2.2) subject to  $\tau_{yx} = 0$  and  $\dot{v} = \partial u / \partial y = 0$  as  $t = 0$ , the strain function is obtained

$$v(t) = \frac{1}{\mu} e^{-(G/\mu)t} \int_0^t \tau_{yx} e^{(G/\mu)t'} dt', \quad (2.8)$$

where  $t'$  is the integration dummy variable.

Equations (2.6) and (2.8) are our governing equations describing the Voigt fluid flow between the parallel surfaces.

### 3. Methodology of Solution

Since the governing equation with boundary conditions and initial condition are known, the problem is well posed. In general, it is not an easy question to solve this kind of equation by the method of separation of variables and eigenfunctions expansion. In this paper, the Laplace transform method is used to reduce the two variables into a single variable. This procedure greatly reduces the difficulties of treating these partial differential [9] and integral equations [11].

The governing equation of motion in  $x$ -direction and the strain function are

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} - \rho \frac{\partial u}{\partial t} - \sigma B_0^2 u, \quad (3.1)$$

$$v(t) = \frac{1}{\mu} e^{-(G/\mu)t} \int_0^t \tau_{yx} e^{(G/\mu)t'} dt'. \quad (3.2)$$

As these two surfaces are  $2h$  apart, the boundary conditions are

$$\begin{aligned} u(h, t) &= 0, \\ \frac{\partial u(0, t)}{\partial y} &= 0. \end{aligned} \quad (3.3)$$

The initial condition is related to the inlet volume flow rate by

$$\int_{-h}^h u(y, t) dy = u_p(t) 2h = Q(t), \quad (3.4)$$

where  $u_p(t)$  is the given average inlet velocity and  $Q(t)$  is the given inlet volume flow rate.

The above governing equation, boundary conditions, and initial condition are prescribed and can be solved by the following calculation of Laplace transform.

Differentiating (3.2) with respect to time and taking Laplace transform, then we have

$$\hat{\tau}_{yx}(y, s) = \frac{\mu s + G}{s} \frac{\partial \hat{u}(y, s)}{\partial y}. \quad (3.5)$$

Taking the Laplace transform of (3.1) and substituting (3.5) into it, we have the governing equation

$$\frac{\partial^2 \hat{u}(y, s)}{\partial y^2} - \frac{\rho s^2 + \sigma B_0^2 s}{\mu s + G} \hat{u}(y, s) = \frac{s}{\mu s + G} \frac{\partial \hat{p}(x, s)}{\partial x}. \quad (3.6)$$

Considering the governing equation as an ordinary differential equation (with respect to  $y$ ) and boundary conditions

$$\hat{u}(h, s) = 0, \quad (3.7)$$

$$\frac{d\hat{u}(0, s)}{dy} = 0, \quad (3.8)$$

and initial condition

$$\int_{-h}^h \hat{u}(y, s) dy = \hat{u}_p(s)2h, \quad (3.9)$$

the general solution of (3.6) is

$$\hat{u}(y, s) = C_1 \sinh my + C_2 \cosh my + \Psi_p, \quad (3.10)$$

where  $\Psi_p$  is the assumed particular solution and  $m = \sqrt{(\rho s^2 + \sigma B_0^2 s) / (\mu s + G)}$ .

The boundary conditions (3.7) and (3.8) are used to solve the two arbitrary coefficients  $C_1$  and  $C_2$ . Substituting  $C_1$  and  $C_2$  into (3.10) gives

$$\hat{u}(y, s) = \Psi_p \left( 1 - \frac{\cosh my}{\cosh mh} \right). \quad (3.11)$$

Substituting (3.11) into the initial condition of (3.9),  $\Psi_p$  is readily obtained as

$$\Psi_p \int_{-h}^h \left( 1 - \frac{\cosh my}{\cosh mh} \right) dy = \hat{u}_p(s)2h \quad (3.12)$$

or

$$\Psi_p = \frac{\hat{u}_p(s)}{(1 - \sinh mh / mh \cosh mh)}. \quad (3.13)$$

Substituting  $\Psi_p$  into (3.11) gives

$$\hat{u}(y, s) = \frac{\hat{u}_p(s)(\cosh mh - \cosh my)}{(\cosh mh - \sinh mh / mh)} \quad (3.14)$$

or

$$\hat{u}(y, s) = \hat{u}_p(s)\hat{\Omega}(y, s), \quad (3.15)$$

where

$$\hat{\Omega}(y, s) = \frac{(\cosh mh - \cosh my)}{(\cosh mh - \sinh mh / mh)}. \quad (3.16)$$

Taking the inverse Laplace transform, the velocity profile is

$$u(y, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{u}_p(s)\hat{\Omega}(y, s)e^{st} ds. \quad (3.17)$$

Furthermore, the pressure gradient is found by substituting (3.11) into (3.6) to give

$$\frac{\partial \hat{p}(x, s)}{\partial x} = -(\rho s + \sigma B_0^2) \Psi_p \quad (3.18)$$

or

$$\frac{\partial \hat{p}(x, s)}{\partial x} = -(\rho s + \sigma B_0^2) \frac{\hat{u}_p(s) \cosh mh}{(\cosh mh - \sinh mh/mh)}. \quad (3.19)$$

Using the inverse transform formula, the pressure gradient distribution can also be obtained.

#### 4. Illustration of Examples

Hereafter, we will solve the cases proposed by Das and Arakeri [10] with the Voigt fluid to understand the different flow characteristics between these two fluids under the same condition.

For the first case, the piston velocity  $u_p(t)$  moves with a constant acceleration and for the second one, the piston starts suddenly from rest and then maintains this velocity. These two solutions are used to assess the trapezoidal motion of the piston, namely, the piston has three stages: constant acceleration of piston starting from rest, a period of constant velocity, and a constant deceleration of the piston to a stop. Finally, the oscillatory piston motion is also considered.

##### 4.1. Constant Acceleration Piston Motion

The piston motion of constant acceleration can be described by the following equation:

$$u_p(t) = a_p t = \left( \frac{U_p}{t_0} \right) t, \quad (4.1)$$

where  $a_p$  is the constant acceleration,  $U_p$  is the final velocity after acceleration, and  $t_0$  is the time period of acceleration.

Taking the Laplace transform of (4.1),

$$\hat{u}_p(s) = \frac{U_p}{t_0 s^2}. \quad (4.2)$$

From (3.17) and (4.2), the velocity profile is

$$u(y, t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{U_p}{t_0 s^2} \frac{(\cosh mh - \cosh my)}{(\cosh mh - \sinh mh/mh)} e^{st} ds. \quad (4.3)$$

From the above expression, the integration is determined using complex variable theory, as discussed by Arpaci [38]. It is easily observed that  $s = 0$  is a pole of order 2.

Therefore, the residue at  $s = 0$  is

$$\text{Res}(0) = U_p \left\{ \frac{3t}{2t_0} \left[ 1 - \left( \frac{y}{h} \right)^2 \right] \right\}. \quad (4.4)$$

The other singular points are the roots of following transcendental equation:

$$mh \cosh mh - \sinh mh = 0. \quad (4.5)$$

Setting  $m = i\alpha$ , we have

$$\alpha h \cos \alpha h - \sin \alpha h = 0 \quad (4.6)$$

or

$$\tan \alpha h = \alpha h. \quad (4.6b)$$

If  $\alpha_n, n = 1, 2, 3, \dots, \infty$ , are zeros of (4.6), then

$$s_{1n} = \frac{-\left(\sigma B_0^2 + \alpha_n^2 \mu\right) + \sqrt{\left(\sigma B_0^2 + \alpha_n^2 \mu\right)^2 - 4\rho \alpha_n^2 G}}{2\rho}, \quad (4.7)$$

$$s_{2n} = \frac{-\left(\sigma B_0^2 + \alpha_n^2 \mu\right) - \sqrt{\left(\sigma B_0^2 + \alpha_n^2 \mu\right)^2 - 4\rho \alpha_n^2 G}}{2\rho},$$

$n = 1, 2, 3, \dots, \infty$ , are these poles. These are simple poles, and residues at all of these poles can be obtained as

$$\text{Res}(s_{1n}) = \frac{U_p}{t_0} \frac{e^{s_{1n}t}}{2s_{1n}Q + s_{1n}^2 Q'(s_{1n})}, \quad (4.8)$$

$$\text{Res}(s_{2n}) = \frac{U_p}{t_0} \frac{e^{s_{2n}t}}{2s_{2n}Q + s_{2n}^2 Q'(s_{2n})}$$

where

$$Q = \alpha_n h \cos \alpha_n h - \sin \alpha_n h, \quad Q'(s_{in}) = \alpha_n m'(s_{in}) h^2 \sin \alpha_n h, \quad i = 1, 2,$$

$$m'(s_{in}) = \frac{\partial m(s_{in})}{\partial s} = \frac{1}{2} \frac{\mu \rho s_{in}^2 + 2G \rho s_{in} + G \sigma B_0^2}{\left(\rho s_{in}^2 + \sigma B_0^2 s_{in}\right)^{0.5} (\mu s_{in} + G)^{1.5}}, \quad i = 1, 2. \quad (4.9)$$

Adding  $\text{Res}(0)$ ,  $\text{Res}(s_{1n})$ , and  $\text{Res}(s_{2n})$ , a complete solution for constant acceleration case is obtained as

$$\frac{u(y, t)t_0}{U_p} = \frac{3t}{2} \left[ 1 - \left( \frac{y}{h} \right)^2 \right] + \sum_{n=1}^{\infty} \left( \frac{e^{s_{1n}t}}{2s_{1n}Q + s_{1n}^2Q'(s_{1n})} + \frac{e^{s_{2n}t}}{2s_{2n}Q + s_{2n}^2Q'(s_{2n})} \right) R(y), \quad (4.10)$$

where  $R(y) = \alpha_n h (\cos \alpha_n h - \cos \alpha_n y)$  and  $Q$ ,  $Q'(s_{1n})$ ,  $Q'(s_{2n})$  are defined in (4.9).

The first term on the right-hand side of (4.10) represents the steady state velocity and the second term, the transient response of the flow to an abrupt change either in the boundary conditions, body forces, pressure gradient, or other external driving force.

Equation (3.19) is used to determine the pressure gradient in this flow field and follows the same procedure for solving velocity profile

$$\begin{aligned} \text{Res}(0) &= -\frac{\rho U_p}{t_0} \left[ \frac{3}{2} t (2\rho + \sigma B_0^2 t) \right], \\ \text{Res}(s_{1n}) &= -\frac{\rho U_p}{t_0} \frac{(\rho s_{1n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{2s_{1n}Q + s_{1n}^2Q'(s_{1n})} e^{s_{1n}t}, \\ \text{Res}(s_{2n}) &= -\frac{\rho U_p}{t_0} \frac{(\rho s_{2n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{2s_{2n}Q + s_{2n}^2Q'(s_{1n})} e^{s_{2n}t}. \end{aligned} \quad (4.11)$$

Therefore, the pressure gradient is

$$\frac{\partial p(x, t)}{\partial x} = -\frac{\rho U_p}{t_0} \left\{ \frac{3}{2} t (2\rho + \sigma B_0^2 t) + \sum_{n=1}^{\infty} \left[ \frac{(\rho s_{1n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{2s_{1n}Q + s_{1n}^2Q'(s_{1n})} e^{s_{1n}t} + \frac{(\rho s_{2n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{2s_{2n}Q + s_{2n}^2Q'(s_{2n})} e^{s_{2n}t} \right] \right\}. \quad (4.12)$$

$Q$ ,  $Q'(s_{1n})$ ,  $Q'(s_{2n})$  are defined in (4.9).

## 4.2. Suddenly Started Flow

For a suddenly started flow between the parallel surfaces,

$$u_p = \begin{cases} 0, & \text{for } t \leq 0, \\ U_p, & \text{for } t > 0, \end{cases} \quad (4.13)$$

where  $U_p$  is the constant velocity.



In which case, the velocity profile is

$$\frac{u(y,t)}{U_p} = \frac{3}{2} \left[ 1 - \left( \frac{y}{h} \right)^2 \right] + \sum_{n=1}^{\infty} \left( \frac{e^{s_{1n}t}}{Q + s_{1n}Q'(s_{1n})} + \frac{e^{s_{2n}t}}{Q + s_{2n}Q'(s_{2n})} \right) R(y), \quad (4.14)$$

where  $R(y) = \alpha_n h (\cos \alpha_n h - \cos \alpha_n y)$ ,  $Q, Q'(s_{1n}), Q'(s_{2n})$  are defined in (4.9), and the pressure gradient is

$$\frac{\partial p(x,t)}{\partial x} = -\rho U_p \left\{ \frac{3}{2} \sigma B_0^2 + \sum_{n=1}^{\infty} \left[ \frac{(\rho s_{1n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{Q + s_{1n}Q'(s_{1n})} e^{s_{1n}t} + \frac{(\rho s_{2n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{Q + s_{2n}Q'(s_{2n})} e^{s_{2n}t} \right] \right\}. \quad (4.15)$$

### 4.3. Linear Acceleration Piston Motion

The piston motion of linear acceleration can be described by the following equation:

$$u_p(t) = a_p t^2 = \left( \frac{U_p}{t_0} \right) t^2, \quad (4.16)$$

where  $a_p$  is the constant acceleration,  $U_p$  is the final velocity after acceleration, and  $t_0$  is the time period of acceleration.

In which case, the velocity profile is

$$\frac{u(y,t)t_0}{U_p} = 3t^2 \left[ 1 - \left( \frac{y}{h} \right)^2 \right] + 2 \sum_{n=1}^{\infty} \left( \frac{e^{s_{1n}t}}{3s_{1n}^2 Q + s_{1n}^3 Q'(s_{1n})} + \frac{e^{s_{2n}t}}{3s_{2n}^2 Q + s_{2n}^3 Q'(s_{2n})} \right) R(y), \quad (4.17)$$

where  $R(y) = \alpha_n h (\cos \alpha_n h - \cos \alpha_n y)$ ,  $Q, Q'(s_{1n}), Q'(s_{2n})$  are defined in (4.9), and the pressure gradient is

$$\frac{\partial p(x,t)}{\partial x} = -\frac{2\rho}{t_0} U_p \left\{ \frac{3}{2} t (2\rho + \sigma B_0^2 t) + 2 \sum_{n=1}^{\infty} \left[ \frac{(\rho s_{1n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{3s_{1n}^2 Q + s_{1n}^3 Q'(s_{1n})} e^{s_{1n}t} + \frac{(\rho s_{2n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{3s_{2n}^2 Q + s_{2n}^3 Q'(s_{2n})} e^{s_{2n}t} \right] \right\}. \quad (4.18)$$

#### 4.4. Oscillatory Piston Motion

The oscillating piston motion starting from rest is considered. The piston motion is described as

$$u_p = \begin{cases} 0, & \text{for } t \leq 0, \\ U_0 \sin(\omega t), & \text{for } t > 0. \end{cases} \quad (4.19)$$

Taking the Laplace transform of (4.19), we have

$$\hat{u}_p(s) = \frac{U_0 \omega}{s^2 + \omega^2} \quad s > 0. \quad (4.20)$$

Substituting (4.20) into (3.17) to find the velocity profile, the poles are simple poles at  $s = \pm i\omega$  and the roots of  $ah \cos ah - \sin ah = 0$ . The solution to the velocity profile is

$$\begin{aligned} \frac{u(y, t)}{U_0} &= \frac{i}{2} \left[ e^{-i\omega t} \hat{\Omega}(y, -i\omega) - e^{i\omega t} \hat{\Omega}(y, i\omega) \right] \\ &+ \sum_{n=1}^{\infty} \left( \frac{e^{s_{1n}t}}{2s_{1n}Q + s_{1n}^2 Q'(s_{1n})} + \frac{e^{s_{2n}t}}{2s_{2n}Q + s_{2n}^2 Q'(s_{2n})} \right) R(y), \end{aligned} \quad (4.21)$$

where  $R(y) = \alpha_n h (\cos \alpha_n h - \cos \alpha_n y)$ ,  $Q, Q'(s_{1n}), Q'(s_{2n})$  are defined in (4.9),  $\hat{\Omega}(y, s)$  is defined by (3.16), and the pressure gradient is obtained as

$$\begin{aligned} \frac{\partial p(x, t)}{\partial x} &= -\frac{\rho U_0}{2} \left\{ (i\rho\omega + \sigma B_0^2) e^{i\omega t} \Gamma(i\omega) + (-i\omega\rho + \sigma B_0^2) e^{-i\omega t} \Gamma(-i\omega) \right. \\ &\left. + \sum_{n=1}^{\infty} \left[ \frac{(\rho s_{1n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{2s_{1n}Q + (s_{1n}^2 + \omega^2) Q'(s_{1n})} e^{s_{1n}t} + \frac{(\rho s_{2n} + \sigma B_0^2) \alpha_n h \cos \alpha_n h}{2s_{2n}Q + (s_{2n}^2 + \omega^2) Q'(s_{1n})} e^{s_{2n}t} \right] \right\}, \end{aligned} \quad (4.22)$$

where  $Q, Q'(s_{1n}), Q'(s_{2n})$  are defined in (4.9), and

$$\Gamma(s) = \frac{\cosh mh}{(\cosh mh - \sinh mh/mh)}, \quad m = \sqrt{\frac{\rho s^2 + \sigma B_0^2 s}{\mu s + G}}. \quad (4.23)$$

## 5. Conclusions

In this paper, the analytical solutions of unsteady unidirectional MHD flow of Voigt fluids under magnetic field effects for different piston motion that provide different volume flow rates are derived and solved by Laplace transform technique. The results are presented in analytical forms.

The pressure gradient for each flow condition is thus being derived from the known function of inlet volume flow rate by using the same method. It is interesting to note that for fully developed flows the relaxation time only appears as the motion is unsteady.

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