

Review Article

Infinite System of Differential Equations in Some *BK* Spaces

M. Mursaleen¹ and Abdullah Alotaibi²

¹ Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

² Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Abdullah Alotaibi, mathker11@hotmail.com

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The first measure of noncompactness was defined by Kuratowski in 1930 and later the Hausdorff measure of noncompactness was introduced in 1957 by Goldenstein et al. These measures of noncompactness have various applications in several areas of analysis, for example, in operator theory, fixed point theory, and in differential and integral equations. In particular, the Hausdorff measure of noncompactness has been extensively used in the characterizations of compact operators between the infinite-dimensional Banach spaces. In this paper, we present a brief survey on the applications of measures of noncompactness to the theory of infinite system of differential equations in some *BK* spaces $c_0, c, \ell_p (1 \leq p \leq \infty)$ and $n(\phi)$.

1. *FK* and *BK* Spaces

In this section, we give some basic definitions and notations about *FK* and *BK* spaces for which we refer to [1–3].

We will write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let $\varphi, \ell_{\infty}, c$ and c_0 denote the sets of all finite, bounded, convergent, and null sequences, respectively, and cs be the set of all convergent series. We write $\ell_p := \{x \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$. By e and $e^{(n)}$ ($n \in \mathbb{N}$), we denote the sequences such that $e_k = 1$ for $k = 0, 1, \dots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ ($k \neq n$). For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^n x_k e^{(k)}$ be its n -section.

Note that ℓ_{∞}, c and c_0 are Banach spaces with the norm $\|x\|_{\infty} = \sup_{k \geq 0} |x_k|$, and $\ell_p (1 \leq p < \infty)$ are Banach spaces with the norm $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a linear metric space X is called *Schauder basis* if for every $x \in X$, there is a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b^{(n)}$. A sequence space X with a linear topology is called a *Kspace* if each of the maps $p_i : X \rightarrow \mathbb{C}$ defined by $p_i(x) = x_i$

is continuous for all $i \in \mathbb{N}$. A K space is called an *FK space* if X is complete linear metric space; a *BK space* is a normed *FK space*. An *FK space* $X \supset \phi$ is said to have *AK* if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $x = \lim_{n \rightarrow \infty} x^{[n]}$.

A linear space X equipped with a translation invariant metric d is called a *linear metric space* if the algebraic operations on X are continuous functions with respect to d . A complete linear metric space is called a *Fréchet space*. If X and Y are linear metric spaces over the same field, then we write $\mathcal{B}(X, Y)$ for the class of all continuous linear operators from X to Y . Further, if X and Y are normed spaces then $\mathcal{B}(X, Y)$ consists of all bounded linear operators $L : X \rightarrow Y$, which is a normed space with the operator norm given by $\|L\| = \sup_{x \in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X, Y)$, where S_X denotes the unit sphere in X , that is, $S_X := \{x \in X : \|x\| = 1\}$. Also, we write $\bar{B}_X := \{x \in X : \|x\| \leq 1\}$ for the closed unit ball in a normed space X . In particular, if $Y = \mathbb{C}$ then we write X^* for the set of all continuous linear functionals on X with the norm $\|f\| = \sup_{x \in S_X} |f(x)|$.

The *theory of FK spaces* is the most powerful and widely used tool in the characterization of matrix mappings between sequence spaces, and the most important result was that matrix mappings between *FK spaces* are continuous.

A sequence space X is called an *FK space* if it is a locally convex Fréchet space with continuous coordinates $p_n : X \rightarrow \mathbb{C} (n \in \mathbb{N})$, where \mathbb{C} denotes the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A normed *FK space* is called a *BK space*, that is, a *BK space* is a Banach sequence space with continuous coordinates.

The famous example of an *FK space* which is not a *BK space* is the space (w, d_w) , where

$$d_w(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\frac{|x_k - y_k|}{1 + |x_k - y_k|} \right); \quad (x, y \in w). \quad (1.1)$$

On the other hand, the classical sequence spaces are *BK spaces* with their natural norms. More precisely, the spaces ℓ_{∞} , c , and c_0 are *BK spaces* with the sup-norm given by $\|x\|_{\ell_{\infty}} = \sup_k |x_k|$. Also, the space ℓ_p ($1 \leq p < \infty$) is a *BK space* with the usual ℓ_p -norm defined by $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p}$. Further, the spaces bs , cs , and cs_0 are *BK spaces* with the same norm given by $\|x\|_{bs} = \sup_n \sum_{k=0}^n |x_k|$, and bv is a *BK space* with $\|x\|_{bv} = \sum_k |x_k - x_{k-1}|$.

An *FK space* $X \supset \phi$ is said to have *AK* if every sequence $x = (x_k) \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $\lim_{n \rightarrow \infty} (\sum_{k=0}^n x_k e^{(k)}) = x$. This means that $(e^{(k)})_{k=0}^{\infty}$ is a Schauder basis for any *FK space* with *AK* such that every sequence, in an *FK space* with *AK*, coincides with its sequence of coefficients with respect to this basis.

Although the space ℓ_{∞} has no Schauder basis, the spaces w , c_0 , c , and ℓ_p all have Schauder bases. Moreover, the spaces w , c_0 , and ℓ_p have *AK*, where $1 \leq p < \infty$.

There are following *BK spaces* which are closely related to the spaces ℓ_p ($1 \leq p \leq \infty$).

Let \mathcal{C} denote the space whose elements are finite sets of distinct positive integers. Given any element σ of \mathcal{C} , we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ such that $c_n(\sigma) = 1$ for $n \in \sigma$, and $c_n(\sigma) = 0$ otherwise. Further

$$\mathcal{C}_s = \left\{ \sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s \right\}, \quad (1.2)$$

that is, C_s is the set of those σ whose support has cardinality at most s , and define

$$\Phi = \{ \phi = (\phi_k) \in \omega : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1}, (n + 1)\phi_n \geq n\phi_{n+1} \}. \tag{1.3}$$

For $\phi \in \Phi$, the following sequence spaces were introduced by Sargent [4] and further studied in [5–8]:

$$\begin{aligned} m(\phi) &= \left\{ x = (x_k) \in \omega : \|x\|_{m(\phi)} = \sup_{s \geq 1} \sup_{\sigma \in C_s} \left(\frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| \right) < \infty \right\}, \\ n(\phi) &= \left\{ x = (x_k) \in \omega : \|x\|_{n(\phi)} = \sup_{u \in S(x)} \left(\sum_{k=1}^{\infty} |u_k| \Delta \phi_k \right) < \infty \right\}, \end{aligned} \tag{1.4}$$

where $S(x)$ denotes the set of all sequences that are rearrangements of x .

Remark 1.1. (i) The spaces $m(\phi)$ and $n(\phi)$ are BK spaces with their respective norms.

(ii) If $\phi_n = 1$ for all $n \in \mathbb{N}$, then $m(\phi) = l_1$, $n(\phi) = l_\infty$, and if $\phi_n = n$ for all $n \in \mathbb{N}$, then $m(\phi) = l_\infty$, $n(\phi) = l_1$.

(iii) $l_1 \subseteq m(\phi) \subseteq l_\infty [l_\infty \supseteq n(\phi) \supseteq l_1]$ for all ϕ of Φ .

2. Measures of Noncompactness

The first measure of noncompactness was defined and studied by Kuratowski [9] in 1930. The Hausdorff measure of noncompactness was introduced by Goldenštejn et al. [10] in 1957, and later studied by Goldenštejn and Markus [11] in 1965.

Here, we shall only consider the Hausdorff measure of noncompactness; it is the most suitable one for our purposes. The basic properties of measures of noncompactness can be found in [12–14].

Let S and M be subsets of a metric space (X, d) and let $\epsilon > 0$. Then S is called an ϵ -net of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x, s) < \epsilon$. Further, if the set S is finite, then the ϵ -net S of M is called a *finite ϵ -net* of M , and we say that M has a finite ϵ -net in X . A subset M of a metric space X is said to be *totally bounded* if it has a finite ϵ -net for every $\epsilon > 0$ and is said to be *relatively compact* if its closure \overline{M} is a compact set. Moreover, if the metric space X is complete, then M is totally bounded if and only if M is relatively compact.

Throughout, we shall write \mathcal{M}_X for the collection of all bounded subsets of a metric space (X, d) . If $Q \in \mathcal{M}_X$, then the *Hausdorff measure of noncompactness* of the set Q , denoted by $\chi(Q)$, is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$ and centers in X . This can equivalently be redefined as follows:

$$\chi(Q) = \inf \{ \epsilon > 0 : Q \text{ has a finite } \epsilon\text{-net} \}. \tag{2.1}$$

The function $\chi : \mathcal{M}_X \rightarrow [0, \infty)$ is called the *Hausdorff measure of noncompactness*.

If Q , Q_1 , and Q_2 are bounded subsets of a metric space X , then we have

$$\begin{aligned} \chi(Q) = 0 & \text{ if and only if } Q \text{ is totally bounded,} \\ Q_1 \subset Q_2 & \text{ implies } \chi(Q_1) \leq \chi(Q_2). \end{aligned} \quad (2.2)$$

Further, if X is a normed space, then the function χ has some additional properties connected with the linear structure, for example,

$$\begin{aligned} \chi(Q_1 + Q_2) & \leq \chi(Q_1) + \chi(Q_2), \\ \chi(\alpha Q) & = |\alpha| \chi(Q), \quad \forall \alpha \in \mathbb{C}. \end{aligned} \quad (2.3)$$

Let X and Y be Banach spaces and χ_1 and χ_2 be the Hausdorff measures of noncompactness on X and Y , respectively. An operator $L : X \rightarrow Y$ is said to be (χ_1, χ_2) -bounded if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$ and there exist a constant $C \geq 0$ such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in \mathcal{M}_X$. If an operator L is (χ_1, χ_2) -bounded then the number $\|L\|_{(\chi_1, \chi_2)} := \inf\{C \geq 0 : \chi_2(L(Q)) \leq C\chi_1(Q) \text{ for all } Q \in \mathcal{M}_X\}$ is called the (χ_1, χ_2) -measure of noncompactness of L . If $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi_1, \chi_2)} = \|L\|_\chi$.

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows: let X and Y be Banach spaces and $L \in B(X, Y)$. Then, the Hausdorff measure of noncompactness of L , denoted by $\|L\|_\chi$, can be determined by

$$\|L\|_\chi = \chi(L(S_X)), \quad (2.4)$$

and we have that L is compact if and only if

$$\|L\|_\chi = 0. \quad (2.5)$$

Furthermore, the function χ is more applicable when X is a Banach space. In fact, there are many formulae which are useful to evaluate the Hausdorff measures of noncompactness of bounded sets in some particular Banach spaces. For example, we have the following result of Goldenštein et al. [10, Theorem 1] which gives an estimate for the Hausdorff measure of noncompactness in Banach spaces with Schauder bases. Before that, let us recall that if $(b_k)_{k=0}^\infty$ is a Schauder basis for a Banach space X , then every element $x \in X$ has a unique representation $x = \sum_{k=0}^\infty \phi_k(x)b_k$, where ϕ_k ($k \in \mathbb{N}$) are called the basis functionals. Moreover, the operator $P_r : X \rightarrow X$, defined for each $r \in \mathbb{N}$ by $P_r(x) = \sum_{k=0}^r \phi_k(x)b_k$ ($x \in X$), is called the projector onto the linear span of $\{b_0, b_1, \dots, b_r\}$. Besides, all operators P_r and $I - P_r$ are equibounded, where I denotes the identity operator on X .

Theorem 2.1. *Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^\infty$, $E \in \mathcal{M}_X$ and $P_n : X \rightarrow X (n \in \mathbb{N})$ be the projector onto the linear span of $\{b_0, b_1, \dots, b_n\}$. Then, one has*

$$\frac{1}{a} \cdot \limsup_{n \rightarrow \infty} \left(\sup_{x \in Q} \|(I - P_n)(x)\| \right) \leq \chi(E) \leq \limsup_{n \rightarrow \infty} \left(\sup_{x \in Q} \|(I - P_n)(x)\| \right), \quad (2.6)$$

where $a = \limsup_{n \rightarrow \infty} \|I - P_n\|$.

In particular, the following result shows how to compute the Hausdorff measure of noncompactness in the spaces c_0 and $\ell_p (1 \leq p < \infty)$ which are BK spaces with AK.

Theorem 2.2. *Let E be a bounded subset of the normed space X , where X is ℓ_p for $1 \leq p < \infty$ or c_0 . If $P_n : X \rightarrow X (n \in \mathbb{N})$ is the operator defined by $P_n(x) = x^{[n]} = (x_0, x_1, \dots, x_n, 0, 0, \dots)$ for all $x = (x_k)_{k=0}^\infty \in X$, then one has*

$$\chi(E) = \lim_{n \rightarrow \infty} \left(\sup_{x \in Q} \|(I - P_n)(x)\| \right). \quad (2.7)$$

It is easy to see that for $E \in \mathcal{M}_{\ell_p}$

$$\chi(E) = \lim_{n \rightarrow \infty} \left(\sup_{x \in Q} \sum_{k \geq n} |x_k|^p \right). \quad (2.8)$$

Also, it is known that $(e, e^{(0)}, e^{(1)}, \dots)$ is a Schauder basis for the space c and every sequence $z = (z_n)_{n=0}^\infty \in c$ has a unique representation $z = \bar{z}e + \sum_{n=0}^\infty (z_n - \bar{z})e^{(n)}$, where $\bar{z} = \lim_{n \rightarrow \infty} z_n$. Thus, we define the projector $P_r : c \rightarrow c (r \in \mathbb{N})$, onto the linear span of $\{e, e^{(0)}, e^{(1)}, \dots, e^{(r)}\}$, by

$$P_r(z) = \bar{z}e + \sum_{n=0}^r (z_n - \bar{z})e^{(n)}; \quad (r \in \mathbb{N}), \quad (2.9)$$

for all $z = (z_n) \in c$ with $\bar{z} = \lim_{n \rightarrow \infty} z_n$. In this situation, we have the following.

Theorem 2.3. *Let $Q \in \mathcal{M}_c$ and $P_r : c \rightarrow c (r \in \mathbb{N})$ be the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, \dots, e^{(r)}\}$. Then, one has*

$$\frac{1}{2} \cdot \lim_{r \rightarrow \infty} \left(\sup_{x \in Q} \|(I - P_r)(x)\|_{\ell_\infty} \right) \leq \chi(Q) \leq \lim_{r \rightarrow \infty} \left(\sup_{x \in Q} \|(I - P_r)(x)\|_{\ell_\infty} \right), \quad (2.10)$$

where I is the identity operator on c .

Theorem 2.4. *Let Q be a bounded subset of $n(\phi)$. Then*

$$\chi(Q) = \limsup_{k \rightarrow \infty} \left(\sup_{x \in Q} \left(\sum_{n=k}^\infty |u_n| \Delta \phi_n \right) \right). \quad (2.11)$$

The idea of *compact operators* between Banach spaces is closely related to the Hausdorff measure of noncompactness, and it can be given as follows.

Let X and Y be complex Banach spaces. Then, a linear operator $L : X \rightarrow Y$ is said to be *compact* if the domain of L is all of X , that is, $D(L) = X$, and for every bounded sequence (x_n) in X , the sequence $(L(x_n))$ has a convergent subsequence in Y . Equivalently, we say that L is compact if its domain is all of X and $L(Q)$ is relatively compact in Y for every $Q \in \mathcal{M}_X$.

Further, we write $\mathcal{C}(X, Y)$ for the class of all compact operators from X to Y . Let us remark that every compact operator in $\mathcal{C}(X, Y)$ is bounded, that is, $\mathcal{C}(X, Y) \subset \mathcal{B}(X, Y)$. More precisely, the class $\mathcal{C}(X, Y)$ is a closed subspace of the Banach space $\mathcal{B}(X, Y)$ with the operator norm.

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Let X and Y be Banach spaces and $L \in \mathcal{B}(X; Y)$. Then, the Hausdorff measure of noncompactness of L , denoted by $\|L\|_\chi$, can be given by

$$\|L\|_\chi = \chi(L(S_X)), \quad (2.12)$$

and we have

$$L \text{ is compact if and only if } \|L\|_\chi = 0. \quad (2.13)$$

Since matrix mappings between *BK* spaces define bounded linear operators between these spaces which are Banach spaces, it is natural to use the Hausdorff measure of noncompactness to obtain necessary and sufficient conditions for matrix operators between *BK* spaces to be compact operators. This technique has recently been used by several authors in many research papers (see, for instance, [5, 6, 15–32]).

3. Applications to Infinite Systems of Differential Equations

This section is mainly based on the work of Banaś and Lecko [33], Mursaleen and Mohiuddine [26], and Mursaleen [32]. In this section, we apply the technique of measures of noncompactness to the theory of infinite systems of differential equations in some Banach sequence spaces c_0 , c , ℓ_p ($1 \leq p < \infty$), and $n(\phi)$.

Infinite systems of ordinary differential equations describe numerous world real problems which can be encountered in the theory of branching processes, the theory of neural nets, the theory of dissociation of polymers, and so on (cf. [34–38], e.g.). Let us also mention that several problems investigated in mechanics lead to infinite systems of differential equations [39–41]. Moreover, infinite systems of differential equations can be also used in solving some problems for parabolic differential equations investigated via semidiscretization [42, 43]. The theory of infinite systems of ordinary differential equation seems not to be developed satisfactorily up to now. Indeed, the existence results concerning systems of such a kind were formulated mostly by imposing the Lipschitz condition on

right-hand sides of those systems (cf. [10, 11, 39, 40, 44–50]). Obviously, the assumptions formulated in terms of the Lipschitz condition are rather restrictive and not very useful in applications. On the other hand, the infinite systems of ordinary differential equations can be considered as a particular case of ordinary differential equations in Banach spaces. Until now several existence results have been obtained concerning the Cauchy problem for ordinary differential equations in Banach spaces [33, 35, 51–53]. A considerable number of those results were formulated in terms of measures of noncompactness. The results of such a type have a concise form and give the possibility to formulate more general assumptions than those requiring the Lipschitz continuity. But in general those results are not immediately applicable in concrete situations, especially in the theory of infinite systems of ordinary differential equations.

In this section, we adopt the technique of measures of noncompactness to the theory of infinite systems of differential equations. Particularly, we are going to present a few existence results for infinite systems of differential equations formulated with the help of convenient and handy conditions.

Consider the ordinary differential equation

$$x' = f(t, x), \tag{3.1}$$

with the initial condition

$$x(0) = x_0. \tag{3.2}$$

Then the following result for the existence of the Cauchy problem (3.1)-(3.2) was given in [34] which is a slight modification of the result proved in [33].

Assume that X is a real Banach space with the norm $\| \cdot \|$. Let us take an interval $I = [0, T]$, $T > 0$ and $B(x_0, r)$ the closed ball in X centered at x_0 with radius r .

Theorem A (see [34]). *Assume that $f(t, x)$ is a function defined on $I \times X$ with values in X such that*

$$\|f(t, x)\| \leq Q + R\|x\|, \tag{3.3}$$

for any $x \in X$, where Q and R are nonnegative constants. Further, let f be uniformly continuous on the set $I_1 \times B(x_0, r)$, where $r = (QT_1 + RT_1\|x_0\|)/(1 - RT_1)$ and $I_1 = [0, T_1] \subset I$, $RT_1 < 1$. Moreover, assume that for any nonempty set $Y \subset B(x_0, s)$ and for almost all $t \in I$ the following inequality holds:

$$\mu(f(t, Y)) \leq q(t)\mu(Y), \tag{3.4}$$

with a sublinear measure of noncompactness μ such that $\{x_0\} \in \ker \mu$. Then problem (3.1)-(3.2) has a solution x such that $\{x(t)\} \in \ker \mu$ for $t \in I_1$, where $q(t)$ is an integrable function on I , and $\ker \mu = \{E \in \mathcal{M}_X : \mu(E) = 0\}$ is the kernel of the measure μ .

Remark 3.1. In the case when $\mu = \chi$ (the Hausdorff measure of noncompactness), the assumption of the uniform continuity on f can be replaced by the weaker one requiring only the continuity of f .

Results of Sections 3.1 and 3.2 are from [33], Section 3.3 from [26], and Section 3.4 from [32].

3.1. Infinite Systems of Differential Equations in the Space c_0

From now on, our discussion is exactly same as in Section 3 of [33].

In this section, we study the solvability of the infinite systems of differential equations in the Banach sequence space c_0 . It is known that in the space c_0 the Hausdorff measure of noncompactness can be expressed by the following formula [33]:

$$\chi(E) = \lim_{n \rightarrow \infty} \left(\sup_{x \in E} \left\{ \sup_{k \geq n} |x_k| \right\} \right), \quad (3.5)$$

where $E \in \mathcal{M}_{c_0}$.

We will be interested in the existence of solutions $x(t) = (x_i(t))$ of the infinite systems of differential equations

$$x'_i = f_i(t, x_0, x_1, x_2, \dots), \quad (3.6)$$

with the initial condition

$$x_i(0) = x_i^0, \quad (3.7)$$

($i = 0, 1, 2, \dots$) which are defined on the interval $I = [0, T]$ and such that $x(t) \in c_0$ for each $t \in I$.

An existence theorem for problem (3.6)-(3.7) in the space c_0 can be formulated by making the following assumptions.

Assume that the functions f_i ($i = 1, 2, \dots$) are defined on the set $I \times \mathbb{R}^\infty$ and take real values. Moreover, we assume the following hypotheses:

- (i) $x_0 = (x_i^0) \in c_0$,
- (ii) the map $f = (f_1, f_2, \dots)$ acts from the set $I \times c_0$ into c_0 and is continuous,
- (iii) there exists an increasing sequence (k_n) of natural numbers (obviously $k_n \rightarrow \infty$ as $n \rightarrow \infty$) such that for any $t \in I$, $x = (x_i) \in c_0$ and $n = 1, 2, \dots$ the following inequality holds:

$$|f_n(t, x_1, x_2, \dots)| \leq p_n(t) + q_n(t) \left(\sup_{i \geq k_n} |x_i| \right), \quad (3.8)$$

where $(p_i(t))$ and $(q_i(t))$ are real functions defined and continuous on I such that the sequence $(p_i(t))$ converges uniformly on I to the function vanishing identically and the sequence $(q_i(t))$ is equibounded on I .

Now, let us denote

$$\begin{aligned} q(t) &= \sup_{n \geq 1} q_n(t), \\ Q &= \sup_{t \in I} q(t), \\ P &= \sup\{p_n(t) : t \in I, n = 1, 2, \dots\}. \end{aligned} \tag{3.9}$$

Then we have the following result.

Theorem 3.2 (see [33]). *Under the assumptions (i)–(iii), initial value problem (3.6)–(3.7) has at least one solution $x = x(t) = (x_i(t))$ defined on the interval $I_1 = [0, T_1]$ whenever $T_1 < T$ and $QT_1 < 1$. Moreover, $x(t) \in c_0$ for any $t \in I_1$.*

Proof. Let $x = (x_i)$ be any arbitrary sequence in c_0 . Then, by (i)–(iii), for any $t \in I$ and for a fixed $n \in \mathbb{N}$ we obtain

$$\begin{aligned} |f_n(t, x)| &= |f_n(t, x_1, x_2, \dots)| \leq p_n(t) + q_n(t) \left(\sup_{i \geq k_n} |x_i| \right) \\ &\leq P + Q \sup_{i \geq k_n} |x_i| \leq P + Q \|x\|_\infty. \end{aligned} \tag{3.10}$$

Hence, we get

$$\|f(t, x)\| \leq P + Q \|x\|_\infty. \tag{3.11}$$

In what follows, let us take the ball $B(x_0, r)$, where r is chosen according to Theorem A. Then, for a subset Y of $B(x_0, r)$ and for $t \in I_1$, we obtain

$$\begin{aligned} \chi(f(t, Y)) &= \limsup_{n \rightarrow \infty} \sup_{x \in Y} \left(\sup_{i \geq n} |f_i(t, x)| \right), \\ \chi(f(t, Y)) &= \limsup_{n \rightarrow \infty} \sup_{x \in Y} \left(\sup_{i \geq n} |f_i(t, x_1, x_2, \dots)| \right) \\ &\leq \limsup_{n \rightarrow \infty} \sup_{x \in Y} \left(\sup_{i \geq n} \left\{ p_i(t) + q_i(t) \sup_{j \geq k_n} |x_j| \right\} \right) \\ &\leq \limsup_{n \rightarrow \infty} p_i(t) + q(t) \lim_{n \rightarrow \infty} \left\{ \sup_{x \in Y} \left(\sup_{i \geq n} \left\{ \sup_{j \geq k_n} |x_j| \right\} \right) \right\}. \end{aligned} \tag{3.12}$$

Hence, by assumptions, we get

$$\chi(f(t, Y)) \leq q(t) \chi(Y). \tag{3.13}$$

Now, using our assumptions and inequalities (3.11) and (3.13), in view of Theorem A and Remark 3.1 we deduce that there exists a solution $x = x(t)$ of the Cauchy problem (3.6)-(3.7) such that $x(t) \in c_0$ for any $t \in I_1$.

This completes the proof of the theorem. \square

We illustrate the above result by the following examples.

Example 3.3 (see [33]). Let $\{k_n\}$ be an increasing sequence of natural numbers. Consider the infinite system of differential equations of the form

$$x'_i = f_i(t, x_1, x_2, \dots) + \sum_{j=k_i+1}^{\infty} a_{ij}(t)x_j, \quad (3.14)$$

with the initial condition

$$x_i(0) = x_i^0, \quad (3.15)$$

($i = 1, 2, \dots; t \in I = [0, T]$).

We will investigate problem (3.14)-(3.15) under the following assumptions:

- (i) $x_0 = (x_0) \in c_0$,
- (ii) the functions $f_i : I \times \mathbb{R}^{k_i} \rightarrow \mathbb{R}$ ($i = 1, 2, \dots$) are uniformly equicontinuous and there exists a function sequence $(p_i(t))$ such that $p_i(t)$ is continuous on I for any $i \in \mathbb{N}$ and $(p_i(t))$ converges uniformly on I to the function vanishing identically. Moreover, the following inequality holds:

$$|f_i(t, x_1, x_2, \dots, x_{k_i})| \leq p_i(t), \quad (3.16)$$

for $t \in I, (x_1, x_2, \dots, x_{k_i}) \in \mathbb{R}^{k_i}$ and $i \in \mathbb{N}$,

- (iii) the functions $a_{ij}(t)$ are defined and continuous on I and the function series $\sum_{j=k_i+1}^{\infty} a_{ij}(t)$ converges absolutely and uniformly on I (to a function $a_i(t)$) for any $i = 1, 2, \dots$,
- (iv) the sequence $(a_i(t))$ is equibounded on I ,
- (v) $QT < 1$, where $Q = \sup\{a_i(t) : i = 1, 2, \dots; t \in I\}$.

It can be easily seen that the assumptions of Theorem 3.2 are satisfied under assumptions (i)–(v). This implies that problem (3.14)-(3.15) has a solution $x(t) = (x_i(t))$ on the interval I belonging to the space c_0 for any fixed $t \in I$.

As mentioned in [33], problem (3.14)-(3.15) considered above contains as a special case the infinite system of differential equations occurring in the theory of dissociation of polymers [35]. That system was investigated in [37] in the sequence space ℓ_∞ under very strong assumptions. The existing result proved in [35] requires also rather restrictive assumptions. Thus, the above result is more general than those quoted above.

Moreover, the choice of the space c_0 for the study of the problem (3.14)-(3.15) enables us to obtain partial characterization of solutions of this problem since we have that $x_n(t) \rightarrow 0$ when $n \rightarrow \infty$, for any fixed $t \in [0, T]$.

On the other hand, let us observe that in the study of the heat conduction problem via the method of semidiscretization we can obtain the infinite systems of form (3.14) (see [42] for details).

Example 3.4 (see [33]). In this example, we will consider some special cases of problem (3.14)-(3.15). Namely, assume that $k_i = i$ for $i = 1, 2, \dots$ and $a_{ij} \equiv 0$ on I for all i, j . Then system (3.14) has the form

$$\begin{aligned} x_1' &= f_1(t, x_1), & x_2' &= f_2(t, x_1, x_2), \dots, \\ x_i' &= f_i(t, x_1, x_2, \dots, x_i), \dots, \end{aligned} \tag{3.17}$$

and is called a row-finite system [35].

Suppose that there are satisfied assumptions from Example 3.3, that is, $x_0 = (x_i^0) \in c_0$ and the functions f_i act from $I \times \mathbb{R}^i$ into \mathbb{R} ($i = 1, 2, \dots$) and are uniformly equicontinuous on their domains. Moreover, there exist continuous functions $p_i(t)$ ($t \in I$) such that

$$|f_i(t, x_1, x_2, \dots, x_{k_i})| \leq p_i(t), \tag{3.18}$$

for $t \in I$ and $x_1, x_2, \dots, x_i \in \mathbb{R}$ ($i = 1, 2, \dots$). We assume also that the sequence $p_i(t)$ converges uniformly on I to the function vanishing identically.

Further, let $|\cdot|_i$ denote the maximum norm in \mathbb{R}^i ($i = 1, 2, \dots$). Take $f^i = (f_1, f_2, \dots, f_i)$. Then we have

$$\begin{aligned} |f^i(t, x)|_i &= \max\{|f_1(t, x_1)|, |f_2(t, x_1, x_2)|, \dots, |f_i(t, x_1, x_2, \dots, x_i)|\} \\ &\leq \max\{p_1(t), p_2(t), \dots, p_i(t)\}. \end{aligned} \tag{3.19}$$

Taking $P_i(t) = \max\{p_1(t), p_2(t), \dots, p_i(t)\}$, then the above estimate can be written in the form

$$|f^i(t, x)|_i \leq P_i(t). \tag{3.20}$$

Observe that from our assumptions it follows that the initial value problem $u' = P_i(t)$, $u(0) = x_i^0$ has a unique solution on the interval I . Hence, applying a result from [33], we infer that Cauchy problem (3.17)-(3.15) has a solution on the interval I . Obviously from the result contained in Theorem 3.2 and Example 3.3, we deduce additionally that the mentioned solution belongs to the space c_0 .

Finally, it is noticed [33] that the result described above for row-finite systems of the type (3.17) can be obtained under more general assumptions.

In fact, instead of inequality (3.18), we may assume that the following estimate holds to be true:

$$|f_i(t, x_1, x_2, \dots, x_i)| \leq p_i(t) + q_i(t) \max\{|x_1|, |x_2|, \dots, |x_i|\}, \tag{3.21}$$

where the functions $p_i(t)$ and $q_i(t)$ ($i = 1, 2, \dots$) satisfy the hypotheses analogous to those assumed in Theorem 3.2.

Remark 3.5 (see [33]). Note that in the birth process one can obtain a special case of the infinite system (3.17) which is lower diagonal linear infinite system [35, 43]. Thus, the result proved above generalizes that from [35, 37].

3.2. Infinite Systems of Differential Equations in the Space c

Now, we will study the solvability of the following perturbed diagonal system of differential equations

$$x'_i = a_i(t)x_i + g_i(t, x_1, x_2, \dots), \quad (3.22)$$

with the initial condition

$$x_i(0) = x_i^0, \quad (3.23)$$

($i = 1, 2, \dots$), where $t \in I = [0, T]$.

From now, we are going exactly the same as in Section 4 of [33].

We consider the following measures μ of noncompactness in c which is more convenient, regular, and even equivalent to the Hausdorff measure of noncompactness [33].

For $E \in \mathcal{M}_c$

$$\begin{aligned} \mu(E) &= \lim_{p \rightarrow \infty} \left\{ \sup_{x \in E} \left\{ \sup_{n, m \geq p} |x_n - x_m| \right\} \right\}, \\ \chi(E) &= \lim_{n \rightarrow \infty} \left(\sup_{x \in Q} \left\{ \sup_{k \geq n} |x_k| \right\} \right). \end{aligned} \quad (3.24)$$

Let us formulate the hypotheses under which the solvability of problem (3.22)-(3.23) will be investigated in the space c . Assume that the following conditions are satisfied.

Assume that the functions f_i ($i = 1, 2, \dots$) are defined on the set $I \times \mathbb{R}^\infty$ and take real values. Moreover, we assume the following hypotheses:

- (i) $x_0 = (x_i^0) \in c$,
- (ii) the map $g = (g_1, g_2, \dots)$ acts from the set $I \times c$ into c and is uniformly continuous on $I \times c$,
- (iii) there exists sequence $(b_i) \in c_0$ such that for any $t \in I$, $x = (x_i) \in c$ and $n = 1, 2, \dots$ the following inequality holds:

$$|g_n(t, x_1, x_2, \dots)| \leq b_i, \quad (3.25)$$

- (iv) the functions $a_i(t)$ are continuous on I such that the sequence $(a_i(t))$ converges uniformly on I .

Further, let us denote

$$\begin{aligned} a(t) &= \sup_{i \geq 1} a_i(t), \\ Q &= \sup_{t \in I} a(t). \end{aligned} \tag{3.26}$$

Observe that in view of our assumptions, it follows that the function $a(t)$ is continuous on I . Hence, $Q < \infty$.

Then we have the following result which is more general than Theorem 3.2.

Theorem 3.6 (see [33]). *Let assumptions (i)–(iv) be satisfied. If $QT < 1$, then the initial value problem (3.12)–(3.13) has a solution $x(t) = (x_i(t))$ on the interval I such that $x(t) \in c$ for each $t \in I$.*

Proof. Let $t \in I$ and $x = (x_i) \in c$ and

$$\begin{aligned} f_i(t, x) &= a_i(t)x_i + g_i(t; x), \\ f(t, x) &= (f_1(t, x), f_2(t; x), \dots) = (f_i(t, x)). \end{aligned} \tag{3.27}$$

Then, for arbitrarily fixed natural numbers n, m we get

$$\begin{aligned} |f_n(t, x) - f_m(t, x)| &= |a_n(t)x_n + g_n(t; x) - a_m(t)x_m - g_m(t; x)| \\ &\leq |a_n(t)x_n + g_n(t; x)| - |a_m(t)x_m + g_m(t; x)| \\ &\leq |a_n(t)x_n - a_m(t)x_m| + |a_n(t)x_m - a_m(t)x_m| + b_n + b_m \\ &\leq |a_n(t)||x_n - x_m| + |a_n(t) - a_m(t)||x_m| + b_n + b_m \\ &\leq |a_n(t)||x_n - x_m| + \|x\|_\infty |a_n(t) - a_m(t)| b_n + b_m. \end{aligned} \tag{3.28}$$

By assumptions (iii) and (iv), from the above estimate we deduce that $(f_i(t, x))$ is a real Cauchy sequence. This implies that $(f_i(t, x)) \in c$.

Also we obtain the following estimate:

$$\begin{aligned} |f_i(t, x)| &\leq |a_i(t)||x_i| + |g_i(t, x)| \\ &\leq Q|x_i| + b_i \leq Q\|x\|_\infty + B, \end{aligned} \tag{3.29}$$

where $B = \sup_{i \geq 1} b_i$. Hence,

$$\|f(t, x)\| \leq Q\|x\|_\infty + B. \tag{3.30}$$

In what follows, let us consider the mapping $f(t, x)$ on the set $I \times B(x_0, r)$, where r is taken according to the assumptions of Theorem A, that is, $r = (BT + QT\|x_0\|_\infty)/(1 - QT)$.

Further, fix arbitrarily $t, s \in I$ and $x, y \in B(x_0, r)$. Then, by our assumptions, for a fixed i , we obtain

$$\begin{aligned} |f_i(t, x) - f_i(s, y)| &= |a_i(t)x_i + g_i(t, x) - a_i(s)y_i - g_i(s, y)| \\ &\leq |a_i(t)x_i - a_i(s)y_i| + |g_i(t, x) - g_i(s, y)| \\ &\leq |a_i(t) - a_i(s)||x_i| + |a_i(s)||x_i - y_i| + |g_i(t, x) - g_i(s, y)|. \end{aligned} \quad (3.31)$$

Then,

$$\begin{aligned} \|f(t, x) - f(s, y)\| &= \sup_{i \geq 1} |f_i(t, x) - f_i(s, y)| \\ &\leq (r + \|x_0\|_\infty) \cdot \sup_{i \geq 1} |a_i(t) - a_i(s)| \\ &\quad + Q\|x - y\|_\infty + \|g(t, x) - g(s, y)\|. \end{aligned} \quad (3.32)$$

Hence, taking into account that the sequence $(a_i(t))$ is equicontinuous on the interval I and g is uniformly continuous on $I \times c$, we conclude that the operator $f(t, x)$ is uniformly continuous on the set $I \times B(x_0, r)$.

In the sequel, let us take a nonempty subset E of the ball $B(x_0, r)$ and fix $t \in I$, $x \in X$. Then, for arbitrarily fixed natural numbers n, m we have

$$\begin{aligned} |f_n(t, x) - f_m(t, x)| &\leq |a_n(t)||x_n - x_m| + |x_m||a_n(t) - a_m(t)| + |g_n(t, x)| + |g_m(t, x)| \\ &\leq a(t)|x_n - x_m| + (r + \|x_0\|_\infty)|a_n(t) - a_m(t)| + b_n + b_m. \end{aligned} \quad (3.33)$$

Hence, we infer the following inequality:

$$\mu(f(t, E)) \leq a(t)\mu(E). \quad (3.34)$$

Finally, combining (3.30), (3.34) and the fact (proved above) that f is uniformly continuous on $I \times B(x_0, r)$, in view of Theorem A, we infer that problem (3.22)-(3.23) is solvable in the space c .

This completes the proof of the theorem. \square

Remark 3.7 (see [33, Remark 4]). The infinite systems of differential equations (3.22)-(3.23) considered above contain as special cases the systems studied in the theory of neural sets (cf. [35, pages 86-87] and [37], e.g.). It is easy to notice that the existence results proved in [35, 37] are obtained under stronger and more restrictive assumptions than our one.

3.3. Infinite Systems of Differential Equations in the Space ℓ_p

In this section, we study the solvability of the infinite systems of differential equations (3.6)-(3.7) in the Banach sequence space ℓ_p ($1 \leq p < \infty$) such that $x(t) \in \ell_p$ for each $t \in I$.

An existence theorem for problem (3.6)-(3.7) in the space ℓ_p can be formulated by making the following assumptions:

- (i) $x_0 = (x_i^0) \in \ell_p$,
- (ii) $f_i : I \times \mathbb{R}^\infty \rightarrow \mathbb{R}$ ($i = 0, 1, 2, \dots$) maps continuously the set $I \times \ell_p$ into ℓ_p ,
- (iii) there exist nonnegative functions $q_i(t)$ and $r_i(t)$ defined on I such that

$$|f_i(t, x)|^p = |f_i(t, x_0, x_1, x_2, \dots)|^p \leq q_i(t) + r_i(t)|x_i|^p, \quad (3.35)$$

for $t \in I; x = (x_i) \in \ell_p$ and $i = 0, 1, 2, \dots$,

- (iv) the functions $q_i(t)$ are continuous on I and the function series $\sum_{i=0}^\infty q_i(t)$ converges uniformly on I ,
- (v) the sequence $(r_i(t))$ is equibounded on the interval I and the function $r(t) = \limsup_{i \rightarrow \infty} r_i(t)$ is integrable on I .

Now, we prove the following result.

Theorem 3.8 (see [26]). *Under the assumptions (i)–(v), problem (3.6)-(3.7) has a solution $x(t) = (x_i(t))$ defined on the interval $I = [0, T]$ whenever $RT < 1$, where R is defined as the number*

$$R = \sup\{r_i(t) : t \in I, i = 0, 1, 2, \dots\}. \quad (3.36)$$

Moreover, $x(t) \in \ell_p$ for any $t \in I$.

Proof. For any $x(t) \in \ell_p$ and $t \in I$, under the above assumptions, we have

$$\begin{aligned} \|f(t, x)\|_p^p &= \sum_{i=0}^\infty |f_i(t, x)|^p \\ &\leq \sum_{i=0}^\infty [q_i(t) + r_i(t)|x_i|^p] \\ &\leq \sum_{i=0}^\infty q_i(t) + \left(\sup_{i \geq 0} r_i(t)\right) \left(\sum_{i=0}^\infty |x_i|^p\right) \\ &\leq Q + R\|x\|_p^p, \end{aligned} \quad (3.37)$$

where $Q = \sup_{t \in I} (\sum_{i=0}^\infty q_i(t))$.

Now, choose the number $s = (QT + RT\|x_0\|_p^p)/(1 - RT)$ as defined in Theorem A. Consider the operator $f = (f_i)$ on the set $I \times B(x_0; s)$. Let us take a set $Y \in \mathcal{M}_{\ell_p}$. Then by using (2.8), we get

$$\begin{aligned} \chi(f(t, Y)) &= \lim_{n \rightarrow \infty} \sup_{x \in Y} \left(\sum_{i \geq n} |f_i(t, x_1, x_2, \dots)|^p \right) \\ &\leq \lim_{n \rightarrow \infty} \left(\sum_{i \geq n} q_i(t) + \left(\sup_{i \geq n} r_i(t)\right) \left(\sum_{i \geq n} |x_i|^p\right) \right). \end{aligned} \quad (3.38)$$

Hence, by assumptions (iv)-(v), we get

$$\chi(f(t, Y)) \leq r(t)\chi(Y), \quad (3.39)$$

that is, the operator f satisfies condition (3.4) of Theorem A. Hence, by Theorem A and Remark 3.1 we conclude that there exists a solution $x = x(t)$ of problem (3.6)-(3.7) such that $x(t) \in \ell_p$ for any $t \in I$.

This completes the proof of the theorem. \square

Remark 3.9. (I) For $p = 1$, we get Theorem 5 of [33].

- (II) It is easy to notice that the existence results proved in [44] are obtained under stronger and more restrictive assumptions than our one.
- (III) We observe that the above theorem can be applied to the perturbed diagonal infinite system of differential equations of the form

$$x'_i = a_i(t)x_i + g_i(t, x_0, x_1, x_2, \dots), \quad (3.40)$$

with the initial condition

$$x_i(0) = x_i^0, \quad (3.41)$$

($i = 0, 1, 2, \dots$) where $t \in I$.

An existence theorem for problem (3.6)-(3.7) in the space ℓ_p can be formulated by making the following assumptions:

- (i) $x_0 = (x_i^0) \in \ell_p$,
- (ii) the sequence $(|a_i(t)|)$ is defined and equibounded on the interval $I = [0, T]$. Moreover, the function $a(t) = \limsup_{i \rightarrow \infty} \sup |a_i(t)|$ is integrable on I ,
- (iii) the mapping $g = (g_i)$ maps continuously the set $I \times \ell_p$ into ℓ_p ,
- (iv) there exist nonnegative functions $b_i(t)$ such that

$$|f_i(t, x_0, x_1, x_2, \dots)|^p \leq b_i(t), \quad (3.42)$$

for $t \in I$; $x = (x_i) \in \ell_p$ and $i = 0, 1, 2, \dots$,

- (v) the functions $b_i(t)$ are continuous on I and the function series $\sum_{i=0}^{\infty} b_i(t)$ converges uniformly on I .

3.4. Infinite Systems of Differential Equations in the Space $n(\phi)$

An existence theorem for problem (3.6)-(3.7) in the space $n(\phi)$ can be formulated by making the following assumptions:

- (i) $x_0 = (x_i^0) \in n(\phi)$,

- (ii) $f_i : I \times \mathbb{R}^\infty \rightarrow \mathbb{R}$ ($i = 1, 2, \dots$) maps continuously the set $I \times n(\phi)$ into $n(\phi)$,
- (iii) there exist nonnegative functions $p_i(t)$ and $q_i(t)$ defined on I such that

$$|f_i(t, u)| = |f_i(t, u_1, u_2, \dots)| \leq p_i(t) + q_i(t)|u_i|, \tag{3.43}$$

for $t \in I; x = (x_i) \in n(\phi)$ and $i = 1, 2, \dots$, where $u = (u_i)$ is a sequence of rearrangement of $x = (x_i)$,

- (iv) the functions $p_i(t)$ are continuous on I and the function series $\sum_{i=1}^\infty p_i(t)\Delta\phi_i$ converges uniformly on I ,
- (v) the sequence $(q_i(t))$ is equibounded on the interval I and the function $q(t) = \limsup_{i \rightarrow \infty} q_i(t)$ is integrable on I .

Now, we prove the following result.

Theorem 3.10 (see [32]). *Under the assumptions (i)–(v), problem (3.6)–(3.7) has a solution $x(t) = (x_i(t))$ defined on the interval $I = [0, T]$ whenever $QT < 1$, where Q is defined as the number*

$$Q = \sup\{q_i(t) : t \in I, i = 1, 2, \dots\}. \tag{3.44}$$

Moreover, $x(t) \in n(\phi)$ for any $t \in I$.

Proof. For any $x(t) \in n(\phi)$ and $t \in I$, under the above assumptions, we have

$$\begin{aligned} \|f(t, x)\|_{n(\phi)} &= \sup_{u \in S(x)} \sum_{i=1}^\infty |f_i(t, u)| \Delta\phi_i \\ &\leq \sup_{u \in S(x)} \sum_{i=1}^\infty [p_i(t) + q_i(t)|u_i|] \Delta\phi_i \\ &\leq \sum_{i=1}^\infty p_i(t) \Delta\phi_i + \left(\sup_i q_i(t) \right) \left(\sup_{u \in S(x)} \sum_{i=1}^\infty |u_i| \Delta\phi_i \right) \\ &\leq P + Q \|x\|_{n(\phi)}, \end{aligned} \tag{3.45}$$

where $P = \sup_{t \in I} \sum_{i=1}^\infty p_i(t) \Delta\phi_i$.

Now, choose the number r defined according to Theorem A, that is, $r = (PT + QT \|x_0\|_{n(\phi)}) / (1 - QT)$. Consider the operator $f = (f_i)$ on the set $I \times B(x_0; r)$. Let us take a set $X \in \mathcal{M}_{n(\phi)}$. Then by using Theorem 2.4, we get

$$\begin{aligned} \chi(f(t, X)) &= \limsup_{k \rightarrow \infty} \sup_{x \in X} \left(\sup_{u \in S(x)} \left(\sum_{n=k}^\infty |f_n(t, u_1, u_2, \dots)| \Delta\phi_n \right) \right) \\ &\leq \lim_{k \rightarrow \infty} \left(\sum_{n=k}^\infty p_n(t) \Delta\phi_n + \left(\sup_{n \geq k} q_n(t) \right) \left(\sup_{u \in S(x)} \sum_{n=k}^\infty |u_n| \Delta\phi_n \right) \right). \end{aligned} \tag{3.46}$$

Hence, by assumptions (iv)-(v), we get

$$\chi(f(t, X)) \leq q(t)\chi(X), \quad (3.47)$$

that is, the operator f satisfies condition (3.4) of Theorem A. Hence, the problem (3.6)-(3.7) has a solution $x(t) = (x_i(t))$.

This completes the proof of the theorem. \square

Remark 3.11. Similarly, we can establish such type of result for the space $m(\phi)$.

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