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Research Article

A Contraction Fixed Point Theorem in Partially Ordered Metric Spaces and Application to Fractional Differential Equations

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We generalize a fixed point theorem in partially ordered complete metric spaces in the study of A. Amini-Harandi and H. Emami (2010). We also give an application on the existence and uniqueness of the positive solution of a multipoint boundary value problem with fractional derivatives.

1. Introduction

Let $\mathcal{S}$ denote the class of those functions $\beta : [0, +\infty) \to [0, 1)$ which satisfies the condition

$$\beta(t_n) \longrightarrow 1 \quad \text{implies } t_n \longrightarrow 0. \tag{1.1}$$

The following generalization of Banach's contraction principle is due to Geraghty [1].

Theorem 1.1. *Let (M, d) be a complete metric space and let $f : M \to M$ be a map. Suppose there exists $\beta \in \mathcal{S}$ such that for each $x, y \in M$,*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y). \tag{1.2}$$

Then f has a unique fixed point $z \in M$, and $\{f^n(x)\}$ converges to z, for each $x \in M$.

And then, Amini-Harandi and Emami [2] proved a version of Theorem 1.1 in the context of partially ordered complete metric spaces.

Theorem 1.2. *Let $(M, \preceq)$ be a partially ordered set and suppose that there exists a metric d in M such that (M, d) is a complete metric space. Let $f : M \to M$ be a nondecreasing mapping, and there exists an element $x_0 \in M$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in \mathcal{S}$ such that*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y), \quad \text{for each } x, y \in M \text{ with } x \succeq y. \tag{1.3}$$

Assume that either f is continuous or M is such that

$$\text{if an increasing sequence } \{x_n\} \longrightarrow x \quad \text{in } M, \text{ then } x_n \preceq x, \ \forall n. \tag{1.4}$$

Besides, if

$$\text{for any } x, y \in M, \quad \text{there exists } z \in M \text{ which is comparable to } x \text{ and } y, \tag{1.5}$$

Then f has a unique fixed point.

Theorem 2.2 was also applied to obtain the existence and uniqueness of the solution of a periodic boundary value problem by Amini-Harandi and Emami [2] and a singular fractional three-point boundary value problem by Cabrera et al. [3].

2. Main Results

In this section, we firstly define a class of functions $\mathcal{A}$ by $\beta : [0, +\infty) \to [0, +\infty)$, and there exists a constant $K > 0$ such that

$$\sup_{t \in [0, +\infty)} \{\beta(t)\} \leq K. \tag{2.1}$$

Remark 2.1. Function classes $\mathcal{A}$ include all bounded functions on $[0, +\infty)$ with upper bound K which are more extensive than those of $\mathcal{S}$. For example, $\beta(t) = K \sin t \in \mathcal{A}$ but not in $\mathcal{S}$.

Now, we give an extended version of Theorem 1.2 in the context of partially ordered complete metric spaces.

Theorem 2.2. *Let $(M, \preceq)$ be a partially ordered set and suppose that there exists a metric d in M such that (M, d) is a complete metric space. Let $f : M \to M$ be a nondecreasing mapping, and there exists s an element $x_0 \in M$ with $x_0 \preceq f(x_0)$. Suppose that there exists a constant $\theta \in (0, 1/K)$ and $\beta \in \mathcal{S}$ such that*

$$d(f(x), f(y)) \leq \theta\beta(\theta d(x, y))d(x, y), \quad \text{for each } x, y \in M \text{ with } x \succeq y. \tag{2.2}$$

Assume that either f is continuous or

$$\text{if an increasing sequence } \{x_n\} \longrightarrow x \quad \text{in } M, \text{ then } x_n \preceq x, \ \forall n. \tag{2.3}$$

Besides, if

$$\text{for any } x, y \in M, \quad \text{there exists } z \in M \text{ which is comparable to } x \text{ and } y, \tag{2.4}$$

then f has a unique fixed point.

Proof. We first show that f has a fixed point. Since $x_0 \preceq f(x_0)$ and f is an increasing function, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq f^3(x_0) \preceq \cdots \preceq f^n(x_0) \preceq \cdots. \tag{2.5}$$

Put $x_{n+1} = f^n(x_0)$, $n = 1, 2, \ldots$ For each integer $n \geq 1$, from (2.5), we have $x_n \preceq x_{n+1}$, then by (2.2)

$$d(x_{n+1}, x_n) = d\left(f^n(x_0), f^{n-1}(x_0)\right) \leq \theta\beta(\theta d(x_n, x_{n-1}))d(x_n, x_{n-1}) \leq K\theta d(x_n, x_{n-1}). \tag{2.6}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0-1}) = 0$, then $x_{n_0} = f^{n_0-1}(x_0) = f(x_{n_0-1}) = x_{n_0-1}$ and x_{n_0-1} is a fixed point; in this case, the proof is finished. Otherwise, for any $n \in \mathbb{N}$, $d(x_n, x_{n-1}) \neq 0$. Then by (2.6), we have

$$d(x_{n+1}, x_n) \leq K\theta d(x_n, x_{n-1}) \leq \cdots \leq (K\theta)^{n+1} d(x_{n_0}, x_{n_0-1}) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \tag{2.7}$$

that is,

$$d(x_{n+1}, x_n) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{2.8}$$

Now, we show that x_n is a Cauchy sequence. By the triangle inequality and (2.2), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + \theta\beta(\theta d(x_n, x_m))d(x_n, x_m) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + \theta K d(x_n, x_m) + d(x_{m+1}, x_m), \end{aligned} \tag{2.9}$$

and then

$$d(x_n, x_m) \leq (1 - K\theta)^{-1}[d(x_n, x_{n+1}) + d(x_{m+1}, x_m)] \longrightarrow 0, \quad \text{as } m, n \longrightarrow +\infty, \tag{2.10}$$

which implies that x_n is a Cauchy sequence in M. Since (M, d) is a complete metric space, then there exists a $z \in M$ such that $\lim_{n \to +\infty} x_n = z$. To prove that z is a fixed point of f, if f is continuous, then

$$z = \lim_{n \to +\infty} x_n = \lim_{n \to +\infty} f^n(x_0) = \lim_{n \to +\infty} f^{n+1}(x_0) = f\left(\lim_{n \to +\infty} f^n(x_0)\right) = f(z), \tag{2.11}$$

hence $z = f(z)$. If case (2.3) holds, then we claim that $f(z) = z$ still holds. In fact,

$$d(f(z), z) \le d(f(z), f(x_n)) + d(f(x_n), z) \le \theta K d(z, x_n) + d(x_{n+1}, z), \tag{2.12}$$

since $d(z, x_n) \to 0$, taking limit as $n \to +\infty$, then $d(f(z), z) \le 0$, this proves that $d(f(z), z) = 0$; consequently, $f(z) = z$.

Let y be another fixed point of f. From (2.4) there exists $x \in M$ which is comparable to y and z. Monotonicity implies that $f^n(x)$ is comparable to $f^n(y) = y$ and $f^n(z) = z$ for $n = 0, 1, 2 \ldots$ Moreover,

$$d(z, f^n(x)) = d(f^n(z), f^n(x)) \le K\theta d\left(z, f^{n-1}(x)\right). \tag{2.13}$$

Taking limit, and then $\lim_{n\to+\infty} d\ (z, f^n(x)) = 0$. Similar to [2], we have $d(z, y) = 0$. The proof of the uniqueness of the fixed point is completed. □

3. Application to Fractional Differential Equations

In this section, we consider the unique positive solution for a general higher order fractional differential equation by using the generalized fixed point theorem

$$\begin{gathered}
-\mathfrak{D}^{\alpha} x(t) = f\left(t, x(t), \mathfrak{D}^{\mu_1} x(t), \mathfrak{D}^{\mu_2} x(t), \ldots, \mathfrak{D}^{\mu_{n-1}} x(t)\right), \\
\mathfrak{D}^{\mu_i} x(0) = 0, \quad 1 \le i \le n-1, \\
\mathfrak{D}^{\mu_{n-1}+1} x(0) = 0, \qquad \mathfrak{D}^{\mu_{n-1}} x(1) = \sum_{j=1}^{m-2} a_j \mathfrak{D}^{\mu_{n-1}} x(\xi_j),
\end{gathered} \tag{3.1}$$

where $n-1 < \alpha \le n$, $n \in \mathbb{N}$ and $n \ge 3$ with $0 < \mu_1 < \mu_2 < \cdots < \mu_{n-2} < \mu_{n-1}$ and $n-3 < \mu_{n-1} < \alpha - 2$, $a_j \in \mathbb{R}$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ satisfying $0 < \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_{n-1}-1} < 1$ and $\mathfrak{D}^{\alpha}$ is the standard Riemann-Liouville derivative, $f \in C([0,1] \times \mathbb{R}^n, [0, +\infty))$. Recently, there has been a significant development in the study of fractional differential equations; for more details we refer the reader to [4–12] and the references cited therein.

For the convenience of the reader, we present some notations and lemmas which will be used in the proof of our results.

Definition 3.1 (see [13, 14]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \tag{3.2}$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 3.2 (see [13, 14]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathfrak{D}_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s) ds, \tag{3.3}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Proposition 3.3 (see [13, 14]). *Consider the following.*

(1) *If* $x \in L^1(0,1)$, $\nu > \sigma > 0$, *then*

$$I^{\nu} I^{\sigma} x(t) = I^{\nu+\sigma} x(t), \qquad \mathfrak{D}_t^{\sigma} I^{\nu} x(t) = I^{\nu-\sigma} x(t), \qquad \mathfrak{D}_t^{\sigma} I^{\sigma} x(t) = x(t). \tag{3.4}$$

(2) *If* $\nu > 0$, $\sigma > 0$, *then*

$$\mathfrak{D}_t^{\nu} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}. \tag{3.5}$$

Proposition 3.4 (see [13, 14]). *Let* $\alpha > 0$, *and* $f(x)$ *is integrable, then*

$$I^{\alpha} \mathfrak{D}_t^{\alpha} x(t) = f(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_n x^{\alpha-n}, \tag{3.6}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, n *is the smallest integer greater than or equal to* α.

Lemma 3.5 (see [15]). *Let* $x(t) = I^{\mu_{n-1}} u(t)$, *then BVP* (3.1) *is equivalent to the following BVP:*

$$\begin{gathered} -\mathfrak{D}^{\alpha-\mu_{n-1}} u(t) = f\left(t, I^{\mu_{n-1}} u(t), I^{\mu_{n-1}-\mu_1} u(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}} u(t), u(t)\right), \\ u(0) = u'(0) = 0 \qquad u(1) = \sum_{j=1}^{m-2} a_j u(\xi_j). \end{gathered} \tag{3.7}$$

Moreover, if $v \in C([0,1], [0, +\infty))$ *is a solution of problem* (3.7), *then the function* $x(t) = I^{\mu_{n-1}} u(t)$ *is a positive solution of problem* (3.1).

Let

$$k(t,s) = \begin{cases} \dfrac{(t(1-s))^{\alpha-\mu_{n-1}-1} - (t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{(t(1-s))^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \le t \le s \le 1, \end{cases} \tag{3.8}$$

from [15], the Green function of (3.7) is

$$H(t,s)=k(t,s)+\frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\mu_{n-1}-1}}\sum_{j=1}^{m-2}a_jk(\xi_j,s), \tag{3.9}$$

and with property

$$0\le H(t,s)\le\frac{1}{\Gamma(\alpha-\mu_{n-1})}\left(1+\frac{\sum_{j=1}^{m-2}a_j}{1-\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\mu_{n-1}-1}}\right)=\eta. \tag{3.10}$$

In our considerations, we will work in the Banach space $E=C([0,1];\mathbb{R})$ with the classical metric given by $d(x,y)=\sup_{0\le t\le 1}|x(t)-y(t)|$. Notice that this space can be equipped with a partial order given by

$$x,y\in C[0,1],\quad x\le y\Longleftrightarrow x(t)\le y(t)\quad\text{for any } t\in[0,1]. \tag{3.11}$$

In [2], it is proved that $(C[0,1],\le)$ satisfies condition (2.3) of Theorem 2.2. Moreover, for $x,y\in C[0,1]$, as the function $\max\{x,y\}\in C[0,1]$, $(C[0,1],\le)$ satisfies condition [15]. Consider the cone

$$P=\{u\in C[0,1]:u(t)\ge 0\}. \tag{3.12}$$

Note that as P is a closed set of $C[0,1]$, P is a complete metric space.

It is well known that the BVP (3.7) is equivalent to the integral equation

$$u(t)=\int_0^1 H(t,s)f\left(s,I^{\mu_{n-1}}u(s),I^{\mu_{n-1}-\mu_1}u(s),\dots,I^{\mu_{n-1}-\mu_{n-2}}u(s),u(s)\right)ds. \tag{3.13}$$

Now, for $u\in P$ we define the operator T by

$$(Tu)(t)=\int_0^1 H(t,s)f\left(s,I^{\mu_{n-1}}u(s),I^{\mu_{n-1}-\mu_1}u(s),\dots,I^{\mu_{n-1}-\mu_{n-2}}u(s),u(s)\right)ds. \tag{3.14}$$

Then from the assumption on f and (3.10), we have

$$T(P)\subset P. \tag{3.15}$$

We also introduce the following class of nondecreasing functions $\mathcal{B}$ by $\phi:[0,+\infty)\to[0,+\infty)$ satisfying the following:

$$\phi(x)\le Kx,\quad\text{for any } x>0. \tag{3.16}$$

Clearly, if $\phi \in \mathcal{B}$, then $\phi(x)/x \in \mathcal{A}$. The standard functions $\phi \in \mathcal{B}$, for example, $\phi(x) = Kx$, $\phi(x) = (2K/\pi)x \arctan x$, and $\phi(x) = Kx^2/(1+x)$.

Theorem 3.6. *Suppose $f(t, x_1, x_2, \ldots, x_n)$ is nondecreasing in x_i on $[0, +\infty)$; moreover, there exist n positive constants ρ_i, $i = 1, 2, \ldots, n$ that satisfy*

$$\max\{\rho_1, \rho_2, \ldots, \rho_n\} \leq (n\eta)^{-1}, \tag{3.17}$$

and there exist a function $\phi \in \mathcal{B}$ and constants $0 < \theta_1 < \Gamma(\mu_{n-1})/K$, $0 < \theta_n < 1/K$, $0 < \theta_i < \Gamma(\mu_{n-1} - \mu_i)/K$, $i = 1, 2, \ldots, n-2$ such that

$$\left| f(t, x_1, x_2, \ldots, x_n) - f(t, y_1, y_2, \ldots, y_n) \right| \leq \sum_{i=1}^{n} \rho_i \phi(\theta_i(x_i - y_i)), \tag{3.18}$$

for $x_i, y_i \in [0, +\infty)$, $i = 1, 2, \ldots, n$ with $x_i \geq y_i$ and $t \in [0, 1]$. Then problem (3.1) has a unique nonnegative solution.

Proof. We check that the hypotheses in Theorem 2.2 are satisfied.

Firstly, the operator T is nondecreasing by the hypothesis. Then for any $u \geq v$, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 H(t,s) f\left(s, I^{\mu_{n-1}} u(s), I^{\mu_{n-1}-\mu_1} u(s), \ldots, I^{\mu_{n-1}-\mu_{n-2}} u(s), u(s)\right) ds \\ &\geq \int_0^1 H(t,s) f\left(s, I^{\mu_{n-1}} v(s), I^{\mu_{n-1}-\mu_1} v(s), \ldots, I^{\mu_{n-1}-\mu_{n-2}} v(s), v(s)\right) ds \\ &= (Tv)(t).\end{aligned} \tag{3.19}$$

Noticing that

$$\begin{gathered} u(s) - v(s) \leq d(u, v), \\ I^{\mu_{n-1}} u(s) - I^{\mu_{n-1}} v(s) \leq \int_0^t \frac{(t-s)^{\mu_{n-1}-1} |u(s) - v(s)|}{\Gamma(\mu_{n-1})} ds \leq \frac{d(u, v)}{\Gamma(\mu_{n-1})}, \\ I^{\mu_{n-1}-\mu_i} u(s) - I^{\mu_{n-1}-\mu_i} v(s) \leq \int_0^t \frac{(t-s)^{\mu_{n-1}-\mu_i-1} |u(s) - v(s)|}{\Gamma(\mu_{n-1} - \mu_i)} ds \\ \leq \frac{d(u, v)}{\Gamma(\mu_{n-1} - \mu_i)}, \quad i = 1, 2, \ldots, n-2, \end{gathered} \tag{3.20}$$

and take

$$\theta = \max\left\{ \frac{\theta_1}{\Gamma(\mu_{n-1})}, \frac{\theta_2}{\Gamma(\mu_{n-1} - \mu_1)}, \ldots, \frac{\theta_{n-1}}{\Gamma(\mu_{n-1} - \mu_{n-2})}, \theta_n \right\}. \tag{3.21}$$

Thus, for any $u \geq v$, we have

$$\begin{aligned}
d(Tu,Tv) &= \max_{t\in[0,1]}|Tu(t)-Tv(t)| \\
&\leq \eta\int_0^1 \big|f\left(s,I^{\mu_{n-1}}u(s),I^{\mu_{n-1}-\mu_1}u(s),\ldots,I^{\mu_{n-1}-\mu_{n-2}}u(s),u(s)\right)ds \\
&\qquad -f\left(s,I^{\mu_{n-1}}v(s),I^{\mu_{n-1}-\mu_1}v(s),\ldots,I^{\mu_{n-1}-\mu_{n-2}}v(s),v(s)\right)\big| \\
&\leq \eta\int_0^1 \left[\rho_1\phi(\theta_1(I^{\mu_{n-1}}u(s)-I^{\mu_{n-1}}v(s)))+\rho_2\phi\left(\theta_2\left(I^{\mu_{n-1}-\mu_1}u(s)-I^{\mu_{n-1}-\mu_1}v(s)\right)\right)\right. \\
&\qquad \left.+\cdots+\rho_{n-1}\phi\left(\theta_{n-1}\left(I^{\mu_{n-1}-\mu_{n-2}}u(s)-I^{\mu_{n-1}-\mu_{n-2}}v(s)\right)\right)+\rho_n\phi(\theta_n(u(s)-v(s)))\right]ds \\
&\leq n\eta\max\{\rho_1,\rho_2,\ldots,\rho_n\}\phi(\theta d(u,v)) \leq \phi(\theta d(u,v)).
\end{aligned} \tag{3.22}$$

For $u \neq v$, we have

$$d(Tu,Tv) \leq \frac{\phi(\theta d(u,v))}{\theta d(u,v)}\theta d(u,v) = \beta(\theta d(u,v))\theta d(u,v), \tag{3.23}$$

and this inequality is obviously satisfied for $u = v$. Thus, we have

$$d(Tu,Tv) \leq \frac{\phi(\theta d(u,v))}{\theta d(u,v)}\theta d(u,v) = \beta(\theta d(u,v))\theta d(u,v), \quad \text{for any } u,v \in P \text{ with } u \geq v. \tag{3.24}$$

Finally, since the zero function satisfies $0 \leq T0$, Theorem 2.2 tells us that the operator T has a unique fixed point in P, or, equivalently, the BVP (3.1) has a unique nonnegative solution x in $C[0,1]$. □

Theorem 3.7. *If the assumptions of Theorem 3.6 are satisfied, and there exists $t_0 \in [0,1]$ such that $f(t_0,0,\ldots,0) \neq 0$, then the unique solution of* (3.1) *is positive (a positive solution means a solution satisfying $x(t) > 0$ for $t \in (0,1)$).*

Proof. By Theorem 3.6, the problem (3.1) has a unique nonnegative solution. We prove the nonnegative solution is also positive.

Otherwise, there exists $0 < t^* < 1$ such that $x(t^*) = 0$, and

$$x(t^*) = \int_0^1 H(t^*,s)f\left(s,I^{\mu_{n-1}}u(s),I^{\mu_{n-1}-\mu_1}u(s),\ldots,I^{\mu_{n-1}-\mu_{n-2}}u(s),u(s)\right)ds = 0. \tag{3.25}$$

Then

$$0 = x(t^*) = \int_0^1 H(t^*, s) f\left(s, I^{\mu_{n-1}} u(s), I^{\mu_{n-1}-\mu_1} u(s), \dots, I^{\mu_{n-1}-\mu_{n-2}} u(s), u(s)\right) ds$$
$$\geq \int_0^1 H(t^*, s) f(s, 0, \dots, 0) ds \geq 0. \tag{3.26}$$

Consequently,

$$\int_0^1 H(t^*, s) f(s, 0, \dots, 0) ds = 0, \tag{3.27}$$

this yields

$$H(t^*, s) f(s, 0, \dots, 0) = 0, \quad \text{a.e. } s \in [0, 1]. \tag{3.28}$$

Note that $H(t^*, s) > 0$, $s \in (0, 1)$, then we have

$$f(s, 0, \dots, 0) = 0, \quad \text{a.e. } s \in [0, 1]. \tag{3.29}$$

But on the other hand, since $f(t_0, 0, \dots, 0) \neq 0$, $t_0 \in [0, 1]$, we have $f(t_0, 0, \dots, 0) > 0$, by the continuity of f, we can find a set $\Omega \subset [0, 1]$ satisfying $t_0 \in \Omega$ and the Lebesgue measure $\mu(\Omega) > 0$ such that $f(t, 0, \dots, 0) > 0$ for any $t \in \Omega$. This contradicts to (3.29). Therefore, $x(t) > 0$, that is, $x(t)$ is positive solution of (3.1). □

Example 3.8. Consider the following fractional boundary value problem:

$$-\mathfrak{D}^{5/2} x(t) = e^t + \frac{1}{10} x(t) + \frac{1}{2} \sin^2\left(\mathfrak{D}^{1/8} x(t)\right) + \frac{1}{3} \cos^2\left(\mathfrak{D}^{1/4} x(t)\right), \quad 0 < t < 1,$$
$$\mathfrak{D}^{1/4} x(0) = \mathfrak{D}^{5/4} x(0) = 0, \qquad \mathfrak{D}^{1/4} x(1) = \frac{1}{4} \mathfrak{D}^{1/4} x\left(\frac{1}{4}\right) + \frac{1}{2} \mathfrak{D}^{1/4} x\left(\frac{3}{4}\right). \tag{3.30}$$

Then the BVP (3.30) has a unique positive solution.

Proof. Since

$$\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_{n-1}-1} = \frac{1}{4}\left(\frac{1}{4}\right)^{5/4} + \frac{1}{2}\left(\frac{3}{4}\right)^{5/4} = 0.39316 < 1,$$
$$\eta = \frac{1}{\Gamma(\alpha - \mu_{n-1})}\left(1 + \frac{\sum_{j=1}^{m-2} a_j}{1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}}\right) = 0.10964. \tag{3.31}$$

Thus,

$$(n\eta)^{-1} = 3.0405. \tag{3.32}$$

Let $K = 15$ and $\phi(x) = 15x$, and take

$$f(t, x_1, x_2, x_3) = e^t + \frac{1}{10}x_1 + \frac{1}{2}\sin^2 x_2 + \frac{1}{3}\cos^2 x_3, \quad (t, x_1, x_2, x_3) \in [0,1] \times [0, +\infty)^3. \tag{3.33}$$

Then, for any $x_1 \geq y_1$, $x_2 \geq y_2$, $x_3 \geq y_3$,

$$\begin{aligned}
|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| &= \left| \frac{x_1 - y_1}{10} + \frac{\sin^2 x_2 - \sin^2 y_2}{2} + \frac{\cos^2 x_3 - \cos^2 y_3}{3} \right| \\
&\leq \frac{x_1 - y_1}{10} + \frac{x_2 - y_2}{2} + \frac{x_3 - y_3}{3} \\
&= \frac{2}{75} \times 15 \times \frac{1}{4}(x_1 - y_1) + \frac{1}{10} \times 15 \times \frac{1}{3}(x_2 - y_3) \\
&\quad + \frac{4}{9} \times 15 \times \frac{1}{20}(x_3 - y_3) \\
&= \frac{2}{75}\phi\left(\frac{1}{4}(x_1 - y_1)\right) + \frac{1}{10}\phi\left(\frac{1}{3}(x_2 - y_3)\right) \\
&\quad + \frac{4}{9}\phi\left(\frac{1}{20}(x_3 - y_3)\right) \\
&= \sum_{i=1}^{3} \rho_i \phi(\theta_i(x_i - y_i)),
\end{aligned} \tag{3.34}$$

where

$$\rho_1 = \frac{2}{75}, \quad \rho_2 = \frac{1}{10}, \quad \rho_3 = \frac{4}{9}, \quad \theta_1 = \frac{1}{4}, \quad \theta_2 = \frac{1}{3}, \quad \theta_3 = \frac{1}{20}. \tag{3.35}$$

Thus, $\phi \in \mathcal{B}$ and all of the conditions of Theorem 3.6 are satisfied.

On the other hand, $f(0, 0, \ldots, 0) = 4/3 \neq 0$, by Theorems 3.6 and 3.7, the BVP (19) has a unique positive solution. □

Remark 3.9. In Example 3.8, $\beta(x) = \phi(x)/x = K = 15$ which does not possess property $\beta : [0, +\infty) \to [0, 1)$ and

$$\beta(t_n) \longrightarrow 1 \quad \text{implies } t_n \longrightarrow 0. \tag{3.36}$$

Thus, the unique positive solution of BVP (3.30) cannot be obtained via Theorem 1.2, but we obtain the unique positive solution of BVP (3.30) by using the generalized fixed point Theorem 2.2, which implies that Theorem 2.2 is an essential promotion of Theorem 1.2.

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