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Research Article

Periodic Points and Fixed Points for the Weaker (ϕ, φ)-Contractive Mappings in Complete Generalized Metric Spaces

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We introduce the notion of weaker (ϕ, φ)-contractive mapping in complete metric spaces and prove the periodic points and fixed points for this type of contraction. Our results generalize or improve many recent fixed point theorems in the literature.

1. Introduction and Preliminaries

Let (X, d) be a metric space, D a subset of X and $f : D \to X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that, for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1.1}$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. In 1969, Boyd and Wong [2] introduced the notion of Φ-contraction. A mapping $f : X \to X$ on a metric space is called Φ-contraction if there exists an upper semicontinuous function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$d(fx, fy) \leq \Phi(d(x, y)) \quad \forall x, y \in X. \tag{1.2}$$

In 2000, Branciari [3] introduced the following notion of a generalized metric space where the triangle inequality of a metric space had been replaced by an inequality involving three terms instead of two. Later, many authors worked on this interesting space (e.g., [4–9]).

Let (X, d) be a generalized metric space. For $\gamma > 0$ and $x \in X$, we define

$$B_\gamma(x) := \{y \in X \mid d(x,y) < \gamma\}. \tag{1.3}$$

Branciari [3] also claimed that $\{B_\gamma(x) : \gamma > 0, x \in X\}$ is a basis for a topology on X, d is continuous in each of the coordinates and a generalized metric space is a Hausdorff space. We recall some definitions of a generalized metric space, as follows.

Definition 1.1 (see [3]). Let X be a nonempty set and $d : X \times X \to [0,\infty)$ a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X$ each of them different from x and y, one has

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \le d(x,u) + d(u,v) + d(v,y)$ (rectangular inequality).

Then (X, d) is called a generalized metric space (or shortly *g.m.s*).

Definition 1.2 (see [3]). Let (X, d) be a *g.m.s*, $\{x_n\}$ a sequence in X, and $x \in X$. We say that $\{x_n\}$ is *g.m.s* convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote by $x_n \to x$ as $n \to \infty$.

Definition 1.3 (see [3]). Let (X, d) be a *g.m.s*, $\{x_n\}$ a sequence in X, and $x \in X$. We say that $\{x_n\}$ is *g.m.s* Cauchy sequence if and only if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $n > m > n_0$.

Definition 1.4 (see [3]). Let (X, d) be a *g.m.s*. Then X is called complete *g.m.s* if every *g.m.s* Cauchy sequence is *g.m.s* convergent in X.

In this paper, we also recall the notion of Meir-Keeler function (see [10]). A function $\phi : [0,\infty) \to [0,\infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,\infty)$ with $\eta \le t < \eta + \delta$, we have $\phi(t) < \eta$. Generalization of the above function has been a heavily investigated branch research. Particularly, in [11, 12], the authors proved the existence and uniqueness of fixed points for various Meir-Keeler-type contractive functions. We now introduce the notion of weaker Meir-Keeler function $\phi : [0,\infty) \to [0,\infty)$, as follows.

Definition 1.5. We call $\phi : [0,\infty) \to [0,\infty)$ a weaker Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0,\infty)$ with $\eta \le t < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(t) < \eta$.

2. Main Results

In the paper, we denote by Φ the class of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ϕ_1) $\phi : [0,\infty) \to [0,\infty)$ is a weaker Meir-Keeler function;

(ϕ_2) $\phi(t) > 0$ for $t > 0$, $\phi(0) = 0$;

(ϕ_3) for all $t \in (0,\infty)$, $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;

(ϕ_4) if $\lim_{n\to\infty} t_n = \gamma$, then $\lim_{n\to\infty}\phi(t_n) \le \gamma$.

And we denote by Θ the class of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(φ_1) φ is continuous;

(φ_2) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$.

Our main result is the following.

Theorem 2.1. *Let (X,d) be a Hausdorff and complete g.m.s, and let $f : X \to X$ be a function satisfying*

$$d(fx, fy) \le \phi(d(x,y)) - \varphi(d(x,y)) \tag{2.1}$$

for all $x, y \in X$ and $\phi \in \Phi$, $\varphi \in \Theta$. Then f has a periodic point μ in X, that is, there exists $\mu \in X$ such that $\mu = f^p\mu$ for some $p \in \mathbb{N}$.

Proof. Given x_0, define a sequence $\{x_n\}$ in X by

$$x_{n+1} = f x_n \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Step 1. We will prove that

$$\begin{aligned} &\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \\ &\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \end{aligned} \tag{2.3}$$

Using inequality (2.1), we have that for each $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f x_{n-1}, f x_n) \\ &\le \phi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) \\ &\le \phi(d(x_{n-1}, x_n)), \end{aligned} \tag{2.4}$$

and so

$$\begin{aligned} d(x_n, x_{n+1}) &\le \phi(d(x_{n-1}, x_n)) \\ &\le \phi(\phi(d(x_{n-2}, x_{n-1}))) = \phi^2(d(x_{n-2}, x_{n-1})) \\ &\le \cdots \\ &\le \phi^n(d(x_0, x_1)). \end{aligned} \tag{2.5}$$

Since $\{\phi^n(d(x_0,x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ϕ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \le d(x_0, x_1) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$

such that $\phi^{n_0}(d(x_0, x_1)) < \eta$. Since $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \phi^p(d(x_0, x_1)) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\phi^{p_0+n_0}(d(x_0, x_1)) < \eta$. So we get a contradiction. Therefore, $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Using inequality (2.1), we also have that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_{n+2}) &= d(f x_{n-1}, f x_{n+1}) \\ &\leq \phi(d(x_{n-1}, x_{n+1})) - \varphi(d(x_{n-1}, x_{n+1})) \\ &\leq \phi(d(x_{n-1}, x_{n+1})), \end{aligned} \tag{2.7}$$

and so

$$\begin{aligned} d(x_n, x_{n+2}) &\leq \phi(d(x_{n-1}, x_{n+1})) \\ &\leq \phi(\phi(d(x_{n-2}, x_n))) = \phi^2(d(x_{n-2}, x_n)) \\ &\leq \cdots \\ &\leq \phi^n(d(x_0, x_2)). \end{aligned} \tag{2.8}$$

Since $\{\phi^n(d(x_0, x_2))\}_{n\in\mathbb{N}}$ is decreasing, by the same proof process, we also conclude

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.9}$$

Next, we claim that $\{x_n\}$ is $g.m.s$ Cauchy. We claim that the following result holds.

Step 2. Claim that for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ then $d(x_p, x_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying

$$d(x_{q_n}, x_{p_n}) \geq \epsilon. \tag{2.10}$$

Further, corresponding to $q_n \geq n$, we can choose p_n in such a way that it the smallest integer with $p_n > q_n \geq n$ and $d(x_{q_n}, x_{p_n}) \geq \epsilon$. Therefore, $d(x_{q_n}, x_{p_n-1}) < \epsilon$. By the rectangular inequality and (2.3), we have

$$\begin{aligned} \epsilon &\leq d(x_{p_n}, x_{q_n}) \\ &\leq d(x_{p_n}, x_{p_n-2}) + d(x_{p_n-2}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n}) \\ &< d(x_{p_n}, x_{p_n-2}) + d(x_{p_n-2}, x_{p_n-1}) + \epsilon. \end{aligned} \tag{2.11}$$

Let $n \to \infty$. Then we get

$$\lim_{n\to\infty} d(x_{p_n}, x_{q_n}) = \epsilon. \tag{2.12}$$

On the other hand, we have

$$\begin{aligned} d(x_{p_n}, x_{q_n}) &\le d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}), \\ d(x_{p_n-1}, x_{q_n-1}) &\le d(x_{p_n-1}, x_{p_n}) + d(x_{p_n}, x_{q_n}) + d(x_{q_n}, x_{q_n-1}). \end{aligned} \tag{2.13}$$

Let $n \to \infty$. Then we get

$$\lim_{n\to\infty} d(x_{p_n-1}, x_{q_n-1}) = \epsilon. \tag{2.14}$$

Using inequality (2.1), we have

$$\begin{aligned} d(x_{p_n}, x_{q_n}) &= d(fx_{p_n-1}, fx_{q_n-1}) \\ &\le \phi(d(x_{p_n-1}, x_{q_n-1})) - \varphi(d(x_{p_n-1}, x_{q_n-1})). \end{aligned} \tag{2.15}$$

Letting $n \to \infty$, using the definitions of the functions ϕ and φ, we have

$$\epsilon \le \epsilon - \varphi(\epsilon), \tag{2.16}$$

which implies that $\varphi(\epsilon) = 0$. By the definition of the function φ, we have $\epsilon = 0$. So we get a contradiction. Therefore $\{x_n\}$ is *g.m.s* Cauchy.

Step 3. We claim that f has a periodic point in X.

Suppose, on contrary, f has no periodic point. Then $\{x_n\}$ is a sequence of distinct points, that is, $x_p \ne x_q$ for all $p, q \in \mathbb{N}$ with $p \ne q$. By Step 2, since X is complete *g.m.s*, there exists $\nu \in X$ such that $x_n \to \nu$. Using inequality (2.1), we have

$$d(fx_n, f\nu) \le \phi(d(x_n, \nu)) - \varphi(d(x_n, \nu)). \tag{2.17}$$

Letting $n \to \infty$, we have

$$d(fx_n, f\nu) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.18}$$

that is,

$$x_{n+1} = fx_n \longrightarrow f\nu, \quad \text{as } n \longrightarrow \infty. \tag{2.19}$$

As (X, d) is Hausdorff, we have $\nu = f\nu$, a contradiction with our assumption that f has no periodic point. Therefore, there exists $\nu \in X$ such that $\nu = f^p(\nu)$ for some $p \in \mathbb{N}$. So f has a periodic point in X. □

Following Theorem 2.1, it is easy to get the below fixed point result.

Theorem 2.2. *Let (X, d) be a Hausdorff and complete g.m.s, and let $f : X \to X$ be a function satisfying*

$$d(fx, fy) \leq \phi(d(x, y)) - \varphi(d(x, y)) \tag{2.20}$$

for all $x, y \in X$, where $\phi \in \Phi$ with $0 < \phi(t) < t$ for all $t > 0$, and $\varphi \in \Theta$. Then f has a unique fixed point in X.

Proof. From Theorem 2.1, we conclude that f has a periodic point $\nu \in X$, that is, there exists $\nu \in X$ such that $\nu = f^p(\nu)$ for some $p \in \mathbb{N}$. If $p = 1$, then we complete the proof, that is, ν is a fixed point of f. If $p > 1$, then we will show that $\mu = f^{p-1}\nu$ is a fixed point of f. Suppose that it is not the case, that is, $f^{p-1}\nu \neq f^p\nu$. Then Using inequality (2.1), we have

$$\begin{aligned} d(\nu, f\nu) &= d\left(f^p\nu, f^{p+1}\nu\right) \\ &\leq \phi\left(d\left(f^{p-1}\nu, f^p\nu\right)\right) - \varphi\left(d\left(f^{p-1}\nu, f^p\nu\right)\right) \\ &< \phi\left(d\left(f^{p-1}\nu, f^p\nu\right)\right) \\ &\leq d\left(f^{p-1}\nu, f^p\nu\right). \end{aligned} \tag{2.21}$$

Using inequality (2.1), we also have

$$\begin{aligned} d\left(f^{p-1}\nu, f^p\nu\right) &\leq \phi\left(d\left(f^{p-2}\nu, f^{p-1}\nu\right)\right) - \varphi\left(d\left(f^{p-2}\nu, f^{p-1}\nu\right)\right) \\ &\leq \phi\left(d\left(f^{p-2}\nu, f^{p-1}\nu\right)\right) \\ &\leq d\left(f^{p-2}\nu, f^{p-1}\nu\right). \end{aligned} \tag{2.22}$$

Continuing this process, we conclude that

$$d(\nu, f\nu) < d\left(f^{p-1}\nu, f^p\nu\right) \leq d\left(f^{p-2}\nu, f^{p-1}\nu\right) \leq \cdots \leq d(\nu, f\nu), \tag{2.23}$$

which implies a contradiction. Thus, $\mu = f^{p-1}\nu$ is a fixed point of f.

Finally, to prove the uniqueness of the fixed point, suppose μ, ν are fixed points of f. Then,

$$d(\mu, \nu) = d(f\mu, f\nu) \leq \phi(d(\mu, \nu)) - \varphi(d(\mu, \nu)), \tag{2.24}$$

which implies that $d(\mu, \nu) = 0$, that is, $\mu = \nu$. So we complete the proof. □

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