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Research Article

Orthogonal Multiwavelet Frames in $L^2(R^d)$

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We characterize the orthogonal frames and orthogonal multiwavelet frames in $L^2(R^d)$ with matrix dilations of the form $(Df)(x) = \sqrt{|\det A|}f(Ax)$, where A is an arbitrary expanding $d \times d$ matrix with integer coefficients. Firstly, through two arbitrarily multiwavelet frames, we give a simple construction of a pair of orthogonal multiwavelet frames. Then, by using the unitary extension principle, we present an algorithm for the construction of arbitrarily many orthogonal multiwavelet tight frames. Finally, we give a general construction algorithm for orthogonal multiwavelet tight frames from a scaling function.

1. Introduction

Wavelets are mathematical functions that take account into the resolutions and the frequencies simultaneously [1–4]. Moreover, wavelets could cut up data into different frequency components such that people can study each component with a resolution matched to its scale.

The classical MRA scaler wavelets are probably the most important class of orthonormal wavelets. However, the scalar wavelets cannot have the orthogonality, compact support, and symmetry at the same time (except the Haar wavelet). It is a disadvantage for signal processing. Multiwavelets have attracted much attention in the research community, since multiwavelets have more desired properties than any scalar wavelet function, such as orthogonality, short compact support, symmetry, and high approximation order [5–7]. It is natural, therefore, to develop the multiwavelets theory that can produce systems having these properties.

Although many compression applications of wavelets use wavelet or multiwavelet bases, the redundant representation offered by wavelet frames has already been put to good use for signal denoising and image compression. In fact, the concept of frame was introduced a long time ago [8] and has received much attention recently due to the development and study of wavelet theory [9, 10]. In particular, inspired by these and other applications, many people are interested in some types of frames, such as tight wavelet frames, dual wavelet frames, and orthogonal frames [11–19].

In [16], Weber proposed orthogonal wavelet frames, which are useful in multiple access communication systems and superframes. Later in [17], authors discussed a pair of orthogonal frames to be orthogonal in a shift-invariant space. In [18], authors presented sufficient conditions for the construction of orthogonal MRA wavelet frames in $L^2(R)$. This led them to a vector-valued discrete wavelet transform. But all these results just base on 2 dilation wavelet transform. In this paper, we present the construction of orthogonal multiwavelet frames in $L^2(R^d)$ with matrix dilation, where the basic ingredients consists of two fixed multiwavelet basis and a paraunitary matrix of an appropriate size. Furthermore, by using the unitary extension principle, we present an algorithm for the construction of orthogonal multiwavelet tight frames from two suitable functions and give a general construction algorithm for orthogonal multiwavelet tight frames from a scaling function. These constructions lead to filter banks in $l^2(Z^d)$ with similar orthogonality relations.

Let us now describe the organization of the material that follows. Section 2 contains some definitions in this paper. Also, we review some relative notations. In Section 3, we describe the construction of orthogonal multiwavelet frames and present different algorithms for the construction of orthogonal multiwavelet tight frames in $L^2(R^d)$ with matrix dilation.

2. Preliminaries

Let us now establish some basic notations.

We denote by T^d the d-dimensional torus. By $L^2(T^d)$, we denote the space of all Z^d-periodic functions f (i.e., f is 1-periodic in each variable) such that $\int_{T^d} |f(x)|^2 dx < +\infty$. The subsets of R^d invariant under Z^d translations and the subsets of T^d are often identified.

We use the Fourier transform in the form

$$\widehat{f}(\omega) = \int_{R^d} f(x) e^{-2\pi i\langle x,\omega\rangle} dx, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in R^d. The Fourier inverse transform is defined by

$$f(x) = \check{\widehat{f}}(\omega) = \int_{R^d} \widehat{f}(\omega) e^{2\pi i\langle x,\omega\rangle} d\omega. \tag{2.2}$$

Let E_d denote the set of all expanding $d \times d$ matrices A with integer coefficients. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For $A \in E_d$, we denote by B the transpose of A. It is obvious that $B \in E_d$.

A collection of elements $\{\phi_j : j \in J\}$ in a Hilbert space H is called a frame if there exist constants a and b, $0 < a \le b < \infty$, such that

$$a\|f\|^2 \le \sum_{j\in J} |\langle f, \phi_j\rangle|^2 \le b\|f\|^2, \quad \text{for all } f \in H. \tag{2.3}$$

If $\{\phi_j : j \in J\}$ satisfies the second inequality, then $\{\phi_j : j \in J\}$ is called a Bessel sequence. Let a_0 the supremum of all such numbers a and b_0 the infimum of all such numbers b, then a_0 and b_0 are called the frame bounds of the frame $\{\phi_j : j \in J\}$. When $a_0 = b_0$, we say that the frame is tight. When $a_0 = b_0 = 1$, we say the frame is a Parseval frame.

In this paper, we will work with two families of unitary operators on $L^2(R^d)$. The first one consists of all translation operators $T_k : L^2(R^d) \to L^2(R^d)$, $k \in Z^d$, defined by $(T_k f)(x) = f(x-k)$. The second one consists of all integer powers of the dilation operator $D_A : L^2(R^d) \to L^2(R^d)$ defined by $(Df)(x) = \sqrt{|A|} f(Ax)$ with $A \in E_d$.

Let us now fix an arbitrary matrix $A \in E_d$. For $\Psi = \{\psi_1, \ldots, \psi_r\} \subset L^2(R^d)$, we will consider the affine system $X(\Psi)$ defined by

$$X(\Psi) = \left\{\psi^l_{j,\,k}(x) \mid \psi^l_{j,k}(x) = |\det A|^{j/2} \psi^l\left(A^j x - k\right) : j \in Z;\ k \in Z^d;\ l = 1, \ldots, r\right\}. \tag{2.4}$$

Then, we define the multiwavelet frame, the multiwavelet tight frame, the multiwavelet tight frame, and the filter.

Definition 2.1. We say that $X(\Psi) \subset L^2(R^d)$ is a multiwavelet frame if the system (2.4) is a frame for $L^2(R^d)$.

Definition 2.2. We say that $X(\Psi) \subset L^2(R^d)$ is a multiwavelet tight frame if the system (2.4) is a tight frame for $L^2(R^d)$.

Definition 2.3. We say that $X(\Psi) \subset L^2(R^d)$ is a multiwavelet tight frame if the system (2.4) is a Parseval frame for $L^2(R^d)$.

We turn to the concept of multiresolution analysis (MRA) in $L^2(R^d)$ which is a useful tool in our study.

Definition 2.4. Let $\{V_m\}_{m \in Z}$ be a sequence of closed subspaces of $L^2(R^d)$ satisfying:

(1) $V_j \subset V_{j+1}$,

(2) $\overline{\bigcup_{j \in Z} V_j} = L^2(R^d)$,

(3) $\bigcap_{j \in Z} V_j = \{0\}$,

(4) $f(x) \in V_j \Leftrightarrow f(Ax) \in V_{j+1},\ j \in Z$, where $A \in E_d$,

(5) There exists a function $\phi(x) \in V_0$ such that $\{\phi(x-k)\}_{k \in Z^d}$ is a frame of V_0.

Then, $\{V_j\}_{j \in Z}$ is called an MRA and the function ϕ in (5) a scaling function.

There is a standard procedure for constructing multiwavelets from a given MRA(V_j). Firstly, one defines $W_j = V_{j+1} \ominus V_j$ for all $j \in Z$. As an easy consequence of conditions (1)–(4) from Definition 2.4, one obtains $L^2(R^d) = \overline{\oplus_{j \in Z} W_j}$ and $W_{j+1} = DW_j$, for all $j \in Z$. Suppose now that there exist functions $\Psi \subset W_0$ such that the system $E(\Psi) := \{\psi(\cdot - k) : k \in Z^d, \psi \in \Psi\}$ is a frame for W_0. Then, $\{D^j T_k \psi : k \in Z^d, \psi \in \Psi\}$ is a frame for W_j, for all $j \in Z$, and, consequently, $\{D^j T_k \psi : j \in Z, k \in Z^d, \psi \in \Psi\}$ is a frame for $L^2(R^d)$.

In the following, we will borrow some notations from [17, 18] which will be used in this paper.

Let X be a (countable) Bessel system for a separable Hilbert space H over the complex field C. The synthesis operator $T_X : l^2(\mathrm{X}) \to H$, which is well known to be bounded, is defined by $T_X a := \sum_{h\in X} a_h h$ for $a = \{a_h\}_{h\in X}$. The adjoint operator T_X^* of T_X, called the analysis operator, is $T_X^* : H \to l^2(X)$; $T_X^* f := \{\langle f, h\rangle\}_{h\in X}$. Recall that X is a frame for H if and only $S_X := T_X T_X^* : H \to H$, the frame operator or dual Gramian, is bounded and has a bounded inverse [20, 21], and it is a tight frame (with frame bound 1) if and only if S_X is the identity operator. The system X is a Riesz system (for $\overline{\operatorname{span}} X$) if and only its Gramian $G_X := T_X^* T_X$ is bounded and has a bounded inverse; it is an orthonormal system of H if and only if G_X is the identity operator.

Definition 2.5. Let X and $Y = RX$, where $R : h \to Rh$ is a bijection between X and Y, be two frame for H. We call X and Y a dual frames for H if $T_Y T_X^* = I$, that is, $\sum_{h\in X}\langle f, h\rangle Rh = f$ for all $f \in H$.

Definition 2.6. Let X and $Y = RX$, where $R : h \to Rh$ is a bijection between X and Y, be two frames for H. We call X and Y a pair of orthogonal frames for H if $T_Y T_X^* = 0$, that is, $\sum_{h\in X}\langle f, h\rangle Rh = 0$ for all $f \in H$.

Definition 2.7. A closed subspace $V \subset L^2(R^d)$ is shift invariant if for all $f \in V$ implies $T_k f \in V$ for any $k \in Z^d$.

We consider orthogonal frames in a shift-invariant subspace of $L^2(R^d)$. Let Φ be a countable subset of $L^2(R^d)$, and $E(\Phi) := \{\phi(\cdot - k) : k \in Z^d, \phi \in \Phi\}$. Define $S(\Phi) := \overline{\operatorname{span}}\, E(\Phi)$, the smallest closed subspace that contains $E(\Phi)$. Throughout the rest of this paper, we assume that $E(\Phi)$ is a Bessel sequence for $S(\Phi)$. This assumption settles most of the convergence issues. The space $S(\Phi)$ is called the shift-invariant space generated by Φ and Φ a generating set for $S(\Phi)$. Shift-invariant spaces have been studied extensively in the literature, for example, [22, 23].

For $\omega \in R^d$, we define the pre-Gramian by

$$J_\Phi(\omega) = \left(\widehat{\phi}(\omega + \alpha)\right)_{\alpha\in Z^d, \phi\in\Phi}, \tag{2.5}$$

where $\widehat{\phi}$ is the Fourier transform of ϕ. Note that the domain of the pre-Gramian matrix as an operator is $l^2(\Phi)$ and its codomain is $l^2(Z^d)$. The pre-Gramian can be regarded as the synthesis operator represented in Fourier domain as it was extensively studied in [22].

Let Φ and $\Psi = R\Phi$, where R is a bijection satisfying $R(\phi(\cdot - k)) = (R\phi)(\cdot - k)$, be countable subsets of $L^2(R^d)$. Suppose that $S(\Phi) = S(\Psi)$ and that both $E(\Phi)$ and $E(\Psi)$ are frames for $S(\Phi)$. Then, by definition, $E(\Phi)$ and $E(\Psi)$ are a pair of orthogonal frames for $S(\Phi)$ if and only if for all $f \in S(\Phi)$,

$$Sf := T_{E(\Psi)} T_{E(\Phi)}^* = 0. \tag{2.6}$$

We define the mixed dual Gramian as $\widetilde{G}(\omega) = J_\Psi(\omega) J_\Phi^*(\omega)$ and Gramians as

$$G_\Phi(\omega) = J_\Phi^*(\omega) J_\Phi(\omega), \qquad G_\Psi(\omega) = J_\Psi^*(\omega) J_\Psi(\omega). \tag{2.7}$$

Then, it is proven in [24] that, for any $f \in L^2(R^d)$,

$$\left(\widehat{Sf}\right)_{|\omega+\alpha} = \widetilde{G}(\omega)\widehat{f}_{|\omega+\alpha}, \tag{2.8}$$

where $\widehat{f}_{|\omega+\alpha}$ is the column vector $(\widehat{f}(\omega+\alpha))^T_{\alpha\in Z^d}$. By (2.8), one can prove easily that $Sf = 0$ for all $f \in L^2(R^d)$ if and only if $\widetilde{G}(\omega) = 0$ for a.e. $\omega \in R^d$.

3. Orthogonal Multiwavelet Frames

In this section, we present a simple construction of a pair of orthogonal multiwavelet frames from two arbitrarily multiwavelet frames and get some interesting properties about the orthogonal multiwavelet frames. We also show different algorithms for the construction of arbitrarily many orthogonal multiwavelet tight frames.

Firstly, we give a lemma, which has been obtained by Weber in [16].

Lemma 3.1. *Let $\Psi_1 := \{\psi_1^1, \psi_2^1, \ldots, \psi_r^1\}$ and $\Psi_2 := \{\psi_1^2, \psi_2^2, \ldots, \psi_r^2\}$. Suppose that $X(\Psi_1)$ and $X(\Psi_2)$ are multiwavelet frames for $L^2(R^d)$. $X(\Psi_1)$ and $X(\Psi_2)$ are a pair of orthogonal frames for $L^2(R^d)$ if and only if the following two equations are satisfied a.e.:*

$$\begin{gathered}\sum_{i=1}^{r}\sum_{j\in Z}\widehat{\psi_i^1}\left(B^j\omega\right)\overline{\widehat{\psi_i^2}(B^j\omega)} = 0, \quad a.e., \\ \sum_{i=1}^{r}\sum_{j=0}^{+\infty}\widehat{\psi_i^1}\left(B^j\omega\right)\overline{\widehat{\psi_i^2}(B^j(\omega+q))} = 0, \quad a.e., \ \forall k \in Z^d, \ q \in Z^d \setminus BZ^d.\end{gathered} \tag{3.1}$$

From Lemma 3.1, by Theorem 2.3 [17], we can construct a pair of orthogonal multiwavelet frames easily.

Theorem 3.2. *Let $\Psi^1 := \{\psi_1^1, \psi_2^1, \ldots, \psi_r^1\}$ and $\Psi^2 := \{\psi_1^2, \psi_2^2, \ldots, \psi_r^2\} \in L^2(R^d)$ for some positive integer r. Suppose that $X(\Psi^1)$ and $X(\Psi^2)$ are multiwavelet frames for $L^2(R^d)$. Let $V := (V_1; V_2)$ be a $2r \times 2r$ constant unitary matrix, where V_1 is the submatrix of the first r columns and V_2 the remaining r columns. Then, $X(\Psi^{11})$ and $X(\Psi^{22})$ are a pair of orthogonal multiwavelet frames for $L^2(R^d)$, where $\Psi^{11} := V_1\Psi^1$ and $\Psi^{22} := V_2\Psi^2$.*

Proof. Assume that V is a constant matrix such that $\widehat{\Psi}^{11} := V_1\widehat{\Psi}^1$ and $\widehat{\Psi}^{22} := V_2\widehat{\Psi}^2$. Then, one can directly calculate the dual Gramians of $X^q(\Psi^{11})$ and $X^q(\Psi^{22})$. It follows from the fact that the double sums in (3.1) are the entries of the dual Gramian of the affine systems [24].

Let $V = (v_{lm})_{1\le l,m\le 2r}$. For a fixed $q \in Z^d \setminus BZ^d$, $i \in 1, 2$, we have

$$\begin{aligned}\sum_{l=1}^{2r}\sum_{m\ge 0}\overline{\widehat{\psi_l^{ii}}}(B^m\omega)\widehat{\psi_l^{ii}}(B^m(\omega+q)) &= \sum_{l=1}^{2r}\sum_{m\ge 0}\sum_{n=1}^{r}\overline{v_{l,n}}\overline{\widehat{\psi_n^i}}(B^m\omega)\sum_{n'=1}^{r}v_{l,n'}\widehat{\psi_{n'}^i}(B^m(\omega+q)) \\ &= \sum_{m\ge 0}\sum_{n=1}^{r}\overline{\widehat{\psi_n^i}}(B^m\omega)\sum_{n'=1}^{r}\widehat{\psi_{n'}^i}(B^m(\omega+q))\sum_{l=1}^{2r}\overline{v_{l,n}}v_{l,n'} \\ &= \sum_{m\ge 0}\sum_{n=1}^{r}\overline{\widehat{\psi_n^i}}(B^m\omega)\widehat{\psi_n^i}(B^m(\omega+q)),\end{aligned} \tag{3.2}$$

where we used the fact that the double sums converge absolutely a.e., $V^*V = I_{2r}$, and that $X(\Psi^1)$ and $X(\Psi^2)$ are frames for $L^2(R^d)$. Moreover,

$$\begin{aligned}
\sum_{l=1}^{2r}\sum_{m\in Z}\overline{\widehat{\varphi_l^{ii}}}(B^m\omega)\widehat{\varphi_l^{ii}}(B^m\omega) &= \sum_{l=1}^{2r}\sum_{m\in Z}\sum_{n=1}^{r}\overline{v_{l,n}}\overline{\widehat{\varphi_n^{i}}}(B^m\omega)\sum_{n'=1}^{r}v_{l,n'}\widehat{\varphi_{n'}^{i}}(B^m\omega)\\
&= \sum_{m\in Z}\sum_{n=1}^{r}\overline{\widehat{\varphi_n^{i}}}(B^m\omega)\sum_{n'=1}^{r}\widehat{\varphi_{n'}^{i}}(B^m\omega)\sum_{l=1}^{2r}\overline{v_{l,n}}v_{l,n'}\\
&= \sum_{m\in Z}\sum_{n=1}^{r}\overline{\widehat{\varphi_n^{i}}}(B^m\omega)\widehat{\varphi_n^{i}}(B^m\omega).
\end{aligned}\tag{3.3}$$

From the above results, by using the dual Gramian characterization of frames in [25, Corollary 5.7], then $X(\Psi^{11})$ and $X(\Psi^{22})$ are frames for $L^2(R^d)$.

We now show that the multiwavelet systems generated by Ψ^{11} and Ψ^{22} are a pair of orthogonal frames for $L^2(R^d)$. We apply Lemma 3.1 to $\Psi^{11} := \{\varphi_1^{11}, \varphi_2^{11}, \ldots, \varphi_{2r}^{11}\}$ and $\Psi^{22} := \{\varphi_1^{22}, \varphi_2^{22}, \ldots, \varphi_{2r}^{22}\}$. Let $V = (v_{lm})_{1\le l,m\le 2r}$. For all $q \in Z^d \setminus BZ^d$, we have

$$\begin{aligned}
\sum_{l=1}^{2r}\sum_{m\ge 0}\overline{\widehat{\varphi_l^{11}}}(B^m\omega)\widehat{\varphi_l^{22}}(B^m(\omega+q)) &= \sum_{l=1}^{2r}\sum_{m\ge 0}\sum_{n=1}^{r}\overline{v_{l,n}}\overline{\widehat{\varphi_n^{1}}}(B^m\omega)\sum_{n'=1}^{r}v_{l,r+n'}\widehat{\varphi_{n'}^{2}}(B^m(\omega+q))\\
&= \sum_{m\ge 0}\sum_{n=1}^{r}\overline{\widehat{\varphi_n^{1}}}(B^m\omega)\sum_{n'=1}^{r}\widehat{\varphi_{n'}^{2}}(B^m(\omega+q))\sum_{l=1}^{2r}\overline{v_{l,n}}v_{l,r+n'}\\
&= \sum_{m\ge 0}\sum_{n=1}^{r}\overline{\widehat{\varphi_n^{1}}}(B^m\omega)\sum_{n'=1}^{r}\widehat{\varphi_{n'}^{2}}(B^m(\omega+q))\times 0\\
&= 0,
\end{aligned}\tag{3.4}$$

where we used the orthogonality of the columns of V.

Moreover,

$$\begin{aligned}
\sum_{l=1}^{2r}\sum_{m\in Z}\overline{\widehat{\varphi_l^{11}}}(B^m\omega)\widehat{\varphi_l^{22}}(B^m(\omega)) &= \sum_{l=1}^{2r}\sum_{m\in Z}\sum_{n=1}^{r}\overline{v_{l,n}}\overline{\widehat{\varphi_n^{1}}}(B^m\omega)\sum_{n'=1}^{r}v_{l,r+n'}\widehat{\varphi_{n'}^{2}}(B^m(\omega))\\
&= \sum_{m\in Z}\sum_{n=1}^{r}\overline{\widehat{\varphi_n^{1}}}(B^m\omega)\sum_{n'=1}^{r}\widehat{\varphi_{n'}^{2}}(B^m(\omega))\sum_{l=1}^{2r}\overline{v_{l,m}}v_{l,r+n'}\\
&= \sum_{m\in Z}\sum_{n=1}^{r}\overline{\widehat{\varphi_n^{1}}}(B^m\omega)\sum_{n'=1}^{r}\widehat{\varphi_{n'}^{2}}(B^m(\omega))\times 0\\
&= 0,
\end{aligned}\tag{3.5}$$

by Lemma 3.1, Ψ^{11} and Ψ^{22} generate a pair of orthogonalframes. □

The following results give some properties of the orthogonal frames.

Proposition 3.3. *Suppose that $E(\psi_i)$ and $E(\psi_j)$ are a pair of orthogonal affine Bessel sequences in $L^2(R^d)$. If $\alpha \in L^2(R^d)$ is a Z^d-periodic function, then $E(\psi_i)$ and $E(\alpha\psi_j)$ are a pair of orthogonal affine Bessel sequences.*

Proof. Suppose that $E(\psi_i)$ and $E(\psi_j)$ are a pair of orthogonal affine Bessel sequences in $L^2(R^d)$. Then, for all $f \in L^2(R^d)$, we have

$$Sf(x) = \sum_{m\in Z^d} \langle f(x), \psi_i(x+m)\rangle \psi_j(x+m) = 0. \tag{3.6}$$

Let $\psi_j' := \alpha\psi_j$. Since α is a Z^d-periodic function, then $E(\psi_j')$ is an affine Bessel sequence for $L^2(R^d)$ from the fact that, for all $f \in L^2(R^d)$,

$$\begin{aligned}\sum_{k\in Z^d} \left|\langle f(x), \alpha(x-k)\psi_j(x-k)\rangle\right|^2 &= \sum_{k\in Z^d} \left|\left\langle \overline{\alpha(x)} f(x), \psi_j(x-k)\right\rangle\right|^2 \\ &\le B\|\overline{\alpha} f\|^2 \\ &\le B\|\overline{\alpha}\|^2\|f\|^2 \\ &= B'\|f\|^2.\end{aligned} \tag{3.7}$$

Again by α being a Z^d-periodic function, we have the following equation:

$$\begin{aligned}Sf(x) &= \sum_{m\in Z^d} \langle f(x), \psi_i(x+m)\rangle \psi_j(x+m)' \\ &= \sum_{m\in Z^d} \langle f(x), \psi_i(x+m)\rangle \psi_j(x+m)\alpha(x+m) \\ &= \alpha(x)\sum_{m\in Z^d} \langle f(x), \psi_i(x+m)\rangle \psi_j(x+m) \\ &= 0.\end{aligned} \tag{3.8}$$

Hence, $E(\psi_i)$ and $E(\alpha\psi_j)$ are a pair of orthogonal affine Bessel sequences in $L^2(R^d)$. □

Proposition 3.4. *Suppose that $E(\psi_i)$ and $E(\psi_j)$ are a pair of orthogonal frames for $H \subset L^2(R^d)$. Let $\alpha \in L^2(R^d)$ be a Z^d-periodic function. If $E(\alpha\psi_j)$ is a frame for H, then $E(\psi_i)$ and $E(\alpha\psi_j)$ are a pair of orthogonal frames for H.*

Proof. Similar to the proof in Proposition 3.3, we have the desired result. □

Then, we recall a result from [26] that characterizes unitary extension principle (UEP) associated with more general matrix dilations in $L^2(R^d)$.

Lemma 3.5. *Suppose $\Phi = (\phi_j)_{j\in J}$ is a refinable vector with a mask Γ such that*

$$\begin{gathered}\sum_{j\in J}\|\varphi_j\|^2 = \int_{R^d}\left\|\widehat{\Phi}(\xi)\right\|^2_{l^2(J)} d\xi < \infty,\\ \lim_{j\to\infty}\left\|\widehat{\Phi}\left(B^{-j}\xi\right)\right\| = 1, \quad \text{for a.e. } \xi \in R^d.\end{gathered} \tag{3.9}$$

Suppose also that $\Psi = (\varphi^j)_{j\in\widetilde{J}}$, where $\widetilde{J} = \{1,\ldots,N\}$ is finite, is given by

$$\widehat{\Psi}(B\xi) = H(\xi)\widehat{\Phi}(\xi), \tag{3.10}$$

where $H = (h_{i,j})_{i\in\widetilde{J},\, i\in J}$ is a Z^d-periodic, measurable matrix function satisfying

$$\Gamma^*(\xi)\Gamma(\xi+d) + H^*(\xi)H(\xi+d) = \Omega(\xi)\delta_{0,d}, \quad \text{for a.e. } \xi, \tag{3.11}$$

and for any $d \in \Upsilon$, where Υ consists of representatives of distinct cosets of $B^{-1}Z^d/Z^d$, then $\Psi \subset L^2(R^d)$ is a multiwavelet tight frame.

We call m a filter if $m \in L^\infty([0,1)^d)$. We shall call m a low-pass filter if $m(0) = 1$, and we shall call m a high-pass filter if $m(0) = 0$. Though not necessary, we will assume that every filter is continuous on a neighborhood of 0, so there will be no ambiguity in these definitions. Given a collection of filter $M' = \{m_0, m_1, \ldots, m_r\} \subset L^\infty([0,1)^d)$, let $M'(\xi)$ and $\widetilde{M'}(\xi)$ be the matrices

$$M'(\xi) = \begin{pmatrix} m_0(\xi) & m_0(\xi+\beta)\\ m_1(\xi) & m_1(\xi+\beta)\\ \vdots & \vdots\\ m_r(\xi) & m_r(\xi+\beta)\end{pmatrix}, \qquad \widetilde{M'}(\xi) = \begin{pmatrix} m_1(\xi) & m_1(\xi+\beta)\\ m_2(\xi) & m_2(\xi+\beta)\\ \vdots & \vdots\\ m_r(\xi) & m_r(\xi+\beta)\end{pmatrix}, \tag{3.12}$$

where $\beta \in \Upsilon$. In the remainder of the paper, the filter banks will be composed of a single low-pass filter (with index 0) and a number of high-pass filters.

With the above definitions, we present an algorithm for the construction of arbitrarily many orthogonal multiwavelet tight frames.

Theorem 3.6. *Suppose that $\phi_1, \phi_2 \in L^2(R^d)$ are refinable functions which satisfy the conditions of the unitary extension principe, and let $m^1(\xi), m^2(\xi)$ be the associated low-pass filter. Let $M = \{m_0(\xi), m_1(\xi), \ldots, m_r(\xi)\}$ and $N = \{n_0(\xi), n_1(\xi), \ldots, n_r(\xi)\}$ be filter banks with $m_0(\xi) = m^1(\xi)$, $n_0(\xi) = m^2(\xi)$. For all $\beta \in \Upsilon$, suppose that the following matrix equations hold:*

(a) $M^*(\xi)M(\xi) = I_2$ *for almost every* ξ,

(b) $N^*(\xi)N(\xi) = I_2$ *for almost every* ξ,

(c) $\widetilde{M}^*(\xi)\widetilde{N}(\xi) = 0$ *for almost every* ξ.

Let $\widehat{\eta}_k(B\xi) = n_k(\xi)\widehat{\phi}_2(\xi)$ *and* $\widehat{\varphi}_k(B\xi) = m_k(\xi)\widehat{\phi}_1(\xi)$, $1 \le k \le r$. *Then,* $\{\psi_1, \dots, \psi_r\}$ *and* $\{\eta_1, \dots, \eta_r\}$ *generate orthogonal multiwavelet tight frames.*

Proof. For Items (a) and (b), by Lemma 3.5, then $\{\psi_1, \dots, \psi_r\}$ and $\{\eta_1, \dots, \eta_r\}$ generate multiwavelet tight frames. We use the characterization equations of Lemma 3.1 to prove orthogonality.

Let us focus on $\sum_{k=1}^{r}\sum_{j\in Z}\widehat{\varphi}_k(B^j\xi)\overline{\widehat{\eta}_k(B^j\xi)}$. For each k, by Hölder's inequality and virtue of the fact that φ_k and η_k generate Bessel sequences [4, Theorem 8.3.2], we have

$$\sum_{j\in Z}\left|\widehat{\varphi}_k\left(B^j\xi\right)\overline{\widehat{\eta}_k(B^j\xi)}\right| \le \sum_{j\in Z}\left|\widehat{\varphi}_k\left(B^j\xi\right)\right|^2\sum_{j\in Z}\left|\widehat{\eta}_k\left(B^j\xi\right)\right|^2 < \infty, \tag{3.13}$$

then the order of summation can be reversed. With this, by Item (c),

$$\begin{aligned}
\sum_{k=1}^{r}\sum_{j\in Z}\widehat{\varphi}_k\left(B^j\xi\right)\overline{\widehat{\eta}_k(B^j\xi)} &= \sum_{k=1}^{r}\sum_{j\in Z}m_k\left(B^j\xi\right)\widehat{\phi}_1\left(B^j\xi\right)\overline{n_k(B^j\xi)\widehat{\phi}_2(B^j\xi)}\\
&= \sum_{j\in Z}\widehat{\phi}_1\left(B^j\xi\right)\overline{\widehat{\phi}_2(B^j\xi)}\sum_{k=1}^{r}m_k\left(B^j\xi\right)\overline{n_k(B^j\xi)}\\
&= 0
\end{aligned} \tag{3.14}$$

holds for almost every ξ.

Likewise, for $q \in Z^d \setminus BZ^d$, by item (c),

$$\begin{aligned}
\sum_{k=1}^{r}\sum_{j=0}^{\infty}\widehat{\varphi}_k\left(B^j\xi\right)\overline{\widehat{\eta}_k(B^j(\xi+q))} &= \sum_{k=1}^{r}\sum_{j=0}^{\infty}m_k\left(B^{j-1}\xi\right)\widehat{\phi}_1\left(B^{j-1}\xi\right)\overline{n_k(B^{j-1}(\xi+q))\widehat{\phi}_2(B^{j-1}(\xi+q))}\\
&= \sum_{j=0}^{\infty}\widehat{\phi}_1\left(B^j\omega\right)\overline{\widehat{\phi}_2(B^j(\omega+B^{-1}q))}\sum_{k=1}^{r}m_k\left(B^j\omega\right)\overline{n_k(B^j\omega+B^{j-1}q))}\\
&= 0,
\end{aligned} \tag{3.15}$$

where $\omega = B^{-1}\xi$. □

The following results show the relationship between a pair of orthogonal MRA multiwavelet frames.

Theorem 3.7. *Suppose that* $X(\Psi_i)$ *and* $X(\Psi_j)$ *are a pair of orthogonal MRA multiwavelet frames, where* $\Psi_i := \{\psi_1^i, \psi_2^i, \dots, \psi_r^i\}$, $\Psi_j := \{\psi_1^j, \psi_2^j, \dots, \psi_r^j\}$. *If* $S(\Psi_i) = S(\Psi_j)$ *and there exist functions* $p, w \in L^2(R^d)$ *such that* $\Psi_i^p := \{\psi_1^{ip}, \psi_2^{ip}, \dots, \psi_r^{ip}\}$ *and* $\Psi_j^p := \{\psi_1^{jw}, \psi_2^{jw}, \dots, \psi_r^{jw}\}$ *are multiwavelet frames, where* ψ_l^p *and* ψ_l^w *defined by* $\widehat{\psi_l^{ip}(\omega)} = \widehat{\psi}_l^i(\omega)\widehat{p}(\omega)$, $\widehat{\psi_l^{jw}(\omega)} = \widehat{\psi}_l^j(\omega)\widehat{w}(\omega)$, $1 \le l \le r$ *respectively, then* $X(\Psi_i^p)$ *and* $X(\Psi_j^w)$ *are a pair of orthogonal multiwavelet frames for* $L^2(R^d)$.

Proof. Suppose that $X(\Psi_i)$, $X(\Psi_j)$ are a pair of orthogonal MRA multiwavelet frames and $S(\Psi_i) = S(\Psi_j)$, then, by the property of MRA multiwavelet frames, for any $n \neq m \in Z$, we have $S(A^m\Psi_i) \perp S(A^n\Psi_i)$. Hence, for all $f_1 \in S(\Psi_i)$

$$\begin{aligned} 0 &= Sf_1(x) \\ &= \sum_{l=1}^{r}\sum_{k\in Z^d}\sum_{s\in Z}\left\langle f_1(x), \varphi_l^i(A^s x - k)\right\rangle \varphi_l^j(A^s x - k) \\ &= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_1(x), \varphi_l^i(x - k)\right\rangle \varphi_l^j(x - k). \end{aligned} \tag{3.16}$$

For any $f \in L^2(R^d)$, define $f = f_1 + f_2$, where $f_1 \in S(\Psi_i)$, $f_2 \in (L^2(R^d) \setminus S(\Psi_i))$, then, $\langle f_1, f_2\rangle = 0$. With this, we get

$$Sf_2(x) = \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_2(x), \varphi_l^i(x - k)\right\rangle \varphi_l^j(x - k) = 0. \tag{3.17}$$

Hence, for all $f \in L^2(R^d)$, the following equation holds:

$$\begin{aligned} Sf(x) &= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \varphi_l^i(x - k)\right\rangle \varphi_l^j(x - k) \\ &= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_1(x), \varphi_l^i(x - k)\right\rangle \varphi_l^j(x - k) + \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_2(x), \varphi_l^i(x - k)\right\rangle \varphi_l^j(x - k) \\ &= 0. \end{aligned} \tag{3.18}$$

Notice that $\Psi_j := \{\varphi_1^j, \ldots, \varphi_r^j\}$, since $\widehat{\varphi_l^{jw}(\xi)} = \widehat{\varphi}_j(\xi)\widehat{w}(\xi), 1 \leq l \leq r$, by $Sf(x) = 0$, then

$$\begin{aligned} 0 &= \widehat{Sf(x)} \\ &= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \varphi_l^i(x - k)\right\rangle \widehat{\varphi_l^j}(\omega)e^{-2\pi ik\omega} \\ &= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_1(x), \varphi_l^i(x - k)\right\rangle \widehat{\varphi_l^j}(\omega)e^{-2\pi ik\omega} + \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_2(x), \varphi_l^i(x - k)\right\rangle \widehat{\varphi_l^j}(\omega)e^{-2\pi ik\omega} \\ &= w(\omega)\left(\sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_1(x), \varphi_l^i(x - k)\right\rangle \widehat{\varphi_l^j}(\omega)e^{-2\pi ik\omega} + \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_2(x), \varphi_l^i(x - k)\right\rangle \widehat{\varphi_l^j}(\omega)e^{-2\pi ik\omega}\right) \end{aligned}$$

$$= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_1(x), \psi_l^i(x-k)\right\rangle \widehat{\psi_l^{jw}}(\omega)e^{-2\pi ik\omega} + \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f_2(x), \psi_l^i(x-k)\right\rangle \widehat{\psi_l^{jw}}(\omega)e^{-2\pi ik\omega}$$

$$= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^i(x-k)\right\rangle \widehat{\psi_l^{jw}}(\omega)e^{-2\pi ik\omega}. \tag{3.19}$$

Applying Fourier inverse transform on (3.19), we have

$$\begin{aligned} 0 &= \check{\widehat{Sf(x)}} \\ &= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^i(x-k)\right\rangle \check{\widehat{\psi_l^{jw}}}(\omega)e^{-2\pi ik\omega} \\ &= \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^i(x-k)\right\rangle \psi_l^{jw}(x-k). \end{aligned} \tag{3.20}$$

From the above result, we get the following equation:

$$\begin{aligned} \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^i(x-k)\right\rangle\left\langle f(x), \psi_l^{jw}(x-k)\right\rangle &= \left\langle f(x), \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^i(x-k)\right\rangle \psi_l^{jw}(x-k)\right\rangle \\ &= \langle f(x), 0\rangle \\ &= \left\langle f(x), \sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^{jw}(x-k)\right\rangle \psi_l^{i}(x-k)\right\rangle, \end{aligned} \tag{3.21}$$

hence,

$$\sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^{jw}(x-k)\right\rangle \psi_l^{i}(x-k) = 0. \tag{3.22}$$

Similar to the calculation of (3.19), clearly

$$\sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^{jw}(x-k)\right\rangle \psi_l^{ip}(x-k) = 0. \tag{3.23}$$

For any $s \in Z$,

$$\sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \psi_l^{jw}(A^s x-k)\right\rangle \psi_l^{ip}(A^s x-k) = A^{-s}\sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(A^{-s}x'), \psi_l^{jw}(x'-k)\right\rangle \psi_l^{ip}(x'-k). \tag{3.24}$$

Let $g(x) := f(A^{-s}x)$. Define operator $T : L^2(R^d) \to L^2(R^d)$; $Tf(x) = g(x)$, obviously T is a surjection operator. If s is fixed, for all $g \in L^2(R^d)$, we get

$$\sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle f(x), \varphi_l^{jw}(A^s x - k)\right\rangle \varphi_l^{ip}(A^s x - k) = A^{-s}\left(\sum_{l=1}^{r}\sum_{k\in Z^d}\left\langle g(x'), \varphi_l^{jw}(x' - k)\right\rangle \varphi_l^{ip}(x' - k)\right) = 0. \tag{3.25}$$

Putting everything together, we have

$$\sum_{l=1}^{r}\sum_{s\in Z}\sum_{k\in Z^d}\left\langle f(x), \varphi_l^{jw}(A^s x - k)\right\rangle \varphi_l^{ip}(A^s x - k) = 0, \tag{3.26}$$

then, $X(\Psi_i^p)$ and $X(\Psi_j^w)$ are a pair of orthogonal multiwavelet frames. □

The following theorem describes a general construction algorithm for orthogonal multiwavelet tight frames.

Theorem 3.8. *Suppose $K(\xi)$ is an $r \times r$ paraunitary matrix with $B^{-1}Z^d$-periodic entries $a_{k,s}(\xi)$; let $K_j(\xi)$ denote the jth column. For all $\beta \in \Upsilon$, suppose $M = \{m_0(\xi), m_1(\xi), \ldots, m_r(\xi)\}$ and $M^*(\xi)M(\xi) = I_2$ hold for almost every ξ, where m_0 and $\{m_1, \ldots, m_l\}$ are low- and high-pass filters, respectively, for a multiwavelet tight frame with scaling function ϕ. For $j = 1, \ldots, r$, define new filters via*

$$\begin{pmatrix} n_{1,1}^j(\xi) \\ \vdots \\ n_{1,r}^j(\xi) \\ \vdots \\ n_{l,1}^j(\xi) \\ \vdots \\ n_{l,r}^j(\xi) \end{pmatrix} = \begin{pmatrix} K_j(\xi)m_1(\xi) \\ \vdots \\ K_j(\xi)m_l(\xi) \end{pmatrix}. \tag{3.27}$$

Then, for $j = 1, \ldots, r$, the affine systems generated by $\Psi^j = \{\varphi_{i,t}^j : i = 1, \ldots, l, t = 1, \ldots, r\}$ obtained via

$$\widehat{\varphi}_{i,t}^j(B\xi) = n_{i,t}^j(\xi)\widehat{\phi}(\xi) \tag{3.28}$$

are multiwavelet tight frames and are pairwise orthogonal.

Proof. Firstly, we prove that $X(\Psi^j), 1 \leq j \leq r$, are multiwavelet tight frames. Assume $M_j = \{m_0(\xi), n_{1,1}^j(\xi), \ldots, n_{1,r}^j(\xi), \ldots, n_{l,1}^j(\xi), \ldots, n_{l,r}^j(\xi)\}$. Define $M_j(\xi)$ according to (3.12):

$$M_j(\xi) = \begin{pmatrix} m_0(\xi) & m_0(\xi+\beta) \\ n_{1,1}^j(\xi) & n_{1,1}^j(\xi+\beta) \\ \vdots & \vdots \\ n_{1,r}^j(\xi) & n_{1,r}^j(\xi+\beta) \\ \vdots & \vdots \\ n_{l,1}^j(\xi) & n_{l,1}^j(\xi+\beta) \\ \vdots & \vdots \\ n_{l,r}^j(\xi) & n_{l,r}^j(\xi+\beta) \end{pmatrix}, \tag{3.29}$$

where $\beta \in \Upsilon$. Then, $M_j^*(\xi)M_j(\xi)$ is a 2×2 matrix. Next, we examine the entries of $M_j^*(\xi)M_j(\xi)$ individually. Note that the columns of $K(\xi)$ have length 1, by $M^*(\xi)M(\xi) = I_2$, it follows that

$$\begin{aligned}
\left[M_j^*(\xi)M_j(\xi)\right]_{1,1} &= |m_0(\xi)|^2 + \sum_{k=1}^{r}\sum_{t=1}^{l}\left|a_{k,j}(\xi)m_t(\xi)\right|^2 \\
&= |m_0(\xi)|^2 + \sum_{k=1}^{r}\left|a_{k,j}(\xi)\right|^2\sum_{t=1}^{l}|m_t(\xi)|^2 \\
&= |m_0(\xi)|^2 + \sum_{t=1}^{l}|m_t(\xi)|^2 \\
&= 1,
\end{aligned} \tag{3.30}$$

where $[M_j^*(\xi)M_j(\xi)]_{1,1}$ means the (1,1) entry of the matrix $M_j^*(\xi)M_j(\xi)$.

Similarly,

$$\begin{aligned}
\left[M_j^*(\xi)M_j(\xi)\right]_{2,2} &= \left|m_0(\xi+\beta)\right|^2 + \sum_{k=1}^{r}\sum_{t=1}^{l}\left|a_{k,j}(\xi+\beta)m_t(\xi+\beta)\right|^2 \\
&= \left|m_0(\xi+\beta)\right|^2 + \sum_{k=1}^{r}\left|a_{k,j}(\xi+\beta)\right|^2\sum_{t=1}^{l}\left|m_t(\xi+\beta)\right|^2 \\
&= \left|m_0(\xi+\beta)\right|^2 + \sum_{t=1}^{l}\left|m_t(\xi+\beta)\right|^2 \\
&= 1.
\end{aligned} \tag{3.31}$$

Now, since the entries of $K(\xi)$ are $B^{-1}Z^d$-periodic, again by $M^*(\xi)M(\xi) = I_2$,

$$\begin{aligned}
\left[M_j^*(\xi)M_j(\xi)\right]_{1,2} &= m_0(\xi+\beta)\overline{m_0(\xi)} + \sum_{k=1}^{r}\sum_{t=1}^{l}\overline{a_{k,j}(\xi)m_t(\xi)}a_{k,j}(\xi+\beta)m_t(\xi+\beta) \\
&= m_0(\xi+\beta)\overline{m_0(\xi)} + \sum_{k=1}^{r}\left|a_{k,j}(\xi)\right|^2\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta) \\
&= m_0(\xi+\beta)\overline{m_0(\xi)} + \sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta) \\
&= 0.
\end{aligned} \tag{3.32}$$

Finally, the (2,1)-entry must be zero by conjugate symmetry of $M_j^*(\xi)M_j(\xi)$. Hence,

$$M_j^*(\xi)M_j(\xi) = I_2, \quad 1 \le j \le r. \tag{3.33}$$

Putting everything together, from Theorem 3.6, the affine systems generated by $\{\psi_{i,t}^j : i = 1,\dots,l, t = 1,\dots,r\}$ obtained via

$$\widehat{\psi}_{i,t}^j(B\xi) = n_{i,t}^j(\xi)\widehat{\phi}(\xi) \tag{3.34}$$

are multiwavelet tight frames.

For orthogonality, according to (3.12), for $j = 1,\dots,r$, we have

$$\widetilde{M}_j(\xi) = \begin{pmatrix} n_{1,1}^j(\xi) & n_{1,1}^j(\xi+\beta) \\ \vdots & \vdots \\ n_{1,r}^j(\xi) & n_{1,r}^j(\xi+\beta) \\ \vdots & \vdots \\ n_{l,1}^j(\xi) & n_{l,1}^j(\xi+\beta) \\ \vdots & \vdots \\ n_{l,r}^j(\xi) & n_{l,r}^j(\xi+\beta) \end{pmatrix} = \begin{pmatrix} K_j(\xi)m_1(\xi) & K_j(\xi+\beta)m_1(\xi+\beta) \\ \vdots & \vdots \\ K_j(\xi)m_l(\xi) & K_j(\xi+\beta)m_l(\xi+\beta) \end{pmatrix}. \tag{3.35}$$

If $1 \le j \ne j' \le r$, then

$$\begin{aligned}
&\widetilde{M}_j^*(\xi)\widetilde{M}_{j'}(\xi) \\
&\quad= \begin{pmatrix} K_j(\xi)m_1(\xi) & K_j(\xi+\beta)m_1(\xi+\beta) \\ \vdots & \vdots \\ K_j(\xi)m_l(\xi) & K_j(\xi+\beta)m_l(\xi+\beta) \end{pmatrix}^* \begin{pmatrix} K_{j'}(\xi)m_1(\xi) & K_{j'}(\xi+\beta)m_1(\xi+\beta) \\ \vdots & \vdots \\ K_{j'}(\xi)m_l(\xi) & K_{j'}(\xi+\beta)m_l(\xi+\beta) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} K_j^*(\xi)K_{j'}(\xi)\sum_{t=1}^{l}|m_t(\xi)|^2 & K_j^*(\xi)K_{j'}(\xi+\beta)\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta) \\ K_j^*(\xi+\beta)K_{j'}(\xi)\sum_{t=1}^{l}\overline{m_t(\xi+\beta)}m_t(\xi) & K_j^*(\xi+\beta)K_{j'}(\xi+\beta)\sum_{t=1}^{l}\overline{m_t(\xi+\beta)}m_t(\xi+\beta) \end{pmatrix} \\
&= \begin{pmatrix} K_j^*(\xi)K_{j'}(\xi)\sum_{t=1}^{l}|m_t(\xi)|^2 & K_j^*(\xi)K_{j'}(\xi)\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta) \\ K_j^*(\xi)K_{j'}(\xi)\sum_{t=1}^{l}\overline{m_t(\xi+\beta)}m_t(\xi) & K_j^*(\xi)K_{j'}(\xi)\sum_{t=1}^{l}\overline{m_t(\xi+\beta)}m_t(\xi+\beta) \end{pmatrix} \\
&= 0,
\end{aligned}
\tag{3.36}
$$

where we use the fact that the product of the two matrices $K_j^*(\xi)K_{j'}(\xi)$ is 0 by the orthogonality of the columns of $K(\xi)$. By Theorem 3.6, we have the desired result. □

The following proposition is directly related to the construction algorithm in Theorem 3.8.

Proposition 3.9. *If ϕ is compactly supported, the paraunitary matrix K in Theorem 3.8 must have entries which are $B^{-1}Z^d$-periodic.*

Proof. The proof will follow the notation of Theorem 3.8. For $1 \le j \le r$, for all $\xi \in B^{-1}Z^d/Z^d$, the matrix

$$
M_j(\xi) = \begin{pmatrix} m_0(\xi) & m_0(\xi+\beta) \\ a_{1,j}(\xi)m_1(\xi) & a_{1,j}(\xi+\beta)m_1(\xi+\beta) \\ \vdots & \vdots \\ a_{r,j}(\xi)m_1(\xi) & a_{r,j}(\xi+\beta)m_1(\xi+\beta) \\ \vdots & \vdots \\ a_{1,j}(\xi)m_l(\xi) & a_{1,j}(\xi+\beta)m_l(\xi+\beta) \\ \vdots & \vdots \\ a_{r,j}(\xi)m_l(\xi) & a_{r,j}(\xi+\beta)m_l(\xi+\beta) \end{pmatrix}
\tag{3.37}
$$

satisfies the equation

$$
M_j^*(\xi)M_j(\xi) = I_2 \quad \text{a.e.}\xi. \tag{3.38}
$$

Then, for almost every ξ, the following equation

$$m_0(\xi+\beta)\overline{m_0(\xi)}+\sum_{k=1}^{r}\overline{a_{k,j}(\xi)}a_{k,j}(\xi+\beta)\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta)=0 \tag{3.39}$$

must hold. Notice that m_0 and $\{m_1,\ldots,m_l\}$ are low- and high-pass filters, respectively, which meet Theorem 3.8. Then,

$$m_0(\xi+\beta)\overline{m_0(\xi)}+\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta)=0. \tag{3.40}$$

Thus, we have

$$m_0(\xi+\beta)\overline{m_0(\xi)}=-\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta). \tag{3.41}$$

From the above results, we get the following equation:

$$\begin{aligned}
0&=m_0(\xi+\beta)\overline{m_0(\xi)}+\sum_{k=1}^{r}\overline{a_{k,j}(\xi)}a_{k,j}(\xi+\beta)\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta)\\
&=-\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta)+\sum_{k=1}^{r}\overline{a_{k,j}(\xi)}a_{k,j}(\xi+\beta)\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta)\\
&=\left(\sum_{k=1}^{r}\overline{a_{k,j}(\xi)}a_{k,j}(\xi+\beta)-1\right)\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta).
\end{aligned} \tag{3.42}$$

Hence, $\sum_{t=1}^{l}\overline{m_t(\xi)}m_t(\xi+\beta)=0$ or $\sum_{k=1}^{r}\overline{a_{k,j}(\xi)}a_{k,j}(\xi+\beta)=1$. If ϕ is compactly supported, then the first possibility is eliminated except possibly on a set of measure 0, whence the second must hold almost everywhere. Now, the sum is precisely the inner product of the two vectors $a_{k,j}(\xi)$ and $a_{k,j}(\xi+\beta)$, each of which has length 1. Applying Cauchy-Schwarz inequation yields that the two vectors must be identical for almost every ξ. □

4. Conclusion

In this paper, motivated by the notion of orthogonal frames, we present the construction of orthogonal multiwavelet frames in $L^2(R^d)$ with matrix dilation, where the basic ingredients consist of two fixed multiwavelet basis and a paraunitary matrix of an appropriate size. The number of orthogonal multiwavelet frames that can be constructed is arbitrary, and is determined by the size of the paraunitary matrix. Moreover, by using the unitary extension principle, we present an algorithm for the construction of orthogonal multiwavelet tight frames and give a general construction algorithm for orthogonal multiwavelet tight frames from a scaling function.

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