

Research Article

The GDTM-Padé Technique for the Nonlinear Lattice Equations

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The GDTM-Padé technique is a combination of the generalized differential transform method and the Padé approximation. We apply this technique to solve the two nonlinear lattice equations, which results in the high accuracy of the GDTM-Padé solutions. Numerical results are presented to show its efficiency by comparing the GDTM-Padé solutions, the solutions obtained by the generalized differential transform method, and the exact solutions.

1. Introduction

The nonlinear differential difference equations (NDDEs) have wide applications in various branches of science, including the mechanical engineering, condensed matter physics, biophysics, mathematical statistics, control theory and so on [1–11]. During the past decades, a large number of solution methods such as the Adomian decomposition method [12, 13], the Jacobian elliptic function method [14], the Exp-function method [15], the (G'/G) -expansion method [16], and the variable-coefficient discrete tanh method [17] were proposed to solve the NDDEs. Recently, the generalized differential transform method [18–20] combined with the Padé technique (named as GDTM-Padé technique) was presented in [21] to construct the numerical or exact solutions of the differential difference equations. Due to the Padé approximation, the convergence and the accuracy of the original series solutions can be improved.

In this paper, we focus on solving two nonlinear lattice equations by applying the GDTM-Padé technique. The first nonlinear equation is the hybrid lattice equation [5] defined by

$$\frac{\partial u_n}{\partial t} = (1 + \alpha u_n + \beta u_n^2)(u_{n-1} - u_{n+1}), \quad (1.1)$$

which was related with the discretization of the KDV equations or the modified KDV equations. The second equation arose in the study of continuum two-boson KP systems [3, 22], which was called as the Volterra lattice equation

$$\begin{aligned}\frac{\partial u_n}{\partial t} &= u_n(v_n - v_{n-1}), \\ \frac{\partial v_n}{\partial t} &= v_n(u_{n+1} - u_n).\end{aligned}\tag{1.2}$$

The rest of this paper is organized as follows. In Section 2, we introduce the idea of the GDTM-Padé technique for the NDDEs. The hybrid lattice equation and the Volterra lattice equation are studied in Section 3. Numerical results are presented to verify the efficiency. Finally, some conclusions are given.

2. The GDTM-Padé Technique

To illustrate the basic idea of the GDTM-Padé technique, we consider the general nonlinear difference differential equation

$$N(u_n(t), u_{n+1}(t), u_{n-1}(t), u_{n+2}(t), \dots) = 0,\tag{2.1}$$

where N is a nonlinear differential operator, $u_n(t)$ is the unknown function with respect to the discrete spatial variable n and the temporal variable t . Applying the one-dimensional differential transform method (GDTM), the differential transform of the k th derivative of the function $u_n(t)$ is defined by

$$U_n(k) = U(n, k) = \frac{1}{k!} \left. \frac{d^k u_n(t)}{dt^k} \right|_{t=t_0}.\tag{2.2}$$

The differential inverse transform of $U_n(k)$ is read as

$$u_n(t) = \sum_{k=0}^{\infty} U_n(k) (t - t_0)^k = \sum_{k=0}^{\infty} U(n, k) (t - t_0)^k.\tag{2.3}$$

Particularly, the function $u_n(t)$ can be formulated as a series when $t_0 = 0$, that is,

$$u_n(t) = \sum_{k=0}^{\infty} U_n(k) t^k = \sum_{k=0}^{\infty} U(n, k) t^k.\tag{2.4}$$

In the real applications, we can determine the coefficients $U(n, k)$ ($k = 1, \dots, m$) and obtain the m th-order approximation of the function $u_n(t)$ given by

$$u_{n,m}(t) = \sum_{k=0}^m U_n(k) t^k = \sum_{k=0}^m U(n, k) t^k.\tag{2.5}$$

The transformed operations for the GDTM are listed in Table 1 [21].

Table 1: The operations for generalized differential transform method.

Original function	Transformed function
$f(n, t) = g(n, t) + h(n, t)$	$F(n, k) = G(n, k) + H(n, k)$
$f(n, t) = \alpha g(n, t)$	$F(n, k) = \alpha G(n, k)$
$f(n, t) = \partial g(n, t) / \partial t$	$F(n, k) = (k + 1)G(n, k + 1)$
$f(n, t) = g(n, t)h(n, t)$	$F(n, k) = \sum_{r=0}^k G(n, r)H(n, k - r)$
$f(n, t) = \partial^m g(n, t) / \partial t^m$	$F(n, k) = (k + m)G(n, k + m)$
$f(n, t) = g(n + s, t)$	$F(n, k) = G(n + s, k)$

To improve the accuracy and convergence of the GDTM solution (2.5), the Padé approximation [23, 24] is used. For simplicity, we denote the $[L, M]$ Padé approximation to $f(x) = \sum_{k=0}^{\infty} a_k t^k$ by

$$f[L, M] = \frac{P_L(x)}{Q_M(x)}, \tag{2.6}$$

where

$$\begin{aligned} P_L(x) &= p_0 + p_1x + p_2x^2 + p_3x^3 + \dots + p_Lx^L, \\ Q_M(x) &= 1 + q_1x + q_2x^2 + q_3x^3 + \dots + q_Mx^M, \end{aligned} \tag{2.7}$$

with the normalization condition $Q_M(0) = 1$. The coefficients of $P_L(x)$ and $Q_M(x)$ can be uniquely determined by comparing the first $(L + M + 1)$ terms of the functions $f[L, M]$ and $f(x)$. In practice, the construction of the $[L, M]$ Padé approximation involves only algebra equations, which are solved by means of the Mathematica or Maple package. We call the solution obtained by the GDTM and the Padé approximation as the GDTM-Padé solution.

3. Numerical Examples

In this section, we will illustrate the validity and advantages of the GDTM-Padé technique for the nonlinear differential difference equations. Two nonlinear lattice equations will be studied, where one is a hybrid lattice and the other is a Volterra lattice.

3.1. The Hybrid Lattice Equation

Consider the hybrid lattice equation (1.1) with the initial condition

$$u_n(0) = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta} \tanh(d)}{2\beta} \tanh(dn). \tag{3.1}$$

The exact solution to (1.1) [2] is of the form

$$u_n(t) = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta} \tanh(d)}{2\beta} \tanh \left[dn + \frac{\alpha^2 - 4\beta}{2\beta} \tanh(d)t \right]. \tag{3.2}$$

Using the GDTM technique, the transformed problem of (1.1) can be expressed in the following recurrence formula:

$$(k+1)U(n, k+1) = U(n-1, k) - U(n+1, k) + \alpha \sum_{s=0}^k U(n, s)(U(n-1, k-s) - U(n+1, k-s)) \\ + \beta \sum_{s=0}^k \sum_{t=0}^s U(n, t)U(n, s-t)(U(n-1, k-s) - U(n+1, k-s)). \quad (3.3)$$

The transformed initial condition is

$$U(n, 0) = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta \tanh(d)}}{2\beta} \tanh(dn). \quad (3.4)$$

One can also easily construct the implicit initial conditions as follows:

$$U(n-1, 0) = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta \tanh(d)}}{2\beta} \tanh(d(n-1)), \\ U(n+1, 0) = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta \tanh(d)}}{2\beta} \tanh(d(n+1)). \quad (3.5)$$

Based on the above initial conditions and the recursive formula (3.3), we can derive the coefficients $U(n, k)$ one by one and obtain the approximate solution $u_{n,m}(t) = \sum_{k=0}^m U(n, k)t^k$. In this example, we set $\alpha = 3$, $\beta = 2$ and $d = 0.5$. The 5th-order approximate solution at $n = 5$ is given by

$$u_{n,5}(t) = -0.6259778749 - 0.0018551343t + 0.0001686607t^2 \\ - 2.3937386136 \times 10^{-6}t^3 - 9.7827591261 \times 10^{-7}t^4 - 4.0803184434 \times 10^{-8}t^5. \quad (3.6)$$

Applying the GDTM-Padé technique to the solution (3.6), we get the [2,2] GDTM Padé approximation:

$$u[2,2] = \frac{-0.6259778749 - 0.059706823t - 0.00445469t^2}{1 + 0.0924181046t + 0.0071119169t^2}. \quad (3.7)$$

For comparison, we plot the GDTM solutions $u_{n,5}(t)$, the GDTM-Padé solutions $u[2,2]$, and the exact solutions of (1.1) in Figure 1. Figure 2 shows the absolute error of the GDTM solutions and the GDTM-Padé solutions. The GDTM solutions are in good agreement with the exact solutions in the small interval ($-5 \leq t \leq 5$), and high errors appear when $t > 5$. By the GDTM-Padé technique, the accuracy of the approximation is improved largely.

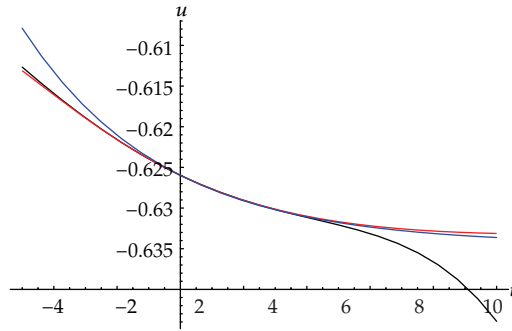


Figure 1: The compared results for the GDTM solutions (black), the GDTM-Padé solutions (red), and the exact solutions (blue) of (1.1).

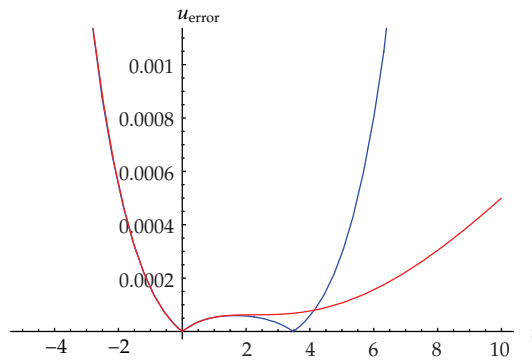


Figure 2: The error curves for the GDTM solutions (blue) and the GDTM-Padé solutions (red).

3.2. The Volterra Lattice Equations

We further consider the two component Volterra lattice equations (1.2) with the initial conditions

$$u_n(0) = -c \coth(d) + c \tanh(dn), \quad v_n(0) = -c \coth(d) - c \tanh(dn). \quad (3.8)$$

We remark that the exact solutions to (1.2) [2] are given by $u_n(t) = -c \coth(d) + c \tanh(dn + ct)$ and $v_n(t) = -c \coth(d) - c \tanh(dn + ct)$, respectively.

Similarly, using the GDTM-Padé technique, we obtain the following transformed problems:

$$(k + 1)U(n, k + 1) = \sum_{s=0}^k U(n, s)(V(n, k - s) - V(n - 1, k - s)), \quad (3.9)$$

$$(k + 1)V(n, k + 1) = \sum_{s=0}^k V(n, s)(U(n + 1, k - s) - U(n, k - s)), \quad (3.10)$$

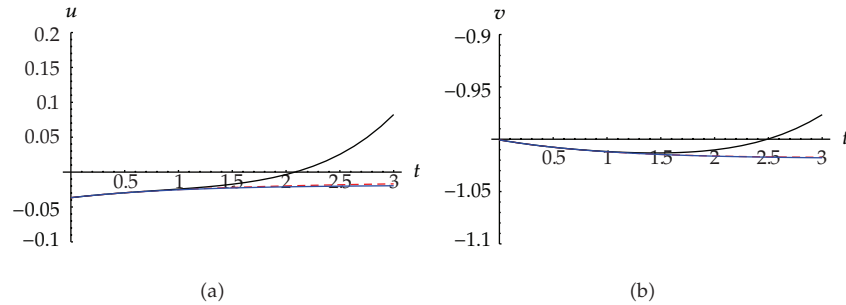


Figure 3: The compared results for the GDTM solutions (black), the GDTM-Padé solutions (red), and the exact solutions (blue) of (1.2) when $c = -0.5$, $d = -2$ and $n = 1$.

with the initial conditions

$$\begin{aligned} U(n, 0) &= -c \coth(d) + c \tanh(dn), & U(n+1, 0) &= -c \coth(d) + c \tanh(d(n+1)), \\ V(n, 0) &= -c \coth(d) - c \tanh(dn), & V(n-1, 0) &= -c \coth(d) - c \tanh(d(n-1)). \end{aligned} \quad (3.11)$$

We first obtain the coefficients $U(n, k+1)$ of $u_n(t)$ by the above initial conditions and (3.9), then derive the coefficients $V(n, k+1)$ of $v_n(t)$, which results in the GDTM solutions $u_{n,m}(t) = \sum_{k=0}^m U(n, k)t^k$ and $v_{n,m}(t) = \sum_{k=0}^m V(n, k)t^k$.

We set $c = -0.5$, $d = -2$ at $n = 1$, and obtain the 6th-order approximations as follows:

$$\begin{aligned} u_{n,6}(t) &= -0.0366435703 + 0.0176627062t - 0.008513668t^2 + 0.0026318132t^3 \\ &\quad + 0.0005448212t^4 - 0.0000317142t^5 + 0.00004837t^6, \\ v_{n,6}(t) &= -1.0006711504 - 0.0176627062t + 0.008513668t^2 - 0.0026318132t^3 \\ &\quad + 0.000559098t^4 + 0.0001457863t^5 - 0.0000126128t^6. \end{aligned} \quad (3.12)$$

The $[2, 2]$ GDTM-Padé solutions to the approximations $u_{n,6}(t)$ and $v_{n,6}(t)$ can be expressed as

$$\begin{aligned} u[2, 2] &= \frac{-0.0366435703 - 0.0274826777t - 0.0030536309t^2}{1 + 1.23201379t + 0.4448436759t^2}, \\ v[2, 2] &= \frac{-1.0006711504 - 0.5t - 0.0833892625t^2}{1 + 0.48201379t + 0.0833333333t^2}. \end{aligned} \quad (3.13)$$

We plot in Figure 3 the curves of the GDTM solutions, the GDTM-Padé solutions, and the exact solutions of (1.2). Figure 4 shows the absolute errors of the GDTM solutions and the GDTM-Padé solutions. The GDTM-Padé method performs better than the GDTM method for this example. We show the absolute errors of $|u_n(t) - u_{n,6}(t)|$ and $|u_n(t) - u[2, 2]|$ in the left column of Table 2. The absolute errors of $|v_n(t) - v_{n,6}(t)|$ and $|v_n(t) - v[2, 2]|$ are shown in the right column. Obviously, the errors of $u[2, 2]$ are reduced significantly, comparing with the approximation $u_{n,6}(t)$ when $t > 1$. This phenomenon also appears in the errors of $v[2, 2]$.

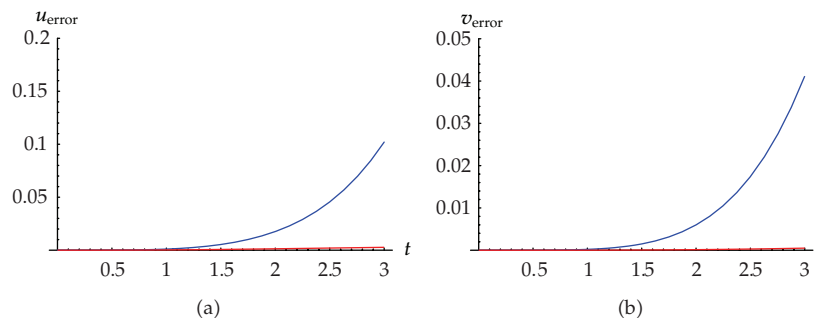


Figure 4: The absolute error curves for the the GDTM solutions (blue) and the GDTM-Padé solutions (red) of (1.2) when $c = -0.5, d = -2$ and $n = 1$.

Table 2: Comparisons of the absolute errors between the GDTM solutions and GDTM-Padé solutions for (1.2) with $c = -0.5, d = -2$ and $n = 1$.

t	GDTM	GDTM-Padé	GDTM	GDTM-Padé
0	0	0	0	0
1	1.05×10^{-3}	2.54×10^{-4}	2.05×10^{-4}	9.79×10^{-6}
2	0.0176	1.33×10^{-3}	5.99×10^{-3}	1.37×10^{-4}
3	0.1020	2.72×10^{-3}	0.0412	4.96×10^{-4}
4	0.3903	4.01×10^{-3}	0.1555	1.07×10^{-3}
5	1.1838	5.08×10^{-3}	0.4214	1.79×10^{-3}
6	3.0662	5.94×10^{-3}	0.9197	2.56×10^{-3}
7	7.0570	6.62×10^{-3}	1.7175	3.35×10^{-3}
8	14.7982	7.17×10^{-3}	2.8349	4.11×10^{-3}
9	28.7776	7.62×10^{-3}	4.2038	4.83×10^{-3}
10	52.5858	7.99×10^{-3}	5.6177	5.51×10^{-3}

If $c = 1, d = 0.5$ at $n = 5$, the 6th-order approximations are given by

$$\begin{aligned}
 u_{n,6}(t) &= -1.1773391156 + 0.0265922267t - 0.0262362711t^2 + 0.0170210046t^3 \\
 &\quad - 0.0071252574t^4 + 0.0028531211t^5 - 0.0001774315t^6, \\
 v_{n,6}(t) &= -3.1505677119 - 0.0265922267t + 0.0262362711t^2 - 0.0170210046t^3 \\
 &\quad + 0.0080477428t^4 - 0.0023581228t^5 + 0.0007216293t^6.
 \end{aligned}
 \tag{3.14}$$

Similarly, the $[2, 2]$ GDTM-Padé solutions to (3.14) are

$$\begin{aligned}
 u[2, 2] &= \frac{-1.1773391156 - 1.2575132091t - 0.5105646518t^2}{1 + 1.0906844246t + 0.4360104081t^2}, \\
 v[2, 2] &= \frac{-3.1505677119 - 3.1349873785t - 1.0501892373t^2}{1 + 0.9866142982t + 0.3333333333t^2}.
 \end{aligned}
 \tag{3.15}$$

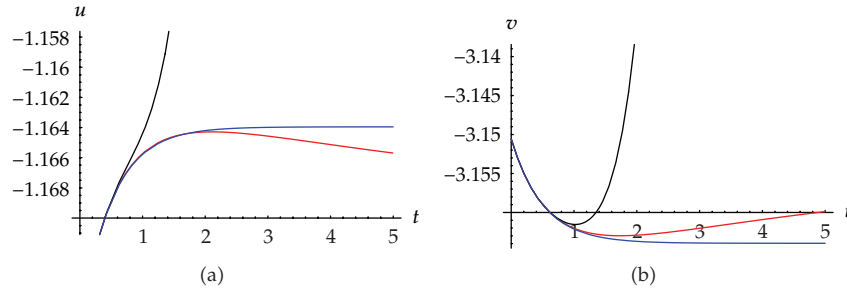


Figure 5: The compared results for the GDTM solutions (black), the GDTM-Padé solutions (red), and the exact solutions (blue) of (1.2) when $c = 1$, $d = 0.5$ and $n = 5$.

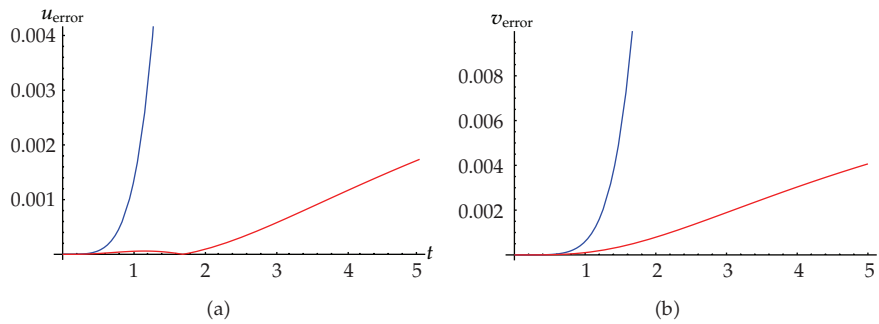


Figure 6: The absolute errors for the the GDTM solutions (blue) and the GDTM-Padé (red) of (1.2) when $c = 1$, $d = 0.5$ and $n = 5$.

Table 3: Numerical results of the absolute errors between the GDTM solutions and the GDTM-Padé solutions for (1.2) with $c = 1$, $d = 0.5$ and $n = 5$.

t	GDTM	GDTM-Padé	GDTM	GDTM-Padé
0	0	0	0	0
1	0.0014	5.62×10^{-5}	5.98×10^{-4}	1.01×10^{-4}
2	0.0372	9.34×10^{-5}	2.82×10^{-2}	7.89×10^{-4}
3	0.2767	5.80×10^{-4}	0.3150	1.89×10^{-3}
4	1.1333	1.17×10^{-3}	1.8388	3.03×10^{-3}
5	3.2816	1.74×10^{-3}	7.3449	4.07×10^{-3}
6	7.5515	2.23×10^{-3}	22.883	4.95×10^{-3}
7	14.695	2.67×10^{-3}	59.863	5.71×10^{-3}
8	25.028	3.03×10^{-3}	137.63	6.35×10^{-3}
9	37.940	3.34×10^{-3}	286.55	6.90×10^{-3}
10	51.278	3.61×10^{-3}	551.65	7.37×10^{-3}

Figure 5 shows the compared results for the solutions including $u_{n,6}(t)$, $u[2,2]$, $u_n(t)$ and $v_{n,6}(t)$, $v[2,2]$, $v_n(t)$. The error curves are plotted in Figure 6. In Table 3, we compare the absolute errors of GDTM solutions and GDTM-Padé solutions. Similar to the previous case, the GDTM-Padé method also outperforms the GDTM method.

4. Conclusions

This paper focused on solving the nonlinear lattice equations by using the GDTM-Padé technique. The numerical results confirmed the effectiveness of this method. In the future work, we will further extend this method to other nonlinear differential difference equations.

Acknowledgments

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