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Research Article

A Geometric Mean of Parameterized Arithmetic and Harmonic Means of Convex Functions

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The notion of the geometric mean of two positive reals is extended by Ando (1978) to the case of positive semidefinite matrices A and B. Moreover, an interesting generalization of the geometric mean $A \# B$ of A and B to convex functions was introduced by Atteia and Raïssouli (2001) with a different viewpoint of convex analysis. The present work aims at providing a further development of the geometric mean of convex functions due to Atteia and Raïssouli (2001). A new algorithmic self-dual operator for convex functions named "the geometric mean of parameterized arithmetic and harmonic means of convex functions" is proposed, and its essential properties are investigated.

1. Introduction

The notion of geometric means is extended by Ando [1] to the case of positive semidefinite matrices A and B as the maximum $A \# B$ of all $X \geq 0$ for which $\left(\begin{smallmatrix} A & X \\ X & B \end{smallmatrix}\right)$ is positive semidefinite. If A is invertible, then $A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$. The geometric mean $A \# B$ appears in the literature with many applications in matrix inequalities, semidefinite programming (scaling point [2, 3]), geometry (geodesic middle [4, 5]), statistical shape analysis (intrinsic mean [6, 7]), and symmetric matrix word equations [8–10]. The most important property of the geometric mean is that it has a Riccati matrix equation as the defining equation. The geometric mean is the unique positive definite solution of the Riccati matrix equation $XA^{-1}X = B$.

An interesting generalization of the geometric mean $A \# B$ to convex functions was introduced by Atteia and Raïssouli [11] with a different viewpoint of the convex analysis. The natural idea to make an extension from positive semidefinite matrices to convex functions is nothing but the association of a positive semidefinite matrix A with the quadratic convex function $q_A(x) = (1/2)\langle Ax, x\rangle$. Atteia and Raïssouli [11] provided a general algorithm to construct the (self-dual) geometric mean and the square root of convex functions. As pointed out in [12], self-dual operators are important in convex analysis and also arise in PDE.

The present work aims at providing a further development of the geometric mean of the convex functions mentioned above. We develop a new algorithmic self-dual operator for convex functions named "the geometric mean of parameterized arithmetic and harmonic means of convex functions" by exploiting the proximal average of convex functions by Bauschke et al. [13] and investigate its essential properties such as limiting behaviors, self-duality, and monotonicity with respect to parameters. While doing so, we will see that the geometric mean due to Atteia and Raïssouli [11] can be interpreted as an element of "the geometric mean of parameterized arithmetic and harmonic means of convex functions" with the particular parameter $\mu = 0$.

In fact, this work is motivated by a recent result due to Kim et al. [14] concerned with a new matrix mean. Actually, the geometric mean of parameterized arithmetic and harmonic means of convex functions is an extension of the new matrix mean to a convex function mean under a standard setting with two convex functions.

2. Geometric Mean and $\mathscr{A}\#\mathscr{H}$-Mean of Parameter μ

We begin with the algorithm of finding the geometric mean of two proper convex lower semicontinuous functions f and g introduced by Atteia and Raïssouli [11, Proposition 4.4] and some comments on the procedure. Let $f, g \in \Gamma$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ where Γ denotes the class of proper convex lower semicontinuous functions from the Euclidean space $\mathbb{R}^n$ to $(-\infty, +\infty]$. Set two sequences of convex functions $\beta_n(f,g)$ and $\beta_n^*(f,g)$ recursively:

$$\beta_0(f,g) = \frac{1}{2}(f+g), \qquad \beta_0^*(f,g) = \left(\frac{1}{2}(f^* + g^*)\right)^*,$$
$$\beta_{n+1}(f,g) = \frac{1}{2}(\beta_n(f,g) + \beta_n^*(f,g)) \quad \text{where } \beta_n^*(f,g) = (\beta_n(f^*, g^*))^*, \tag{2.1}$$

where f^* stands for the Fenchel conjugate of f.

It is claimed that all the $\beta_n(f,g)$ and $\beta_n^*(f,g)$ do belong to Γ [11, Proposition 4.4]. However, to ensure this property, we need more. Indeed, we see

$$\beta_0^*(f,g) = \left(\frac{1}{2}(f^* + g^*)\right)^* = \left(\frac{1}{2}(f \,\square\, g)^*\right)^*, \tag{2.2}$$

where $\square$ stands for the infimal convolution. As is well known, $f \,\square\, g$ can take $-\infty$ as a value so it may not be proper. This happens for two simple linear functionals $f(x) = x$ and $g(x) = -x$ in the one-dimensional case. So the properness of $\beta_0^*(f,g)$ equivalent to that of $f \,\square\, g$ is not safe. Exactly the same problem may occur whenever $\beta_n^*(f,g)$ is defined. Moreover, it is not sure that $\beta_{n+1}(f,g)$ is proper because $\operatorname{dom}\beta_n(f,g) \cap \operatorname{dom}\beta_n^*(f,g)$ can be empty. Thus the basic necessity that $\beta_n(f,g)$ and $\beta_n^*(f,g)$ belong to Γ is not guaranteed under the general assumption only that $f, g \in \Gamma$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ in [11]. Hence it is necessary to impose a suitable condition to meet this demand. For that purpose, recall that a function $f \in \Gamma$ is called *cofinite* if the recession function $f0^+$ of f satisfies $(f0^+)(y) = +\infty$, for all $y \neq 0$ (see [15, page 116]). Then f is cofinite if and only if $\operatorname{dom} f^* = \mathbb{R}^n$ by means of [15, Corollary 13.3.1]. The terminology "cofinite" is renewed as "coercive" in [16, 3.26 Theorem].

Now we take a look at Atteia and Raïssouli [11, Proposition 4.4] with a refined proof.

Proposition 2.1 (See Atteia and Raïssouli [11, Proposition 4.4]). *Let* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *If either* f *or* g *is cofinite, then all* $\beta_n(f,g)$ *and* $\beta_n^*(f,g)$ *belong to* Γ *and* $\beta_n(f,g)$ *is cofinite for all* $n \geq 0$. *Hence the geometric mean* $f \# g$ *due to Atteia and Raïssouli [11], that is, the limit*

$$f \# g = \lim_{n\to\infty} \beta_n(f,g), \tag{2.3}$$

is well defined and proper convex on $\operatorname{dom} \beta_0(f,g)$. *In particular, it belongs to* Γ *under the assumption that either* $\operatorname{dom}\beta_0(f,g) = \operatorname{dom}\beta_0^*(f,g)$ *or* $\operatorname{dom}\beta_0(f,g)$ *is closed. Moreover,* $f \# g = (f^* \# g^*)^*$ *under the condition* $\operatorname{dom}\beta_0(f,g) = \operatorname{dom}\beta_0^*(f,g)$.

Proof. Without loss of generality, we may assume that g is cofinite. Clearly, $\beta_0(f,g) = (1/2)(f+g) \in \Gamma$ since $\operatorname{dom}\beta_0(f,g) = \operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. In addition, $\beta_0(f,g)$ is still cofinite by [15, Theorem 9.3]. Then $\beta_0^*(f,g) = ((1/2)(f^*+g^*))^* = (1/2) \star (f \,\square\, g) \in \Gamma$ by virtue of [15, Corollary 9.2.2]. Thus $\operatorname{dom}\beta_0^*(f,g) = (1/2)(\operatorname{dom} f + \operatorname{dom} g) \supseteq \operatorname{dom}\beta_0(f,g)$. By induction, assume that

$$\beta_n(f,g),\ \beta_n^*(f,g) \in \Gamma, \quad \beta_n(f,g) \text{ is cofinite}, \qquad \operatorname{dom}\beta_n(f,g) \subseteq \operatorname{dom}\beta_n^*(f,g). \tag{2.4}$$

Then $\operatorname{dom}\beta_{n+1}(f,g) = \operatorname{dom}\beta_n(f,g) \cap \operatorname{dom}\beta_n^*(f,g) = \operatorname{dom}\beta_n(f,g)$, so $\beta_{n+1}(f,g) \in \Gamma$. Moreover, $\beta_{n+1}(f,g)$ is cofinite because $\beta_n(f,g)$ is cofinite. It is readily checked that

$$\beta_{n+1}^*(f,g) = (\beta_{n+1}(f^*,g^*))^* = \left(\frac{1}{2}(\beta_n(f,g))^* + \frac{1}{2}(\beta_n^*(f,g))^*\right)^*. \tag{2.5}$$

Hence $\beta_{n+1}^*(f,g) = (1/2) \star (\beta_n(f,g) \,\square\, \beta_n^*(f,g)) \in \Gamma$. In this case, $\operatorname{dom}\beta_{n+1}^*(f,g) = (1/2)(\operatorname{dom}\beta_n(f,g) + \operatorname{dom}\beta_n^*(f,g)) \supseteq \operatorname{dom}\beta_n(f,g) = \operatorname{dom}\beta_{n+1}(f,g)$. Thus we obtain that

$$\begin{aligned} \forall n, \quad \operatorname{dom}\beta_n(f,g) &= \operatorname{dom} f \cap \operatorname{dom} g = \operatorname{dom}\beta_0(f,g), \\ \forall n, \quad \operatorname{dom}\beta_n^*(f,g) &\supseteq \operatorname{dom}\beta_n(f,g) = \operatorname{dom}\beta_0(f,g). \end{aligned} \tag{2.6}$$

According to Atteia and Raïssouli [11, Proposition 4.4], we have

$$\begin{gathered} \beta_{n+1}(f,g) - \beta_{n+1}^*(f,g) \leq \frac{1}{2}(\beta_n(f,g) - \beta_n^*(f,g)), \quad \forall n \geq 0; \\ \beta_0^*(f,g) \leq \cdot \leq \beta_n^*(f,g) \leq \beta_{n+1}^*(f,g) \leq \cdot \leq \beta_{n+1}(f,g) \leq \beta_n(f,g) \leq \cdot \leq \beta_0(f,g). \end{gathered} \tag{2.7}$$

Hence the geometric mean $f \# g$ is well defined and belongs to Γ under the given hypothesis. (If $\operatorname{dom}\beta_0(f,g)$ is closed, we define an increasing sequence $\gamma_n(f,g) \in \Gamma$ by

$$\gamma_n(f,g) = \beta_n^*(f,g) + \delta_C, \tag{2.8}$$

where δ_C denotes the indicator function of the closed convex set $C = \operatorname{dom}\beta_0(f,g)$. Obviously, $f\,\#\,g$ is the common limit of $\beta_n(f,g)$ and $\gamma_n(f,g)$, hence, belongs to Γ.)

For the equality $f\,\#\,g = (f^*\,\#\,g^*)^*$, we have

$$\begin{aligned}
(f^*\,\#\,g^*)^*(x) &= \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - (f^*\,\#\,g^*)(y)\right] \\
&= \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - \lim_{n\to\infty}\beta_n(f^*,g^*)(y)\right] \\
&= \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - \lim_{n\to\infty}\beta_n^*(f^*,g^*)(y)\right] \\
&= \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - \lim_{n\to\infty}(\beta_n(f,g))^*(y)\right] \\
&\le \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - (\beta_n(f,g))^*(y)\right], \quad \forall n \\
&= (\beta_n(f,g))^{**}(x) = \beta_n(f,g)(x), \quad \forall n.
\end{aligned} \tag{2.9}$$

Hence

$$(f^*\,\#\,g^*)^*(x) \le \lim_{n\to\infty}\beta_n(f,g)(x) = (f\,\#\,g)(x). \tag{2.10}$$

On the other hand,

$$\begin{aligned}
(f^*\,\#\,g^*)^*(x) &= \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - (f^*\,\#\,g^*)(y)\right] \\
&= \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - \lim_{n\to\infty}\beta_n(f^*,g^*)(y)\right] \\
&\ge \sup_{y\in\mathbb{R}^n}\left[\langle y,x\rangle - \beta_n(f^*,g^*)(y)\right], \quad \forall n \\
&= (\beta_n(f^*,g^*))^*(x) = \beta_n^*(f,g)(x), \quad \forall n.
\end{aligned} \tag{2.11}$$

Thus

$$(f^*\,\#\,g^*)^*(x) \ge \lim_{n\to\infty}\beta_n^*(f,g)(x) = (f\,\#\,g)(x). \tag{2.12}$$

Therefore we get

$$f\,\#\,g = (f^*\,\#\,g^*)^*. \tag{2.13}$$

□

Remark 2.2. (1) The well definedness of $f^* \# g^*$ is readily checked by the assumption g is cofinite. (Without this condition, $f^* \# g^*$ may not be well defined so that the identity $f \# g = (f^* \# g^*)^*$ breaks down.) With the additional property that $\operatorname{dom} f^*$ is closed, we have $f^* \# g^* \in \Gamma$. Hence

$$(f \# g)^* = f^* \# g^*. \tag{2.14}$$

(2) Proposition 2.1 provides a sufficient condition to entail the validity of [11, Proposition 4.4]. It is also mentioned in [11, Remark 4.5] that if f and g are finite-valued, $\operatorname{dom} \beta_0(f, g) = \operatorname{dom} \beta_0^*(f, g)$ is satisfied. But even though it is true, $\beta_0^*(f, g)$ can be identically $-\infty$ as shown in the case of $f(x) = x$ and $g(x) = -x$ in $\mathbb{R}$ so that the limiting process using (2.7) may not be available any more. So some restrictions should be imposed to properly define the geometric mean of two convex functions f and $g \in \Gamma$. Of course, for an $f \in \Gamma$, the geometric mean $f \# f$ and the convex square root $f^{1/2}$ of f (see [11, Definition 4.7]) are always well defined because q is cofinite. What is a minimal assumption? That is a question to be answered.

Throughout this paper, we adopt the following modified definition of proximal average for the convenience of presentation. For $\mu \geq 0$, with $\mathrm{q} = (1/2)\|\cdot\|^2$,

$$p_\mu(\mathbf{f}, \lambda) = \left(\lambda_1 (f_1 + \mu \mathrm{q})^* + \cdots + \lambda_m (f_m + \mu \mathrm{q})^*\right)^* - \mu \mathrm{q}, \tag{2.15}$$

where $\mathbf{f} = (f_1, \ldots, f_m)$, $\mathbf{g} = (g_1, \ldots, g_m)$, each $f_i : \mathbb{R}^n \to (-\infty, +\infty]$ belongs to Γ, and λ_i's are positive real numbers with $\lambda_1 + \cdots + \lambda_m = 1$.

From now on, we consider the simple case where $m = 2$, $\lambda_1 = \lambda_2 = 1/2$, and $f, g \in \Gamma$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Define two sequences of convex functions $\alpha_n(f, g)$ and $\alpha_n^\bullet(f, g)$ recursively as follows:

$$\begin{gathered}
\alpha_0(f, g) = \frac{1}{2}(f + g), \qquad \alpha_0^\bullet(f, g) = p_\mu\left(f, g; \frac{1}{2}, \frac{1}{2}\right), \\
\alpha_{n+1}(f, g) = \frac{1}{2}\left(\alpha_n(f, g) + \alpha_n^\bullet(f, g)\right), \qquad \alpha_{n+1}^\bullet(f, g) = p_\mu\left(\alpha_n(f, g), \alpha_n^\bullet(f, g); \frac{1}{2}, \frac{1}{2}\right).
\end{gathered} \tag{2.16}$$

Theorem 2.3. *For $\mu > 0$, one has*

(i) $\alpha_n(f, g) \in \Gamma$ *and* $\alpha_n^\bullet(f, g) \in \Gamma$, *for all* $n \geq 0$;

(ii) $\alpha_n^\bullet(f, g) \leq \alpha_n(f, g)$, $\alpha_{n+1}(f, g) \leq \alpha_n(f, g)$ *and* $\alpha_n^\bullet(f, g) \leq \alpha_{n+1}^\bullet(f, g)$, *for all* $n \geq 0$;

(iii) $\alpha_{n+1}(f, g) - \alpha_{n+1}^\bullet(f, g) \leq (1/2)(\alpha_n(f, g) - \alpha_n^\bullet(f, g))$, *for all* $n \geq 0$;

(iv) *there exists a limit* $\tau_\mu(f, g) = \lim_{n \to \infty} \alpha_n(f, g)$ *which is a proper convex function with* $\operatorname{dom} \tau_\mu(f, g) = \operatorname{dom} f \cap \operatorname{dom} g = \operatorname{dom} \alpha_0(f, g)$. *Furthermore, if either* $\operatorname{dom} \alpha_0(f, g) = \operatorname{dom} \alpha_0^\bullet(f, g)$ *or* $\operatorname{dom} \alpha_0(f, g)$ *is closed,* $\tau_\mu(f, g)$ *is the common limit of* $\alpha_n(f, g)$ *and* $\gamma_n(f, g)$ *for some increasing sequence* $\gamma_n(f, g) \in \Gamma$. *In this case,* $\tau_\mu(f, g) \in \Gamma$.

Proof. (i) Since $\alpha_0^\bullet(f,g) = p_\mu(f,g;\ 1/2,1/2)$, by Bauschke et al. [13, Theorem 4.6],

$$\begin{aligned}\operatorname{dom}\alpha_0^\bullet(f,g) &= \frac{1}{2}\ \operatorname{dom} f + \frac{1}{2}\ \operatorname{dom} g \\ &\supseteq \frac{1}{2}(\operatorname{dom} f \cap \operatorname{dom} g) + \frac{1}{2}(\operatorname{dom} f \cap \operatorname{dom} g) \\ &= \operatorname{dom} f \cap \operatorname{dom} g = \operatorname{dom}\alpha_0(f,g)\end{aligned} \tag{2.17}$$

because $\operatorname{dom} f \cap \operatorname{dom} g$ is a convex set. By induction, assume that $\operatorname{dom}\alpha_n^\bullet(f,g) \supseteq \operatorname{dom}\alpha_n(f,g)$. Then

$$\begin{aligned}\operatorname{dom}\alpha_{n+1}(f,g) &= \operatorname{dom}\alpha_n(f,g) \cap \operatorname{dom}\alpha_n^\bullet(f,g) = \operatorname{dom}\alpha_n(f,g), \\ \operatorname{dom}\alpha_{n+1}^\bullet(f,g) &= \frac{1}{2}\operatorname{dom}\alpha_n(f,g) + \frac{1}{2}\ \operatorname{dom}\alpha_n^\bullet(f,g) \\ &\supseteq \operatorname{dom}\alpha_n(f,g) = \operatorname{dom}\alpha_{n+1}(f,g).\end{aligned} \tag{2.18}$$

Thus we obtain that

$$\begin{aligned}&\forall n, \quad \operatorname{dom}\alpha_n(f,g) = \operatorname{dom} f \cap \operatorname{dom} g = \operatorname{dom}\alpha_0(f,g), \\ &\forall n, \quad \operatorname{dom}\alpha_n^\bullet(f,g) \supseteq \operatorname{dom}\alpha_n(f,g) = \operatorname{dom}\alpha_0(f,g).\end{aligned} \tag{2.19}$$

This implies that, for all $n \geq 0$, $\alpha_n(f,g) \in \Gamma$ and $\alpha_n^\bullet(f,g) \in \Gamma$ with the help of [13, Corollary 5.2].

(ii) The first assertion $\alpha_n^\bullet(f,g) \leq \alpha_n(f,g)$ is a direct consequence of [13, Theorem 5.4]. For the second, by definition and the first assertion, we see

$$\alpha_{n+1}(f,g) = \frac{1}{2}(\alpha_n(f,g) + \alpha_n^\bullet(f,g)) \leq \frac{1}{2}(\alpha_n(f,g) + \alpha_n(f,g)) = \alpha_n(f,g). \tag{2.20}$$

For the last, observe that

$$\begin{aligned}\alpha_n^\bullet(f,g) \leq \alpha_{n+1}^\bullet(f,g) &\Longleftrightarrow \alpha_n^\bullet(f,g) + \mu\mathrm{q} \leq \alpha_{n+1}^\bullet(f,g) + \mu\mathrm{q} \\ &\Longleftrightarrow (\alpha_{n+1}^\bullet(f,g) + \mu\mathrm{q})^* \leq (\alpha_n^\bullet(f,g) + \mu\mathrm{q})^* \\ &\Longleftrightarrow \frac{1}{2}(\alpha_n^\bullet(f,g) + \mu\mathrm{q})^* + \frac{1}{2}(\alpha_n(f,g) + \mu\mathrm{q})^* \leq (\alpha_n^\bullet(f,g) + \mu\mathrm{q})^* \\ &\Longleftrightarrow (\alpha_n(f,g) + \mu\mathrm{q})^* \leq (\alpha_n^\bullet(f,g) + \mu\mathrm{q})^* \\ &\Longleftrightarrow \alpha_n^\bullet(f,g) + \mu\mathrm{q} \leq \alpha_n(f,g) + \mu\mathrm{q} \\ &\Longleftrightarrow \alpha_n^\bullet(f,g) \leq \alpha_n(f,g),\end{aligned} \tag{2.21}$$

which is nothing but the first assertion. Note that all the arithmetics are safe because both $(\alpha_n(f,g) + \mu\mathrm{q})^*$ and $(\alpha_n^\bullet(f,g) + \mu\mathrm{q})^*$ are finite-valued.

(iii) By (ii) and the extended arithmetic $\infty + (-\infty) = (-\infty) + \infty = \infty$ (see [16]), we get

$$\begin{aligned} \alpha_{n+1}(f,g) - \alpha^{\bullet}_{n+1}(f,g) &\leq \frac{1}{2}\left(\alpha_n(f,g) + \alpha^{\bullet}_n(f,g)\right) - \alpha^{\bullet}_n(f,g) \\ &= \frac{1}{2}\left(\alpha_n(f,g) - \alpha^{\bullet}_n(f,g)\right). \end{aligned} \tag{2.22}$$

(iv) From (ii), we have

$$\alpha^{\bullet}_0(f,g) \leq \cdot \leq \alpha^{\bullet}_n(f,g) \leq \alpha^{\bullet}_{n+1}(f,g) \leq \cdot \leq \alpha_{n+1}(f,g) \leq \alpha_n(f,g) \leq \cdot \leq \alpha_0(f,g). \tag{2.23}$$

Hence if $x \in \operatorname{dom}\alpha_0(f,g) = \operatorname{dom} f \cap \operatorname{dom} g = \operatorname{dom}\alpha_n(f,g)$ by (2.19), $\alpha_n(f,g)(x)$ converges to a real number r. If $x \notin \operatorname{dom}\alpha_0(f,g)$, $\alpha_n(f,g)(x) = \infty$. Let the limit function be $\tau_\mu(f,g)$. Clearly, $\tau_\mu(f,g)$ is proper convex because $\alpha_n(f,g)$ is convex. Moreover, if $\operatorname{dom}\alpha_0(f,g) = \operatorname{dom}\alpha^{\bullet}_0(f,g)$, by (iii) and (2.23), it is the common limit of $\alpha_n(f,g)$ and $\alpha^{\bullet}_n(f,g)$, so $\tau_\mu(f,g) \in \Gamma$ since it is a supremum of $\alpha^{\bullet}_n(f,g) \in \Gamma$. If $\operatorname{dom}\alpha_0(f,g)$ is closed, we define an increasing sequence $\gamma_n(f,g) \in \Gamma$ by

$$\gamma_n(f,g) = \alpha^{\bullet}_n(f,g) + \delta_C, \tag{2.24}$$

where δ_C denotes the indicator function of the closed convex set $C = \operatorname{dom}\alpha_0(f,g)$. Obviously, $\tau_\mu(f,g)$ is the common limit of $\alpha_n(f,g)$ and $\gamma_n(f,g)$, hence belongs to Γ. □

Remark 2.4. If both f and g are finite-valued, the condition $\operatorname{dom}\alpha_0(f,g) = \operatorname{dom}\alpha^{\bullet}_0(f,g)$ is automatically satisfied.

Corollary 2.5. *For $\mu > 0$ and $f, g \in \Gamma$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$,*

(i) $\tau_\mu(f,g) = \tau_\mu(g,f)$,

(ii) $((1/2)(f^* + g^*))^* \leq \alpha^{\bullet}_0(f,g) \leq \tau_\mu(f,g) \leq \alpha_0(f,g) = (1/2)(f+g)$.

Proof. (i) Trivially, $\alpha_0(f,g) = \alpha_0(g,f)$ and $\alpha^{\bullet}_0(f,g) = \alpha^{\bullet}_0(g,f)$. Again using the induction argument yields that

$$\alpha_n(f,g) = \alpha_n(g,f), \qquad \alpha^{\bullet}_n(f,g) = \alpha^{\bullet}_n(g,f), \quad \forall n \geq 0. \tag{2.25}$$

Hence $\tau_\mu(f,g) = \tau_\mu(g,f)$.

(ii) This is immediate from (2.23) and [13, Theorem 5.4]. □

Now we express $\tau_\mu(f,g)$ in terms of a geometric mean.

Theorem 2.6. *Let $\mu > 0$. For $f, g \in \Gamma$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, one has*

$$\begin{aligned} \tau_\mu(f,g) &= \left(\frac{1}{2}(f+\mu \mathrm{q}) + \frac{1}{2}(g+\mu \mathrm{q})\right) \# \left(\frac{1}{2}(f+\mu \mathrm{q})^* + \frac{1}{2}(g+\mu \mathrm{q})^*\right)^* - \mu \mathrm{q} \\ &= (f+\mu \mathrm{q}) \# (g+\mu \mathrm{q}) - \mu \mathrm{q}. \end{aligned} \tag{2.26}$$

Proof. Claim 1. We have

$$\tau_\mu(f,g) = \left(\frac{1}{2}(f+\mu \mathrm{q}) + \frac{1}{2}(g+\mu \mathrm{q})\right) \# \left(\frac{1}{2}(f+\mu \mathrm{q})^* + \frac{1}{2}(g+\mu \mathrm{q})^*\right)^* - \mu \mathrm{q}. \tag{2.27}$$

Indeed, put $f_0 = (1/2)(f+\mu \mathrm{q}) + (1/2)(g+\mu \mathrm{q})$ and $g_0 = ((1/2)(f+\mu \mathrm{q})^* + (1/2)(g+\mu \mathrm{q})^*)^*$. Then $f_0, g_0 \in \Gamma$ because $(f+\mu \mathrm{q})^*$ and $(g+\mu \mathrm{q})^*$ are finite-valued, and f_0 is cofinite by [15, Theorem 9.3]. By Proposition 2.1, we obtain

$$\lim_{n\to\infty} \beta_n(f_0, g_0) = f_0 \# g_0, \tag{2.28}$$

where $\beta_n(f_0,g_0)$ and $\beta_n^*(f_0,g_0)$ are defined as in (2.1). Set, for each $n \geq 0$,

$$\beta_n'(f_0,g_0) = \beta_n(f_0,g_0) - \mu \mathrm{q}, \qquad (\beta_n^*)'(f_0,g_0) = \beta_n^*(f_0,g_0) - \mu \mathrm{q}. \tag{2.29}$$

Then by (2.5)

$$\begin{aligned}
\beta_{n+1}'(f_0,g_0) &= \beta_{n+1}(f_0,g_0) - \mu \mathrm{q} = \frac{\beta_n(f_0,g_0) + \beta_n^*(f_0,g_0)}{2} - \mu \mathrm{q} \\
&= \frac{\beta_n(f_0,g_0) - \mu \mathrm{q} + \beta_n^*(f_0,g_0) - \mu \mathrm{q}}{2} = \frac{\beta_n'(f_0,g_0) + (\beta_n^*)'(f_0,g_0)}{2} \\
(\beta_{n+1}^*)'(f_0,g_0) &= \beta_{n+1}^*(f_0,g_0) \ - \mu \mathrm{q} = \left(\frac{1}{2}(\beta_n(f_0,g_0))^* + \frac{1}{2}(\beta_n^*(f_0,g_0))^*\right)^* - \mu \mathrm{q} \\
&= \left(\frac{1}{2}(\beta_n'(f_0,g_0) + \mu \mathrm{q})^* + \frac{1}{2}\left((\beta_n^*)'(f_0,g_0) + \mu \mathrm{q}\right)^*\right)^* - \mu \mathrm{q} \\
&= p_\mu\left(\beta_n'(f_0,g_0), (\beta_n^*)'(f_0,g_0); \frac{1}{2}, \frac{1}{2}\right).
\end{aligned} \tag{2.30}$$

Put $\alpha_0(f,g) = (1/2)(f+g)$ and $\alpha_0^\bullet(f,g) = p_\mu(f,g,;\ 1/2, 1/2)$. Also define

$$\alpha_{n+1}(f,g) = \beta_n'(f_0,g_0), \qquad \alpha_{n+1}^\bullet(f,g) = (\beta_n^*)'(f_0,g_0), \quad \forall n \geq 0. \tag{2.31}$$

Then we have

$$\begin{aligned}\alpha_1(f,g) &= \beta_0'(f_0,g_0) = \beta_0(f_0,g_0) - \mu\mathrm{q} = \frac{1}{2}(f_0 - \mu\mathrm{q} + g_0 - \mu\mathrm{q})\\ &= \frac{1}{2}\left(\frac{1}{2}(f+g) + p_\mu\left(f,g,;\frac{1}{2},\frac{1}{2}\right)\right) = \frac{1}{2}(\alpha_0(f,g) + \alpha_0^\bullet(f,g)),\\ \alpha_1^\bullet(f,g) &= (\beta_0^*)'(f_0,g_0) = \beta_0^*(f_0,g_0) - \mu\mathrm{q} = \left(\frac{1}{2}(f_0^* + g_0^*)\right)^* - \mu\mathrm{q}\\ &= \left(\frac{1}{2}\left(\frac{1}{2}(f+g)+\mu\mathrm{q}\right)^* + \frac{1}{2}\left(\frac{1}{2}(f+\mu\mathrm{q})^* + \frac{1}{2}(g+\mu\mathrm{q})^*\right)\right)^* - \mu\mathrm{q}\\ &= \left(\frac{1}{2}(\alpha_0(f,g)+\mu\mathrm{q})^* + \frac{1}{2}(\alpha_0^\bullet(f,g)+\mu\mathrm{q})^*\right)^* - \mu\mathrm{q}\\ &= p_\mu\left(\alpha_0(f,g),\alpha_0^\bullet(f,g);\frac{1}{2},\frac{1}{2}\right).\end{aligned} \tag{2.32}$$

Moreover, it follows from (2.30) that $\alpha_n(f,g)$ and $\alpha_n^\bullet(f,g)$ satisfy the recursion formula in (2.1). From Theorem 2.3 and (2.28), we get

$$\tau_\mu(f,g) = \lim_{n\to\infty}\alpha_n(f,g) = \lim_{n\to\infty}\beta_n'(f_0,g_0) = \lim_{n\to\infty}\beta_n(f_0,g_0) - \mu\mathrm{q} = f_0 \# g_0 - \mu\mathrm{q}. \tag{2.33}$$

Claim 2. $\tau_\mu(f,g) = (f+\mu\mathrm{q})\#(g+\mu\mathrm{q}) - \mu\mathrm{q}$.
Set two cofinite functions $f_1 = f + \mu\mathrm{q}$ and $g_1 = g + \mu\mathrm{q}$. It sufficies to check that

$$\left(\frac{1}{2}(f_1+g_1)\right)\#\left(\frac{1}{2}(f_1^*+g_1^*)\right)^* = f_1 \# g_1. \tag{2.34}$$

In fact, let $F = \beta_0(f_1,g_1)$ and $G = \beta_0^*(f_1,g_1)$. Then F and G belong to Γ, and F is cofinite by Proposition 2.1. Clearly, we have

$$\beta_n(F,G) = \beta_{n+1}(f_1,g_1), \qquad \beta_n^*(F,G) = \beta_{n+1}^*(f_1,g_1), \quad \forall n \ge 0. \tag{2.35}$$

Again appealing to (2.6) yields that

$$f_1 \# g_1 = \lim_{n\to\infty}\beta_n(f_1,g_1) = \lim_{n\to\infty}\beta_n(F,G) = F \# G = \left(\frac{1}{2}(f_1+g_1)\right)\#\left(\frac{1}{2}(f_1^*+g_1^*)\right)^*. \tag{2.36}$$

This completes the proof. □

Now we give the following name to $\tau_\mu(f,g)$ by Theorem 2.6 above.

Definition 2.7. For $f, g \in \Gamma$, one defines

$$\tau_\mu(f,g) = \left(\tau_{-\mu}(f^*,g^*)\right)^*, \quad \text{for } \mu < 0,$$
$$\tau_0(f,g) = f \# g, \quad \text{for } \mu = 0. \tag{2.37}$$

This $\tau_\mu(f,g)$ is called *the geometric mean of parameterized arithmetic and harmonic means of* f and g and abbreviated by "$\mathscr{A}\#\mathscr{H}$-mean of parameter μ".

3. Properties of $\mathscr{A}\#\mathscr{H}$-Mean of Parameter μ

To deal with $\tau_\mu(f,g)$ (for all $\mu \in \mathbb{R}$), in what follows, we assume the following for the simplicity of arguments.

3.1. Constraint Qualifications

Consider

($CQ1$) $f, g \in \Gamma$ with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$,

($CQ2$) $\operatorname{dom} \alpha_0(f,g) = \operatorname{dom} \alpha_0^\bullet(f,g)$,

($CQ3$) either f is cofinite and $\operatorname{dom} g^*$ is closed or g is cofinite and $\operatorname{dom} f^*$ is closed.

With these hypotheses, for all $\mu \in \mathbb{R}$, $\tau_\mu(f,g)$ is well-defined and is in Γ.

Theorem 3.1. *One has the limiting property:*

$$\lim_{\mu\to\infty} \tau_\mu(f,g) = \frac{1}{2}(f+g), \qquad \lim_{\mu\to-\infty} \tau_\mu(f,g) = \left(\frac{1}{2}(f^*+g^*)\right)^*. \tag{3.1}$$

Proof. For $\mu > 0$, by Corollary 2.5, we get

$$\lim_{\mu\to\infty} \alpha_0^\bullet(f,g) \le \lim_{\mu\to\infty} \tau_\mu(f,g) \le \lim_{\mu\to\infty} \alpha_0(f,g) = \frac{1}{2}(f+g). \tag{3.2}$$

By Bauschke et al. [13, Theorem 8.5],

$$\lim_{\mu\to\infty} \alpha_0^\bullet(f,g) = \lim_{\mu\to\infty} p_\mu\left(f, g; \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(f+g). \tag{3.3}$$

Thus

$$\lim_{\mu\to\infty} \tau_\mu(f,g) = \frac{1}{2}(f+g). \tag{3.4}$$

Again appealing to Corollary 2.5 yields that

$$\alpha_0^{\bullet}(f^*,g^*) \le \tau_\mu(f^*,g^*) \le \alpha_0(f^*,g^*) = \frac{1}{2}(f^*+g^*); \quad \text{that is,}$$
$$\left(\frac{1}{2}(f^*+g^*)\right)^* \le (\tau_\mu(f^*,g^*))^* \le (\alpha_0^{\bullet}(f^*,g^*))^*. \tag{3.5}$$

By the self-duality of the proximal average [13, Theorem 5.1], we have

$$(\alpha_0^{\bullet}(f^*,g^*))^* = \left(p_\mu\left(f^*,g^*;\frac{1}{2},\frac{1}{2}\right)\right)^* = p_{\mu^{-1}}\left(f,g;\frac{1}{2},\frac{1}{2}\right). \tag{3.6}$$

Taking the limit in (3.5), we see from (3.6) that

$$\left(\frac{1}{2}(f^*+g^*)\right)^* \le \lim_{\mu\to\infty}(\tau_\mu(f^*,g^*))^* \le \lim_{\mu\to\infty} p_{\mu^{-1}}\left(f,g;\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}\star(f\,\Box\,g), \tag{3.7}$$

where the equality comes from [13, Theorem 8.5]. By $(CQ3)$, $f\,\Box\,g \in \Gamma$; hence we have

$$\frac{1}{2}\star(f\,\Box\,g) = \left(\frac{1}{2}(f^*+g^*)\right)^*. \tag{3.8}$$

Therefore it follows from (3.7) and (3.8) that

$$\lim_{\mu\to-\infty}\tau_\mu(f,g) = \lim_{\mu\to\infty}(\tau_\mu(f^*,g^*))^* = \left(\frac{1}{2}(f^*+g^*)\right)^*. \tag{3.9}$$

This completes the proof. □

Theorem 3.2. *One has*

(i) $p_\mu(f,g;\ 1/2,1/2) \le \tau_\mu(f,g)$, *for* $\mu \ge 0$,

(ii) *(self-duality)* $(\tau_\mu(f,g))^* = \tau_{-\mu}(f,g)$, *for all* $\mu \in \mathbb{R}$.

Proof. (i) According to Corollary 2.5 (ii), $p_\mu(f,g;\ 1/2,1/2) = \alpha_0^{\bullet}(f,g) \le \tau_\mu(f,g)$ for $\mu > 0$. For $\mu = 0$, $p_\mu(f,g;\ 1/2,1/2) = ((1/2)(f^*+g^*))^* = \beta_0^*(f,g) \le f\,\#\,g = \tau_0(f,g)$ by Definition 2.7.

(ii) If $-\infty < \mu < 0$, by definition, $\tau_\mu(f,g) = (\tau_{-\mu}(f^*,g^*))^*$, so $(\tau_\mu(f,g))^* = \tau_{-\mu}(f^*,g^*)$ because $\tau_{-\mu}(f^*,g^*) \in \Gamma$. If $\mu = 0$, then $(\tau_0(f,g))^* = (f\,\#\,g)^* = f^*\,\#\,g^* = \tau_0(f^*,g^*)$ by virtue of Proposition 2.1 and Remark 2.2. Let $\mu > 0$. Then by definition, $(\tau_\mu(f,g))^* = \tau_{-\mu}(f^*,g^*)$, as desired. □

Proposition 3.3. *Let* $f_i, g_i \in \Gamma$ *and* $f_i \le g_i$ *for each* $i = 1,\ldots,m$. *Then, for* $\mu \ge 0$,

$$p_\mu(\mathbf{f},\lambda) \le p_\mu(\mathbf{g},\lambda), \tag{3.10}$$

where $\mathbf{f} = (f_1,\ldots,f_m)$, $\mathbf{g} = (g_1,\ldots,g_m)$ *and* λ_i*'s are positive real numbers with* $\lambda_1 + \cdots + \lambda_m = 1$.

Proof. For each i, clearly

$$\begin{aligned} f_i + \mu \mathrm{q} \le g_i + \mu \mathrm{q} &\Longrightarrow \lambda_i (f_i + \mu \mathrm{q})^* \ge \lambda_i (g_i + \mu \mathrm{q})^* \\ &\Longrightarrow \sum_{i=1}^{m} \lambda_i (f_i + \mu \mathrm{q})^* \ge \sum_{i=1}^{m} \lambda_i (g_i + \mu \mathrm{q})^* \\ &\Longrightarrow \left(\sum_{i=1}^{m} \lambda_i (f_i + \mu \mathrm{q})^* \right)^* \le \left(\sum_{i=1}^{m} \lambda_i (g_i + \mu \mathrm{q})^* \right)^* \\ &\Longrightarrow p_\mu(\mathbf{f}, \lambda) \le p_\mu(\mathbf{g}, \lambda). \end{aligned} \tag{3.11}$$

□

Theorem 3.4 (monotonicity). *One has, for $-\infty \le \mu \le \nu \le \infty$,*

$$\left(\frac{1}{2}(f^* + g^*) \right)^* = \tau_{-\infty}(f, g) \le \tau_\mu(f, g) \le \tau_\nu(f, g) \le \tau_\infty(f, g) = \frac{1}{2}(f + g). \tag{3.12}$$

Proof. Let $0 < \mu \le \nu < \infty$. Clearly

$$\begin{gathered} \frac{1}{2}(f + g) = \left(\alpha_0^\mu \right)(f, g) \le \alpha_0^\nu (f, g) = \frac{1}{2}(f + g), \\ p_\mu \left(f, g; \frac{1}{2}, \frac{1}{2} \right) = (\alpha_0^\bullet)^\mu (f, g) \le (\alpha_0^\bullet)^\nu (f, g) = p_\nu \left(f, g; \frac{1}{2}, \frac{1}{2} \right) \end{gathered} \tag{3.13}$$

by [13, Theorem 8.5]. To use induction, assume that

$$\alpha_n^\mu (f, g) \le \alpha_n^\nu (f, g), \qquad (\alpha_n^\bullet)^\mu (f, g) \le (\alpha_n^\bullet)^\nu (f, g). \tag{3.14}$$

Then

$$\begin{aligned} \alpha_{n+1}^\mu (f, g) &= \frac{1}{2} \left(\alpha_n^\mu (f, g) + (\alpha_n^\bullet)^\mu (f, g) \right) \le \frac{1}{2} (\alpha_n^\nu (f, g) + (\alpha_n^\bullet)^\nu (f, g)) = \alpha_{n+1}^\nu (f, g), \\ (\alpha_{n+1}^\bullet)^\mu (f, g) &= p_\mu \left(\alpha_n^\mu (f, g), (\alpha_n^\bullet)^\mu (f, g); \frac{1}{2}, \frac{1}{2} \right) \le p_\mu \left(\alpha_n^\nu (f, g), (\alpha_n^\bullet)^\nu (f, g); \frac{1}{2}, \frac{1}{2} \right) \\ &\le p_\nu \left(\alpha_n^\nu (f, g), (\alpha_n^\bullet)^\nu (f, g); \frac{1}{2}, \frac{1}{2} \right) = (\alpha_{n+1}^\bullet)^\nu (f, g) \end{aligned} \tag{3.15}$$

by (3.14), Proposition 3.3, and [13, Theorem 8.5]. Thus (3.14) holds for all n. Hence, we get

$$\tau_\mu (f, g) = \lim_{n \to \infty} \alpha_n^\mu (f, g) \le \lim_{n \to \infty} \alpha_n^\nu (f, g) = \tau_\nu (f, g). \tag{3.16}$$

On the other hand, for $-\infty < -\mu \leq -\nu < 0$,

$$\tau_{-\mu}(f,g) = (\tau_\mu(f^*,g^*))^* \leq (\tau_\nu(f^*,g^*))^* = \tau_{-\nu}(f,g) \tag{3.17}$$

by means of (3.16). Now let $\mu > 0$. Recall that $\alpha_0(f,g) = \beta_0(f,g)$ and $\alpha_0^\bullet(f,g) \geq \beta_0^*(f,g)$ (see (2.16), (2.1), and Corollary 2.5 (ii)). Assume that

$$\alpha_n(f,g) \geq \beta_n(f,g), \qquad \alpha_n^\bullet(f,g) \geq \beta_n^*(f,g). \tag{3.18}$$

Then

$$\begin{aligned}
\alpha_{n+1}(f,g) &= \frac{1}{2}(\alpha_n(f,g) + \alpha_n^\bullet(f,g)) \geq \frac{1}{2}(\beta_n(f,g) + \beta_n^*(f,g)) = \beta_{n+1}(f,g), \\
\alpha_{n+1}^\bullet(f,g) &= p_\mu\left(\alpha_n(f,g), \alpha_n^\bullet(f,g); \frac{1}{2}, \frac{1}{2}\right) \geq p_\mu\left(\beta_n(f,g), \beta_n^*(f,g); \frac{1}{2}, \frac{1}{2}\right) \\
&\geq \left(\frac{1}{2}(\beta_n(f,g))^* + \frac{1}{2}(\beta_n^*(f,g))^*\right)^* = \beta_{n+1}^*(f,g)
\end{aligned} \tag{3.19}$$

by virtue of (3.18), Proposition 3.3, [13, Theorem 5.4], and (2.5). Hence (3.18) holds for all n. This implies that

$$f \# g = \tau_0(f,g) = \lim_{n\to\infty} \beta_n(f,g) \leq \lim_{n\to\infty} \alpha_n(f,g) = \tau_\mu(f,g). \tag{3.20}$$

So, we get

$$\tau_{-\mu}(f,g) = (\tau_\mu(f^*,g^*))^* \leq (\tau_0(f^*,g^*))^* = \tau_0(f,g) \tag{3.21}$$

by (3.20) and Proposition 2.1. Therefore, the result follows from (3.16), (3.17), (3.20), (3.21), and Theorem 3.1. □

Corollary 3.5. *Let A and B be two (symmetric) positive definite matrices. Then, for $0 \leq \mu \leq \nu < \infty$, one has*

$$\mathcal{L}_\mu(A,B) \leq \mathcal{L}_\nu(A,B), \tag{3.22}$$

where

$$\mathcal{L}_\mu(A,B) = \left[\frac{1}{2}(A+\mu I) + \frac{1}{2}(B+\mu I)\right] \# \left[\frac{1}{2}(A+\mu I)^{-1} + \frac{1}{2}(B+\mu I)^{-1}\right]^{-1} - \mu I. \tag{3.23}$$

Here # denotes the matrix geometric mean of two positive definite matrices.

Proof. For a positive definite matrix A, define the convex quadratic function

$$q_A(x) = \frac{1}{2}\langle Ax, x\rangle. \tag{3.24}$$

Put $f(x) = q_A(x)$ and $g(x) = q_B(x)$, then q_A and q_B clearly satisfy the constraint qualifications $(CQ1)$–$(CQ3)$. Applying Theorem 2.6 to these functions yields that

$$\begin{aligned}
\tau_\mu(f,g) &= q_{(1/2)(A+\mu I)+(1/2)(B+\mu I)} \ \# \ q_{[(1/2)(A+\mu I)^{-1}+(1/2)(B+\mu I)^{-1}]^{-1}} - \mu q_I \\
&= q_{[(1/2)(A+\mu I)+(1/2)(B+\mu I)]\,\#\,[(1/2)(A+\mu I)^{-1}+(1/2)(B+\mu I)^{-1}]^{-1}} - \mu q_I \\
&= q_{[(1/2)(A+\mu I)+(1/2)(B+\mu I)]\,\#\,[(1/2)(A+\mu I)^{-1}+(1/2)(B+\mu I)^{-1}]^{-1}-\mu I} \\
&= q_{\mathcal{L}_\mu(A,B)},
\end{aligned} \tag{3.25}$$

where the second equality comes from Atteia and Raïssouli [11, Proposition 3.5 (v) and (vii)]. Since $\tau_\mu(f,g) \le \tau_\nu(f,g)$ by Theorem 3.4, we have

$$q_{\mathcal{L}_\mu(A,B)} \le q_{\mathcal{L}_\nu(A,B)}, \quad \text{which is equivalent to } \mathcal{L}_\mu(A,B) \le \mathcal{L}_\nu(A,B). \tag{3.26}$$

□

Remark 3.6. Corollary 3.5 is a particular case of Kim et al. [14, Theorem 3.6] and is based on a different proof using a convex analytic technique in the case of two variables with no weights. To prove the monotonicity of $\mathcal{L}_\mu$ w.r.t. the parameter μ, Kim et al. [14] exploited a well-known variational characterization of the geometric mean of two positive definite matrices.

We close this section with one more observation.

Definition 3.7 (See Bauschke et al. [13, Definition 9.1]). Let g and $(g_k)_k \in \mathbb{N}$ be functions from $\mathbb{R}^n$ to $(\infty, +\infty]$. Then $(g_k)_k \in \mathbb{N}$ *epiconverges to* g, in symbols, $g_k \xrightarrow{e} g$, if the following hold for every $x \in X$:

(i) (for all $(x_k)_{k\in\mathbb{N}}$) $x_k \to x \Rightarrow g(x) \le \liminf g_k(x_k)$,

(ii) $(\exists (y_k)_{k\in\mathbb{N}})$ $y_k \to x$ and $\limsup g_k(y_k) \le g(x)$,

The epitopology is the topology induced by epiconvergence.

Proposition 3.8. *One has*

$$\begin{aligned}
\tau_\mu(f,g) &\xrightarrow{e} \frac{1}{2}(f+g) \quad \text{as } \mu \longrightarrow +\infty, \\
\tau_\mu(f,g) &\xrightarrow{e} \left(\frac{1}{2}(f^*+g^*)\right)^* \quad \text{as } \mu \longrightarrow -\infty.
\end{aligned} \tag{3.27}$$

Proof. By Theorems 3.1 and 3.4 with [16, 7.4 Proposition] or the proof of [13, Corollary 9.6], we can easily get the result. □

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