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Research Article

Positive Solutions for Fractional Differential Equations from Real Estate Asset Securitization via New Fixed Point Theorem

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We study a fractional differential equation dynamics model arising from the analysis of real estate asset securitization by using the generalized fixed point theorem for weakly contractive mappings in partially ordered sets. Based on the analysis for the existence and uniqueness of the solution and scientific numerical calculation of the solution, in further study, some optimization schemes for traditional risk control process will be obtained, and then the main results of this paper can be applied to the forefront of research of real estate asset securitization.

1. Introduction

Real estate asset securitization is a kind of important financial derivatives in the world. In the international capital market, by combining with different development pattern of the financial industry and real estate industry, real estate asset securitization has become a class of financial products with rapid development and great vitality. Recently, by SWOT analysis method, one has found that many mathematical models arising from real estate asset securitization can be interpreted by fractional-order differential or difference equations, under suitable initial conditions or boundary conditions, the existence and uniqueness of solution of the fractional-order mechanical model are important and useful. Especially, by examining the numerical simulation and analysis of solution, one can undertake macroscopical analysis and comparative research for real estate securitization process advantages and disadvantages, and find real estate asset securitization may exist problems and risks, and then one can put forward to optimize the views on traditional risk control process. In recent years, fractional-order models have been proved to be more accurate than integer order models;

that is, there are more degrees of freedom in the fractional-order models, see [1, 2]. For recent works about fractional-order models, we refer the reader to [3–9] and the references therein.

In this paper, we discuss the existence and uniqueness of positive solutions for the following fractional differential equation with nonlocal Riemann-Stieltjes integral condition arising from the real estate asset securitization

$$-\mathfrak{D}_t{}^{\alpha}x(t) = f\left(t, x(t), -\mathfrak{D}_t{}^{\beta}x(t)\right), \quad t \in (0,1),$$
$$\mathfrak{D}_t{}^{\beta}x(0) = \mathfrak{D}_t{}^{\beta+1}x(0) = 0, \qquad \mathfrak{D}_t{}^{\beta}x(1) = \int_0^1 \mathfrak{D}_t{}^{\beta}x(s)dA(s), \tag{1.1}$$

where $2 < \alpha \leq 3$, $0 < \beta < 1$, and $\alpha - \beta > 2$, $\mathfrak{D}_t$ is the standard Riemann-Liouville derivative.

In (1.1), $\int_0^1 \mathfrak{D}_t{}^{\beta}x(s)dA(s)$ denotes the Riemann-Stieltjes integral, and A is a function of bounded variation, which implies dA can be a signed measure. Thus the BVP (1.1) reveal that positive or negative values of the linear functionals $\int_0^1 x(s)dA(s)$ are viable in some cases. But if $A(s) = s$ or $dA(s) = h(s)ds$, the BVP become a multipoint boundary value problems or the integral boundary value problems, some kind of positivity on the functionals $\int_0^1 x(s)dA(s)$ is often required, for example, in $\int_0^1 x(s)dA(s) = \sum_{i=1}^{m-2} \mu_i u(\eta_i)$, $\mu_i > 0$ is often required. Thus the BVP (1.1) include more generalized boundary value conditions. For a detailed description of multipoint and integral boundary conditions on fractional differential equation, we refer the reader to some recent papers (see [10–15]).

Note that the problem (1.1) has been considered by Wang and Li [16], and the authors obtained the existence of one positive solution for (1.1) by using Krasnoselskii's fixed point theorem on a cone under semipositone case, but the uniqueness of the solution is not treated.

The paper is organized as follows. In Section 2, we extend and improve some fixed point theorems for weakly contractive mappings in partially ordered sets by Harjani and Sadarangani [17], we omit many redundant conditions of Theorems 2 and 3 in [17] and obtain better results than those of [17]. In Section 3, the existence and uniqueness of a positive solution for the problem (1.1) are obtained by using a fixed point theorem in partially ordered sets. An example is also given to illuminate the application of the main result.

2. Preliminaries and Lemmas

Definition 2.1 (see [18, 19]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}x(s)ds \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 (see [18, 19]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathfrak{D}_t{}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}x(s)ds, \tag{2.2}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Remark 2.3. If $x, y : (0, +\infty) \to \mathbb{R}$ with order $\alpha > 0$, then

$$\mathfrak{D}_t{}^{\alpha}\big(x(t) + y(t)\big) = \mathfrak{D}_t{}^{\alpha}x(t) + \mathfrak{D}_t{}^{\alpha}y(t). \tag{2.3}$$

Proposition 2.4 (see [18, 19]). (1) *If $x \in L^1(0,1)$, $\nu > \sigma > 0$, then*

$$I^{\nu}I^{\sigma}x(t) = I^{\nu+\sigma}x(t), \qquad \mathfrak{D}_t{}^{\sigma}I^{\nu}x(t) = I^{\nu-\sigma}x(t), \qquad \mathfrak{D}_t{}^{\sigma}I^{\sigma}x(t) = x(t). \tag{2.4}$$

(2) *If $\nu > 0$, $\sigma > 0$, then*

$$\mathfrak{D}_t{}^{\nu}t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)}t^{\sigma-\nu-1}. \tag{2.5}$$

Proposition 2.5 (see [18, 19]). *Let $\alpha > 0$, and $f(x)$ is integrable, then*

$$I^{\alpha}\mathfrak{D}_t{}^{\alpha}f(x) = f(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + \cdots + c_nx^{\alpha-n}, \tag{2.6}$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$), n is the smallest integer greater than or equal to α.

Let $x(t) = I^{\beta}y(t)$, $y(t) \in C[0,1]$, by standard discuss, we easily reduce the BVP (1.1) to the following modified problems:

$$\begin{gathered} -\mathfrak{D}_t{}^{\alpha-\beta}y(t) = f\Big(t, I^{\beta}y(t), -y(t)\Big), \\ y(0) = y'(0) = 0, \qquad y(1) = \int_0^1 y(s)dA(s), \end{gathered} \tag{2.7}$$

and the BVP (2.7) is equivalent to the BVP (1.1).

Lemma 2.6 (see [3]). *Given $y \in L^1(0,1)$, then the problem*

$$\begin{gathered} \mathfrak{D}_t{}^{\alpha-\beta}x(t) + y(t) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \qquad x(1) = 0, \end{gathered} \tag{2.8}$$

has the unique solution

$$x(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.9}$$

where $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha-\beta)}\begin{cases} [t(1-s)]^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \\ [t(1-s)]^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \le s \le t \le 1, \end{cases} \tag{2.10}$$

which is the Green function of the BVP (2.8).

Lemma 2.7 (see [3]). *For any $t, s \in [0,1]$, $G(t,s)$ satisfies*

$$\frac{t^{\alpha-\beta-1}(1-t)s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \leq G(t,s) \leq \frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta-1)}. \tag{2.11}$$

By Lemma 2.6, the unique solution of the problem

$$\begin{gathered} \mathfrak{D}_{\mathbf{t}}{}^{\alpha} x(t) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \qquad x(1) = 1, \end{gathered} \tag{2.12}$$

is $t^{\alpha-\beta-1}$. Let

$$\mathcal{C} = \int_0^1 t^{\alpha-\beta-1} dA(t), \tag{2.13}$$

and define

$$\mathcal{G}_A(s) = \int_0^1 G(t,s) dA(t), \tag{2.14}$$

as in [4], we can get that the Green function for the nonlocal BVP (2.7) is given by

$$H(t,s) = \frac{t^{\alpha-\beta-1}}{1-\mathcal{C}} \mathcal{G}_A(s) + G(t,s). \tag{2.15}$$

Throughout the paper, we always assume the following holds.

(**H0**) A is a increasing function of bounded variation such that $\mathcal{G}_A(s) \geq 0$ for $s \in [0,1]$ and $0 \leq \mathcal{C} < 1$, where $\mathcal{C}$ is defined by (2.13).

Lemma 2.8. *Let $1 < \alpha - \beta \leq 2$ and (**H0**) hold, then $H(t,s)$ satisfies*

$$0 \leq H(t,s) \leq \frac{1}{(1-\mathcal{C})\Gamma(\alpha-\beta-1)}. \tag{2.16}$$

Proof. By (2.11) and that $A(t)$ is a increasing function of bounded variation, we have

$$\mathcal{G}_A(s) = \int_0^1 G(t,s) dA(t) \leq \int_0^1 \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta-1)} dA(t) = \frac{\mathcal{C}}{\Gamma(\alpha-\beta-1)}. \tag{2.17}$$

Consequently,

$$H(t,s) \leq \frac{\mathcal{G}_A(s)}{1-\mathcal{C}} + \frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta-1)} \leq \frac{1}{(1-\mathcal{C})\Gamma(\alpha-\beta-1)}. \tag{2.18}$$

□

Now, we present a result about the fixed point theorems which is improvement of [17].

Theorem 2.9. *Let* $(X, \geq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Assume that* X *satisfies the following condition: if* x_n *is a nondecreasing sequence in* X *such that* $x_n \to x$ *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$. *Let* $T : X \to X$ *be a nondecreasing mapping, and there exists a constant* $\lambda \in (0,1)$ *such that*

$$d(Tx, Ty) \leq \lambda d(x, y) - \psi(\lambda d(x, y)), \quad \text{for } x \geq y, \tag{2.19}$$

where $\psi : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing function. If there exists* $x \in X$ *with* $x_0 \leq Tx_0$, *then* T *has a fixed point.*

Proof. If $T(x_0) = x_0$, then the proof is finished. Suppose that $x_0 < T(x_0)$. Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 < T(x_0) \leq T^2(x_0) \leq T^3(x_0) \leq \cdots \leq T^n(x_0) \leq \cdots. \tag{2.20}$$

Put $x_{n+1} = T^n(x_0) = Tx_n$. Then for each integer $n \geq 1$, from (2.20), the elements x_n and x_{n+1} are comparable, we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \lambda d(x_n, x_{n-1}) - \psi(\lambda d(x_n, x_{n-1})) \leq \lambda d(x_n, x_{n-1}). \tag{2.21}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0-1}) = 0$ then $x_{n_0} = T^{n_0-1}(x_0) = x_{n_0-1}$ and x_{n_0-1} is a fixed point and the proof is finished. In other case, suppose that $d(x_{n_0}, x_{n_0-1}) \neq 0$ for all $n \in \mathbb{N}$. Then by (2.21), we have

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \cdots \leq \lambda^{n-n_0+1} d(x_{n_0}, x_{n_0-1}) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \tag{2.22}$$

that is,

$$\rho_n = d(x_{n+1}, x_n) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{2.23}$$

Similar to [17], x_n is a Cauchy sequence and then there exists $z \in X$ such that $\lim_{n \to +\infty} x_n = z$ since (X, d) is a complete metric space.

We claim that $T(z) = z$. In fact,

$$\begin{aligned} d(Tz, z) &\leq d(Tz, Tx_n) + d(Tx_n, z) \leq \lambda d(z, x_n) - \psi(\lambda d(z, x_n)) + d(x_{n+1}, z) \\ &\leq \lambda d(z, x_n) + d(x_{n+1}, z) \end{aligned} \tag{2.24}$$

and taking limit as $n \to +\infty$, $d(Tz, z) \leq 0$, this proves that $d(Tz, z) = 0$, consequently, $T(z) = z$. □

Remark 2.10. In Theorem 2.9, We do not require that ψ is continuous and ψ is positive in $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t \to +\infty} \psi(t) = +\infty$. This in essence improve and generalize the corresponding results of Theorem 2 and Theorem 3 in paper [17].

If we consider that $(X, \leq)$ satisfies the following condition:

$$\text{for } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \tag{2.25}$$

then we have the following theorem, see [17].

Theorem 2.11. *Adding condition* (2.25) *to the hypotheses of Theorem 2.9, one obtains uniqueness of the fixed point of T.*

In our considerations, we will work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbb{R} \text{ is continuous}\}$ with the standard norm $\|x\| = \max_{0\leq t\leq 1}|x(t)|$.

Note that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \text{for } t \in [0,1]. \tag{2.26}$$

In [20, 21], it is proved that $(C[0,1], \leq)$ with the classic metric given by

$$d(x,y) = \max_{0\leq t\leq 1}\{|x(t) - y(t)|\} \tag{2.27}$$

satisfies the following condition.

If x_n is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \leq x$ for all $n \in \mathbb{N}$. Moreover, for $x, y \in C[0,1]$, as the function $\max\{x, y\}$ is continuous in [0, 1], and $(C[0,1], \leq)$ satisfies condition (2.25).

3. Main Results

Define the class of function $\mathcal{A}$, if $\phi \in \mathcal{A}$, then $\phi : [0,+\infty) \to [0,+\infty)$ is nondecreasing, and $\psi(x) = x - \phi(x)$ satisfies $\psi : [0,+\infty) \to [0,+\infty)$ is nondecreasing.

The standard functions $\phi \in \mathcal{A}$, for example, $\phi(x) = \arctan x$, $\phi(x) = \ln(1 + x)$ and $\phi(x) = x/(1 + x)$, and so forth.

Theorem 3.1. *Suppose (**H0**) holds, $f : [0,1] \times [0,+\infty) \times (-\infty,0,] \to [0,+\infty)$ is continuous, and for any fixed $t \in [0,1]$, $f(t,x,y)$ is nondecreasing in x on $[0,+\infty)$ and nonincreasing in y on $(-\infty,0]$; moreover, there exist two positive constants ρ_1, ρ_2 that satisfy*

$$\rho_1 + \rho_2 \leq \Gamma(\alpha - \beta - 1)(1 - C), \tag{3.1}$$

and there exist a function $\phi \in \mathcal{A}$ and constants $0 < \theta_1 < 1/(\Gamma(\beta + 1))$, $0 < \theta_2 < 1$ such that

$$f(t, x_1, y_1) - f(t, x_2, y_2) \leq \rho_1\phi(\theta_1(x_1 - x_2)) + \rho_2\phi(\theta_2(y_2 - y_1)) \tag{3.2}$$

for $x_i, y_i \in [0,+\infty)$, $i = 1,2$ with $x_1 \geq x_2$, $y_1 \leq y_2$ and $t \in [0,1]$.

Then problem (1.1) *has a unique nonnegative solution.*

Proof. Consider the cone

$$P = \{y \in C[0,1] : y(t) \geq 0\}. \tag{3.3}$$

Note that, as P is a closed set of $C[0,1]$, P is a complete metric space.

Now, for $y \in P$, we define the operator T by

$$(Ty)(t) = \int_0^1 H(t,s) f\left(s, I^\beta y(s), -y(s)\right) ds. \tag{3.4}$$

Then from the assumption on f and Lemma 2.8, we have

$$T(P) \subset P. \tag{3.5}$$

In what follows, we check that hypotheses in Theorems 2.9 and 2.11 are satisfied.

Firstly, the operator T is nondecreasing since, by hypothesis, for any $u \geq v$

$$(Tu)(t) = \int_0^1 H(t,s) f\left(s, I^\beta u(s), -u(s)\right) ds \geq \int_0^1 H(t,s) f\left(s, I^\beta v(s), -v(s)\right) ds = (Tv)(t). \tag{3.6}$$

Besides, for $u \geq v$

$$\begin{aligned} d(Tu, Tv) &= \max_{t\in[0,1]} |u(t) - v(t)| \\ &= \max_{t\in[0,1]} \left[\int_0^1 H(t,s) \left(f\left(s, I^\beta u(s), -u(s)\right) - f\left(s, I^\beta v(s), -v(s)\right) \right) ds \right] \\ &\leq \max_{t\in[0,1]} \left\{ \int_0^1 H(t,s) \left[\rho_1 \phi\left(\theta_1 \left(I^\beta u(s) - I^\beta v(s)\right)\right) + \rho_2 \phi(\theta_2 (u(s) - v(s))) \right] ds \right\}. \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned} I^\beta u(s) - I^\beta v(s) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (u(s) - v(s)) ds \\ &\leq \frac{d(u,v)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds \leq \frac{d(u,v)}{\Gamma(\beta+1)}, \end{aligned} \tag{3.8}$$

and ϕ is nondecreasing, for $u \geq v$, we have

$$\begin{aligned} &\rho_1 \phi\left(\theta_1 \left(I^\beta u(s) - I^\beta v(s)\right)\right) + \rho_2 \phi(\theta_2 (u(s) - v(s))) \\ &\quad \leq \rho_1 \phi\left(\frac{\theta_1}{\Gamma(\beta+1)} d(u,v)\right) + \rho_2 \phi(\theta_2 d(u,v)) \\ &\quad \leq (\rho_1 + \rho_2) \phi(\theta d(u,v)), \end{aligned} \tag{3.9}$$

where

$$\theta = \max\left\{ \frac{\theta_1}{\Gamma(\beta+1)}, \theta_2 \right\} \in (0,1). \tag{3.10}$$

Note that $\phi \in \mathcal{A}$, then $\psi(x) = x - \phi(x)$, and $\psi : [0,+\infty) \to [0,+\infty)$ is nondecreasing. Thus by Lemma 2.8, for $u \geq v$, we have

$$\begin{aligned} d(Tu,Tv) &\leq (\rho_1 + \rho_2)\phi(\theta d(u,v)) \max_{t\in[0,1]} \int_0^1 H(t,s)ds \\ &\leq \phi(\theta d(u,v)) = \theta d(u,v) - (\theta d(u,v) - \phi(\theta d(u,v))) \\ &= \theta d(u,v) - \psi(\theta d(u,v)). \end{aligned} \tag{3.11}$$

Finally, taking into account that the zero function, $0 \leq T0$, by Theorem 2.11, problem (1.1) has a unique nonnegative solution. □

Theorem 3.2. *If the assumptions of Theorem 3.1 are satisfied, and there exists $t_0 \in [0,1]$ such that $f(t_0, 0, 0) \neq 0$, then the unique solution of* (1.1) *is positive. (Positive solution means a solution satisfying $x(t) > 0$ for $t \in (0,1)$.)*

Proof. By Theorem 3.1, the problem (1.1) has a unique nonnegative solution, which also is positive.

In fact, otherwise, there exists $0 < t^* < 1$ such that $x(t^*) = 0$, and

$$x(t^*) = \int_0^1 H(t^*,s) f\left(s, I^\beta x(s), -x(s)\right) ds = 0. \tag{3.12}$$

Then

$$0 = x(t^*) = \int_0^1 H(t^*,s) f\left(s, I^\beta x(s), -x(s)\right) ds \geq \int_0^1 H(t^*,s) f(s,0,0) ds \geq 0. \tag{3.13}$$

Consequently,

$$\int_0^1 H(t^*,s) f(s,0,0) ds = 0, \tag{3.14}$$

this yields

$$H(t^*,s) f(s,0,0) = 0, \text{ a.e.} \tag{3.15}$$

Note that $H(t^*,s) > 0$, $s \in (0,1)$, then we have

$$f(s,0,0) = 0, \text{ a.e.} \tag{3.16}$$

But on the other hand, since $f(t_0,0,0) \neq 0$, $t_0 \in [0,1]$, we have $f(t_0,0,0) > 0$, by the continuity of f, we can find a set $\Omega \subset [0,1]$ satisfying $t_0 \in \Omega$ and the Lebesgue measure $\mu(\Omega) > 0$ such that $f(t,0) > 0$ for any $t \in \Omega$. This contradicts (3.16). Therefore, $x(t) > 0$, that is, $x(t)$ is positive solution of (1.1). □

Example 3.3. Consider the following boundary value problem with fractional order $\alpha = 21/8$:

$$\begin{gathered} -\mathfrak{D}^{21/8}x(t) = e^t + \frac{x(t)}{100(1+x(t))} + \frac{\mathfrak{D}^{1/8}x(t)}{300\left(1+\mathfrak{D}^{1/8}x(t)\right)}, \quad 0 < t < 1, \\ \mathfrak{D}^{1/8}x(0) = \mathfrak{D}^{9/8}x(0) = 0, \qquad \mathfrak{D}^{1/8}x(1) = \int_0^1 \mathfrak{D}^{1/8}x(s)\,dA(s), \end{gathered} \tag{3.17}$$

where

$$A(t) = \begin{cases} 0, & t \in \left[0, \dfrac{1}{4}\right), \\ 1, & t \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right), \\ 2, & t \in \left[\dfrac{3}{4}, 1\right]. \end{cases} \tag{3.18}$$

Thus the BVP (3.17) becomes the 4-Point BVP with coefficients

$$\begin{gathered} -\mathfrak{D}^{21/8}x(t) = e^t + \frac{x(t)}{100(1+x(t))} + \frac{\mathfrak{D}^{1/8}x(t)}{300\left(1+\mathfrak{D}^{1/8}x(t)\right)}, \quad 0 < t < 1, \\ \mathfrak{D}^{1/8}x(0) = \mathfrak{D}^{9/8}x(0) = 0, \qquad \mathfrak{D}^{1/8}x(1) = \mathfrak{D}^{1/8}x\left(\frac{1}{4}\right) + \mathfrak{D}^{1/8}x\left(\frac{3}{4}\right). \end{gathered} \tag{3.19}$$

Then the BVP (3.17) has a unique positive solution.

Proof. Obviously, $\alpha = 21/8$, $\beta = 1/8$, and

$$0 \le \mathcal{C} = \int_0^1 t^{3/2}dA(t) = \left(\frac{1}{4}\right)^{3/2} + \left(\frac{3}{4}\right)^{3/2} \approx 0.7745 < 1, \qquad \Gamma(\alpha-\beta-1)(1-\mathcal{C}) = 0.2014. \tag{3.20}$$

On the other hand, we have

$$G(t,s) = \begin{cases} G_1(t,s) = \dfrac{[t(1-s)]^{3/2}}{\Gamma(5/2)}, & 0 \le t \le s \le 1, \\ G_2(t,s) = \dfrac{[t(1-s)]^{3/2} - (t-s)^{3/2}}{\Gamma(5/2)}, & 0 \le s \le t \le 1, \end{cases}$$

$$\mathcal{G}_A(s)=\begin{cases} G_2\left(\frac{1}{4},s\right)+2G_2\left(\frac{3}{4},s\right), & 0\le s<\frac{1}{4},\\ G_1\left(\frac{1}{4},s\right)+2G_2\left(\frac{3}{4},s\right), & \frac{1}{4}\le s<\frac{3}{4},\\ G_1\left(\frac{1}{4},s\right)+2G_1\left(\frac{3}{4},s\right), & \frac{3}{4}\le s\le 1. \end{cases} \tag{3.21}$$

Thus $\mathcal{G}_A(s)\ge 0,\ 0\le \mathcal{C}<1$ and $A(s)$ is increasing. So (**H0**) holds.

Take

$$f(t,u,v)=e^t+\frac{u}{100(1+u)}-\frac{v}{300(1-v)},(t,u,v)\in[0,1]\times[0,+\infty)\times(-\infty,0]. \tag{3.22}$$

Then

$$f\left(t,x(t),-\mathfrak{D}_t{}^{1/8}x(t)\right)=e^t+\frac{x(t)}{100(1+x(t))}+\frac{\mathfrak{D}^{1/8}x(t)}{300\left(1+\mathfrak{D}^{1/8}x(t)\right)}, \tag{3.23}$$

and for any $x_1\ge x_2,\ y_1\le y_2$,

$$\begin{aligned} &f(t,x_1,y_1)-f(t,x_2,y_2)\\ &\quad=\frac{x_1}{100(1+x_1)}-\frac{y_1}{300(1-y_1)}-\frac{x_2}{100(1+x_2)}+\frac{y_2}{300(1-y_2)}\\ &\quad=\frac{x_1-x_2}{100(1+x_1)(1+x_2)}+\frac{y_2-y_1}{300(1-y_1)(1-y_2)}\le\frac{x_1-x_2}{100(1+x_1-x_2)}+\frac{y_2-y_1}{300(1+y_2-y_1)}\\ &\quad\le\frac{1}{100}\times\frac{x_1-x_2}{1+1/2(x_1-x_2)}+\frac{1}{300}\times\frac{y_2-y_1}{1+1/3(y_2-y_1)}\\ &\quad=\frac{1}{50}\times\frac{(1/2)\ (x_1-x_2)}{1+(1/2)(x_1-x_2)}+\frac{1}{100}\times\frac{(1/3)(y_2-y_1)}{1+(1/3)(y_2-y_1)}\\ &\quad=\frac{1}{50}\phi\left(\frac{1}{2}(x_1-x_2)\right)+\frac{1}{100}\phi\left(\frac{1}{3}(y_2-y_1)\right), \end{aligned} \tag{3.24}$$

where

$$\begin{gathered} \phi(x)=\frac{x}{1+x},\qquad \rho_1+\rho_2=\frac{1}{50}+\frac{1}{100}=0.03<0.2014,\\ \theta_1=\frac{1}{2}<\frac{1}{\Gamma(\beta+1)}=\frac{1}{\Gamma}\left(\frac{9}{8}\right)=1.0619,\qquad \theta_2=\frac{1}{3}. \end{gathered} \tag{3.25}$$

Thus $\phi\in\mathcal{A}$ and all of condition of Theorem 3.1 are satisfied.

On the other hand, $f(t,0,0)=e^t\neq 0$ for any $t\in[0,1]$, by Theorem 3.2, the BVP (3.17) has a unique positive solution. □

In the end, we claim this process and conclusion of paper can be applied to the forefront of research of the real estate asset securitization.

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