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Research Article

Existence of Solutions for the Evolution $p(x)$-Laplacian Equation Not in Divergence Form

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The existence of weak solutions is studied to the initial Dirichlet problem of the equation $u_t = u\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$, with $\inf p(x) > 2$. We adopt the method of parabolic regularization. After establishing some necessary uniform estimates on the approximate solutions, we prove the existence of weak solutions.

1. Introduction

In this paper, we investigate the existence of solutions for the $p(x)$-Laplacian equation

$$\frac{\partial u}{\partial t} = u\,\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right), \quad (x,t) \in Q_T. \tag{1.1}$$

The equation is supplemented the boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $Q_T = \Omega \times (0,T)$, $\inf p(x) > 2$, $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $0 \le u_0(x) \in C(\overline{\Omega}) \cap W_0^{1,p(x)}(\Omega)$.

In the case when p is a constant, there have been many results about the existence, localization and extendibility and of weak solutions. We refer the readers to the bibiography given in [1–5] and the references therein.

A new interesting kind of fluids of prominent technological interest has recently emerged, the so-called electrorheological fluids. This model includes parabolic equations which are nonlinear with respect to the gradient of the thought solution, and with variable exponents of nonlinearity. The typical case is the so-called evolution p-Laplace equation with exponent p as a function of the external electromagnetic field (see [6–12] and the references therein). In [6], the authors studied the regularity for the parabolic systems related to a class of non-Newtonian fluids, and the equations involved are nondegenerated.

On the other hand, there are also many results to the corresponding elliptic $p(x)$-Laplace equations [13–15].

In the present work, we will study the existence of the solutions to problem (1.1)–(1.3). As we know, when p is a constant, the nondegenerate problems have classical solutions, and hence the weak solutions exist. But in the case of $p(x)$-Laplace type, there are no results to the corresponding non-degenerate problems. Since (1.1) degenerates whenever $u = 0$ and $\nabla u = 0$, we need to regularize the problem in two aspects corresponding to two different degeneracy: the first is the initial and boundary value and the second is the equation. We will first consider the non-degenerate problems. Based on the uniform Schauder estimates and using the method of continuity, we obtain the existence of classical solutions for non-degenerate problems. After establishing some necessary uniform estimates on the approximate solutions, we prove the existence of weak solutions.

This paper is arranged as follows. We first state some auxiliary lemmas in Section 2, and then we study a general quasilinear equation in Section 3. Subsequently, we discuss the existence of weak solutions in Section 4.

2. Preliminaries

Denote that

$$p_+ = \operatorname{ess}\sup_{\overline{\Omega}} p(x), \qquad p_- = \operatorname{ess}\inf_{\overline{\Omega}} p(x). \tag{2.1}$$

Throughout the paper, we assume that

$$2 < p_- \le p(x) \le p_+ < \infty, \quad \forall (x,t) \in \Omega \times [0,T], \tag{2.2}$$

where p_-, p_+ are given constants.

To study our problems, we need to introduce some new function spaces. Denote that

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

$$|u|_{p(x)} = \inf\left\{ \lambda : \lambda > 0, \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\},$$

$$L^{p(x)}(Q_T) = \left\{ u : |u|^{p(x)} \in L^1(Q_T) \right\},$$

$$L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right) = \left\{u : |u|^{p(x)} \in L^1(Q_T), |\nabla u|^{p(x)} \in L^1(Q_T),\, u|_{\partial\Omega} = 0\right\},$$
$$C^{2k+\alpha,\,k+\alpha/2}(Q_T) = \left\{u : u \in C^{2k,k}(Q_T),\ D_t^r D_x^s u \in C^{\alpha,\alpha/2},\ 2r+s=2k,\ 0<\alpha\le 1\right\},$$
$$C^{2k,\,k}(Q_T) = \left\{u : D_t^r D_x^s u \in C(Q_T),\ 0 \le 2r+s \le 2k\right\},$$
$$W^{1,p(x)}(\Omega) = \left\{u : u \in L^{p(x)}(\Omega),\ |\nabla u| \in L^{p(x)}(\Omega)\right\},$$
$$|u|_{1,p(x)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}, \qquad \forall u \in W^{1,p(x)}(\Omega). \tag{2.3}$$

We use $W_0^{1,p(x)}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}$.

Remark 2.1. In [16, 17], Zhikov showed

$$W_0^{1,p(x)}(\Omega) \ne \left\{v : v \in W^{1,p(x)}(\Omega),\, v|_{\partial\Omega} = 0\right\}. \tag{2.4}$$

Hence, the property of the space is different from the case when p is a constant. This will bring us some difficulties in taking the limit of the weak solutions. Luckily, our approximating solutions are in $W_0^{1,p(x)}$, and hence the limit function is also in $W_0^{1,p(x)}$ which avoids the above difficulties.

We now give the definition of the solutions to our problem.

Definition 2.2. A nonnegative function $u \in L^\infty(Q_T)$, $|\nabla u| \in L^{p(x)}(Q_T)$, and $u_t \in L^2(Q_T)$ is said to be a weak solution of (1.1)–(1.3), if for all $\varphi \in C_0^\infty(\overline{Q}_T)$ satisfies the following:

$$\iint_{Q_T}\left(u\varphi_t - u|\nabla u|^{p(x)-2}\nabla u\nabla\varphi - |\nabla u|^{p(x)}\varphi\right)dx\,dt = 0,$$
$$\lim_{t\to 0}\int_\Omega |u(x,t) - u_0(x)|dx = 0. \tag{2.5}$$

In the following, we state some of the properties of the function spaces introduced as above.

Proposition 2.3 (see [15, 18]).

(i) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p_-} + \frac{1}{q_-}\right)|u|_{p(x)}|v|_{q(x)}. \tag{2.6}$$

(ii) *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(x)} \hookrightarrow L^{p_1(x)}$ and the imbedding continuous.*

(iii) *There is a constant $C > 0$, such that*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{2.7}$$

This implies that $|\nabla u|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms of $W_0^{1,p(x)}$.

(iv) *We have $\int_\Omega |u|^{p(x)}dx \geq |u|_{p(x)}^{p^-} - 1$, for all $u \in L^{p(x)}(\Omega)$.*

Proposition 2.4 (see [18]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}dx, \quad \forall u \in L^{p(x)}(\Omega), \tag{2.8}$$

then

(i) $|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$,

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p_-} \leq \rho(u) \leq |u|_{p(x)}^{p_+}$,

$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p_-} \geq \rho(u) \geq |u|_{p(x)}^{p_+}$,

(iii) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$; $|u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Lemma 2.5 (see [4]). *Let $\theta \geq 0$, $A(\eta) = (A_1(\eta), \ldots, A_N(\eta)) = (|\eta|^2 + \theta)^{(p-2)/2}$, $\eta = (\eta_1, \ldots, \eta_N) \in R^N$. Then*

$$[A(\eta) - A(\eta')] \cdot [\eta - \eta'] \geq C|\eta - \eta'|^p, \quad \forall \eta, \eta' \in R^N, \tag{2.9}$$

where $C = C(p)$ is a positive constant depending on p.

3. A General Quasilinear Equation

Here, we will consider the general quasilinear equations

$$u_t - a_{ij}(x,t,u,u_x)u_{x_i,x_j} + a(x,t,u,u_x) = 0, \tag{3.1}$$

$$u|_{\Gamma_T} = \varphi|_{\Gamma_T}, \tag{3.2}$$

where $\Gamma_T = \partial\Omega \times (0,T] \cup \overline{\Omega} \times \{0\}$.

Proposition 3.1 (see [19, Theorem 2.9 of Chapter I]). *Let $u(x,t)$ be a classical solution of* (3.1) *in Q_T. Suppose that the functions $a_{ij}(x,t,u,p)$ and $a(x,t,u,t)$ take finite value for any finite u, p, and $(x,t) \in \overline{Q}_T$, and that for $(x,t) \in Q_T$ and arbitrary u*

$$\begin{gathered} a_{ij}(x,t,u,0)\xi_i\xi_j \geq 0, \\ ua(x,t,u,0) \geq -b_1u^2 - b_2, \end{gathered} \tag{3.3}$$

where b_1 and b_2 are nonnegative constants. Then

$$\max_{Q_T}|u(x,t)| \le M, \tag{3.4}$$

where M depends only on b_1, b_2, T, and $\max_{\Gamma_T}|u|$.

We suppose that for $(x,t) \in \overline{Q}_T, \max_{Q_T}|u(x,t)| \le M$ and arbitrary q the functions $a_{ij}(x,t,u,q)$, $a(x,t,u,q)$ are continuous in x,t,u,q, continuously differentiable with respect to x,u, and q, and satisfy the inequalities

$$\begin{gathered}
\nu(1+|q|)^{m(x,t)-2}\xi^2 \le a_{ij}(x,t,u,q)\xi_i\xi_j \le \mu_0(1+|q|)^{m(x,t)-2}\xi^2, \\
\left|\frac{\partial a_{ij}}{\partial p_k}\right|(1+|q|)^3 + |a| + \left|\frac{\partial a}{\partial q_k}\right|(1+|q|) \le \mu_1(1+|q|)^{m(x,t)}, \\
\left|\frac{\partial a_{ij}}{\partial x_k}\right|(1+|q|)^2 + \left|\frac{\partial a}{\partial x_k}\right| \le (\varepsilon + P(|q|))(1+|q|)^{m(x,t)+1}, \\
\left|\frac{\partial a_{ij}}{\partial u}\right| \le (\varepsilon + P(|q|))(1+|q|)^{m(x,t)-2}, \\
-\frac{\partial a}{\partial u} \le (\varepsilon + P(|q|))(1+|q|)^{m(x,t)},
\end{gathered} \tag{3.5}$$

where $P(\rho)$ is a nonnegative continuous function that tends to zero for $\rho \to \infty$ and $1 < m(x,t) \in C^1(\overline{Q}_T)$ is an arbitrary function.

Lemma 3.2. *Let $u(x,t)$ be a classical solution of* (3.1) *in Q_T. Suppose that the conditions of Proposition 3.1 hold and satisfy* (3.5) *with a sufficiently small ε determined by the numbers M, ν, μ, μ_1, and*

$$\widehat{P} = \max_{\rho \ge 0} P(\rho). \tag{3.6}$$

Then

$$\max_{Q_T}|u_x(x,t)| \le M_1. \tag{3.7}$$

The proof of Lemma 3.2 is quite similar to the Theorem 4.1, chapter VI of [19]; one only has to replace m with $m(x,t)$ and remark that the constants in the proof are depending only on $\inf m(x,t)$ and $\sup m(x,t)$; we omit the details.

Theorem 3.3. *Suppose that the following conditions are fulfilled.*

(a) *For $(x,t) \in \overline{Q}_T$ and arbitrary u either conditions* (3.3) *are fulfilled.*

(b) *For $(x,t) \in \overline{Q}_T$, $|u| \le M$ (where M is taken from estimate* (3.4)*) and arbitrary p, the functions $a_{ij}(x,t,u,p)$ and $a(x,t,u,p)$ are continuous and differentiable with respect to x,*

u, and p and satisfy inequalities (3.5) *with a sufficiently small ε determined by the numbers M, ν, μ, μ_1, and*

$$\widehat{P} = \max_{\rho \geq 0} P(\rho). \tag{3.8}$$

(c) *For $(x,t) \in \overline{Q}_T$ $|u| \leq M$ and $|p| \leq M_1$ (where M_1 is taken from estimate* (3.7)*), the functions $a_{ij}(x,t,u,p)$ and $a(x,t,u,p)$ are continuously differentiable with respect to all of their arguments.*

(d) *The boundary condition* (3.2) *is given by a function $\psi(x,t)$ belonging to $C^{2+\beta,1+\beta/2}(\overline{Q}_T)$ and satisfying on $S_0 = \{(x,t) : x \in \partial\Omega, t = 0\}$* (3.1)*, that is,*

$$\psi_t - a_{ij}(x,0,\psi(x,0),\psi_x(x,0))\psi_{x_i x_j} + a(x,0,\psi(x,0))\big|_{x \in \partial\Omega} = 0 \tag{3.9}$$

(in other words, the compatibility conditions of zero and first orders are assumed to be fulfilled).

(e) *We have $\partial\Omega \in C^{2+\beta}$.*

Then there exists a unique solution of problem (3.1) *and* (3.2) *in the space $C^{2+\beta,1+\beta/2}(\overline{Q}_T)$. This solution has derivatives u_{tx_i} from $L^2(Q_T)$.*

Proof. We consider problem (3.1) and (3.2) along with a one-parameter family of problems of the same type

$$\begin{aligned} L_\tau u = u_t &- \left[\tau a_{ij}(x,t,u,u_x) + (1-\tau)\left(1+u_x^2\right)^{m(x)/2-1}\delta_i^j\right]u_{x_i x_j} \\ &+ \tau a(x,t,u,u_x) - (1-\tau)\left[\psi_t - \left(1+\psi_x^2\right)^{m(x)/2-1}\Delta\psi\right] = 0, \\ u|_{\Gamma_T} = \psi|_{\Gamma_T}&, \quad 0 \leq \tau \leq 1, \text{ assuming } \psi \in C^{2+\beta,1+\beta/2}\left(\overline{Q}_T\right). \end{aligned} \tag{3.10}$$

Define the Banach space

$$X = \left\{ w \in C^{1+\alpha,(1+\alpha)/2}\left(\overline{Q_T}\right) \,\middle|\, w|_{\Gamma_T} = 0 \right\}. \tag{3.11}$$

For any $w \in X$, let $v = w + \psi$. Using Schauder theory, the linear problem

$$\begin{aligned} v_t &- \left[\tau a_{ij}(x,t,w,w_x) + (1-\tau)\left(1+w_x^2\right)^{m(x)/2-1}\delta_i^j\right]v_{x_i x_j} \\ &+ \tau a(x,t,w,w_x) - (1-\tau)\left[\psi_t - \left(1+\psi_x^2\right)^{m(x)/2-1}\Delta\psi\right] = 0, \\ u|_{\Gamma_T} &= \psi|_{\Gamma_T} \end{aligned} \tag{3.12}$$

admits a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$. Let $z = u - \psi$, clearly $z \in X$, and define the map $G : X \to X$ such that $z = G(w)$. By [19], we know that G is continuous and compact.

By Proposition 3.1, Lemma 3.2, and the Leray-Schauder fixed point principle, the operator G has a fixed point u. □

4. Existence

In this section, we are going to prove the existence of solutions of the problem (1.1)–(1.3).

Theorem 4.1. *Assume that* $p(x) \in C^1(\overline{\Omega})$, $\inf p(x) > 2$, *and* $0 \le u_0(x) \in C(\overline{\Omega}) \cap W_0^{1,p(x)}(\Omega)$. *Then the problem* (1.1)–(1.3) *admits a weak solution* u.

Consider the following problem:

$$\frac{\partial u_{\varepsilon,\eta}}{\partial t} = u_{\varepsilon,\eta} \operatorname{div}\left(\left(\left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta}\right), \tag{4.1}$$

$$u_{\varepsilon,\eta}\big|_{S_T} = \varepsilon, \qquad u_{\varepsilon,\eta}\big|_{t=0} = u_0 + \varepsilon, \tag{4.2}$$

where $S_T = \partial\Omega \times (0,T]$, $\varepsilon \in (0,1)$, and $\eta \in (0,\varepsilon)$. Roughly speaking, here we use to regularize the initial-boundary value and use η to regularize the equation. Thus, we have to carry out two limit processes, that is, first let $\eta \to 0$ (along a certain subsequence) and then let $\varepsilon \to 0$.

We first change (4.1) into the form

$$u_t - a_{ij}(x,t,u,u_x)u_{x_i,x_j} + a(x,t,u,u_x) = 0, \tag{4.3}$$

where

$$\begin{aligned} a_{ij}(x,t,u,u_x) &= u\delta_{ij}\left(|\nabla u|^2 + \eta\right)^{(p(x)-2)/2} + u(p(x)-2)\left(|\nabla u|^2 + \eta\right)^{(p(x)-4)/2} \partial_i u\, \partial_j u, \\ a(x,t,u,u_x) &= u\frac{1}{2}\left(|\nabla u|^2 + \eta\right)^{(p(x)-2)/2} \partial_i p(x) \partial_i u \ln\left(|\nabla u|^2 + \eta\right). \end{aligned} \tag{4.4}$$

It is easily seen that (4.4) satisfies (3.3) and (3.5), where $p(x)$ instead of $m(x,t)$. By Theorem 3.3, we know that (4.1)-(4.2) has a classical solution $u_{\varepsilon,\eta}$.

Proposition 4.2. *We have*

$$\varepsilon \le u_{\varepsilon,\eta} \le |u_0|_\infty + \varepsilon, \qquad u_{\varepsilon_1,\eta} \le u_{\varepsilon_2,\eta}, \quad \textit{for } \varepsilon_1 \le \varepsilon_2. \tag{4.5}$$

Proof. By the maximum principle, we know that $\varepsilon \le u_{\varepsilon,\eta} \le |u_0|_\infty + \varepsilon$.

A simple calculation shows that

$$\begin{aligned} &\frac{\partial u_{\varepsilon_1,\eta}}{\partial t} - a_{ij}\left(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta}\right)\frac{\partial^2 u_{\varepsilon_1,\eta}}{\partial x_i\,\partial x_j} + a\left(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta}\right) = 0, \\ &\frac{\partial u_{\varepsilon_2,\eta}}{\partial t} - a_{ij}\left(x,t,u_{\varepsilon_2,\eta},\nabla u_{\varepsilon_2,\eta}\right)\frac{\partial^2 u_{\varepsilon_2,\eta}}{\partial x_i\,\partial x_j} + a\left(x,t,u_{\varepsilon_2,\eta},\nabla u_{\varepsilon_2,\eta}\right) = 0, \end{aligned} \tag{4.6}$$

where $a_{ij}(x,t,u,u_x)$ and $a(x,t,u,u_x)$ are defined as (4.4).

It is easy to prove that

$$\xi_i a_{ij}(x,t,u_{\varepsilon_i,\eta},\nabla u_{\varepsilon_i,\eta})\xi_j \geq \varepsilon_i \eta^{(p(x)-2)/2}|\xi|^2, \quad \forall \xi \in R^N. \tag{4.7}$$

Hence, we have

$$\begin{aligned}\frac{\partial u_{\varepsilon_1,\eta}}{\partial t}-\frac{\partial u_{\varepsilon_2,\eta}}{\partial t} &= a_{ij}(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta})\frac{\partial^2(u_{\varepsilon_1,\eta}-u_{\varepsilon_2,\eta})}{\partial x_i\,\partial x_j}\\ &\quad+\left[a_{ij}(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta})-a_{ij}(x,t,u_{\varepsilon_2,\eta},\nabla u_{\varepsilon_2,\eta})\right]\frac{\partial^2 u_{\varepsilon_2,\eta}}{\partial x_i\,\partial x_j}\\ &\quad+\left[a(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta})-a(x,t,u_{\varepsilon_2,\eta},\nabla u_{\varepsilon_2,\eta})\right].\end{aligned} \tag{4.8}$$

Let $w=u_{\varepsilon_1,\eta}-u_{\varepsilon_2,\eta}$, $a_{ij}(x,t)=a_{ij}(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta})$. Using the mean value theorem, we have

$$\begin{aligned}&\left[a_{ij}(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta})-a_{ij}(x,t,u_{\varepsilon_2,\eta},\nabla u_{\varepsilon_2,\eta})\right]\frac{\partial^2 u_{\varepsilon_2,\eta}}{\partial x_i\,\partial x_j}\\ &\quad=\left[a_{ij}(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta})-a_{ij}(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_2,\eta})\right]\frac{\partial^2 u_{\varepsilon_2,\eta}}{\partial x_i\,\partial x_j}\\ &\qquad+\left[a_{ij}(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_2,\eta})-a_{ij}(x,t,u_{\varepsilon_2,\eta},\nabla u_{\varepsilon_2,\eta})\right]\frac{\partial^2 u_{\varepsilon_2,\eta}}{\partial x_i\,\partial x_j}\\ &\quad=d_k(x,t)\frac{\partial w}{\partial x_k}+e(x,t)w.\end{aligned} \tag{4.9}$$

Similarly, we get

$$a(x,t,u_{\varepsilon_1,\eta},\nabla u_{\varepsilon_1,\eta})-a(x,t,u_{\varepsilon_2,\eta},\nabla u_{\varepsilon_2,\eta})=f_k(x,t)\frac{\partial w}{\partial x_k}+g(x,t)w, \tag{4.10}$$

where $d_k(x,t), e(x,t), f_k(x,t)$, and $g(x,t)$ are bounded functions. Hence, we see that

$$\frac{\partial w}{\partial t}=a_{ij}(x,t)\frac{\partial^2 w}{\partial x_i\,\partial x_j}+b_k(x,t)\frac{\partial w}{\partial x_k}+c(x,t)w. \tag{4.11}$$

Since $w \leq 0$ on Γ_T, by comparison principle of linear parabolic equation, we have $w \leq 0$. □

Lemma 4.3. *For all $\alpha \in [0,1)$ and $\eta \in (0,\varepsilon)$, there hold*

$$\begin{aligned}&(1)\quad \iint_{Q_T}\frac{\left(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\right)^{p(x)/2}}{u_{\varepsilon,\eta}^{\alpha}}dx\,dt \leq C,\\ &(2)\quad \iint_{Q_T}\left(\frac{\partial u_{\varepsilon,\eta}}{\partial t}\right)^2 dx\,dt \leq C.\end{aligned} \tag{4.12}$$

Proof. Multiplying (4.1) by $u_{\varepsilon,\eta}^{-\alpha}$, integrating both sides of the equality over Q_T and integrating by parts, we derive

$$\begin{aligned}\iint_{Q_T}\frac{\partial u_{\varepsilon,\eta}}{\partial t}u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt &= \frac{1}{1-\alpha}\int_\Omega\Big(u_{\varepsilon,\eta}^{1-\alpha}(x,T)-u_{\varepsilon,\eta}^{1-\alpha}(x,0)\Big)dx\\ &=\iint_{Q_T}u_{\varepsilon,\eta}^{1-\alpha}\operatorname{div}\Big(\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}\nabla u_{\varepsilon,\eta}\Big)dx\,dt\\ &=\int_0^T\int_{\partial\Omega}\left[u_{\varepsilon,\eta}^{1-\alpha}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}\frac{\partial u_{\varepsilon,\eta}}{\partial\nu}\right]d\sigma\,dt\\ &\quad-(1-\alpha)\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}\left|\nabla u_{\varepsilon,\eta}\right|^2u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt,\end{aligned}\tag{4.13}$$

where ν denotes the outward normal to $\partial\Omega\times(0,T)$. Since from (4.5), $u_{\varepsilon,\eta}\ge\varepsilon$, we have $\partial u_{\varepsilon,\eta}/\partial\nu\le 0$ on $\partial\Omega\times(0,T)$. Hence

$$\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}\left|\nabla u_{\varepsilon,\eta}\right|^2u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt\le\frac{1}{(1-\alpha)^2}\int_\Omega u_{\varepsilon,\eta}^{1-\alpha}(x,0)dx\le C,\tag{4.14}$$

where $C=C(\alpha,\Omega,|u_0|_\infty)$.

Using $u_{\varepsilon,\eta}\ge\varepsilon>\eta$ and Young's inequality, we have

$$\begin{aligned}&\eta\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt\\ &\quad\le\iint_{Q_T}\frac{\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}}{u_{\varepsilon,\eta}^{(p(x)-2)\alpha/p(x)}}\frac{\eta^{2\alpha/p(x)}}{u_{\varepsilon,\eta}^{2\alpha/p(x)}}dx\,dt\\ &\quad\le\frac{1}{2}\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{p(x)/2}u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt+C(p_-,p_+,T,\Omega).\end{aligned}\tag{4.15}$$

Combining (4.14) with (4.15) yields

$$\begin{aligned}\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{p(x)/2}u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt&=\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}u_{\varepsilon,\eta}^{-\alpha}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)dx\,dt\\ &\le\eta\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{(p(x)-2)/2}u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt+C\\ &\le\frac{1}{2}\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{p(x)/2}u_{\varepsilon,\eta}^{-\alpha}\,dx\,dt+C(p,T,\Omega).\end{aligned}\tag{4.16}$$

Hence,

$$\iint_{Q_T}\Big(\left|\nabla u_{\varepsilon,\eta}\right|^2+\eta\Big)^{p(x)/2}u_{\varepsilon,\eta}^{-\alpha}dx\,dt\le C.\tag{4.17}$$

Multiplying (4.1) by $(\partial u_{\varepsilon,\eta}/\partial t)\ u_{\varepsilon,\eta}^{-1}$, integrating both sides of the equality over Q_T and integrating by parts and noticing that $(u_{\varepsilon,\eta})_t = 0$ on $\partial\Omega \times (0,T)$, we derive

$$\begin{aligned}
&\iint_{Q_T} \left(\frac{\partial u_{\varepsilon,\eta}}{\partial t}\right)^2 u_{\varepsilon,\eta}^{-1} dx\, dt \\
&\quad= \iint_{Q_T} \operatorname{div}\left(\left(|\nabla u_{\varepsilon,\eta}|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta}\right) \frac{\partial u_{\varepsilon,\eta}}{\partial t} dx\, dt \\
&\quad= \iint_{Q_T} \operatorname{div}\left(\left(|\nabla u_{\varepsilon,\eta}|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta} \frac{\partial u_{\varepsilon,\eta}}{\partial t}\right) dx\, dt \\
&\qquad - \iint_{Q_T} \left(\left(|\nabla u_{\varepsilon,\eta}|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta}\right) \nabla\left(\frac{\partial u_{\varepsilon,\eta}}{\partial t}\right) dx\, dt \\
&\quad= -\iint_{Q_T} \left(\left(|\nabla u_{\varepsilon,\eta}|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta}\right) \nabla\left(\frac{\partial u_{\varepsilon,\eta}}{\partial t}\right) dx\, dt \\
&\quad= -\int_\Omega \frac{1}{p(x)} \left(|\nabla u_{\varepsilon,\eta}(x,T)|^2 + \eta\right)^{p(x)/2} dx \\
&\qquad + \int_\Omega \frac{1}{p(x)} \left(|\nabla u_{\varepsilon,\eta}(x,0)|^2 + \eta\right)^{p(x)/2} dx \\
&\quad\le \int_\Omega \frac{1}{p(x)} \left(|\nabla u_0|^2 + \eta\right)^{p(x)/2} dx.
\end{aligned} \tag{4.18}$$

□

Equation (4.5), Lemma 4.3, and Proposition 2.3 imply that, for any $\varepsilon \in (0,1)$, there exists a subsequence of $u_{\varepsilon,\eta}$, denoted by u_{ε,η_k}, and a function $u_\varepsilon \in L^\infty(Q_T)\ \nabla u \in L^{p(x)}(Q_T)$, such that, as $\eta = \eta_k \to 0$,

$$u_{\varepsilon,\eta} \longrightarrow u_\varepsilon, \quad \text{a.e. in } Q_T, \tag{4.19}$$

$$\nabla u_{\varepsilon,\eta} \longrightarrow \nabla u_\varepsilon, \quad \text{weakly in } L^{p(x)}(Q_T), \tag{4.20}$$

$$\frac{\partial u_{\varepsilon,\eta}}{\partial t} \longrightarrow \frac{\partial u_\varepsilon}{\partial t}, \quad \text{weakly in } L^2(Q_T). \tag{4.21}$$

Lemma 4.4. *As $\eta = \eta_k \to 0$, we have*

$$\begin{aligned}
&(1)\quad \iint_{Q_T} |\nabla u_{\varepsilon,\eta} - \nabla u_\varepsilon|^{p(x)} dx\, dt \longrightarrow 0, \\
&(2)\quad \iint_{Q_T} \left| |\nabla u_{\varepsilon,\eta}|^{p(x)} - |\nabla u_\varepsilon|^{p(x)} \right| dx\, dt \longrightarrow 0, \\
&(3)\quad \iint_{Q_T} \left| \left(|\nabla u_{\varepsilon,\eta}|^2 + \eta\right)^{(p(x)-2)/2} |\nabla u_{\varepsilon,\eta}|^2 - |\nabla u_\varepsilon|^{p(x)} \right| dx\, dt \longrightarrow 0, \\
&(4)\quad \iint_{Q_T} \left| u_{\varepsilon,\eta}\left(|\nabla u_{\varepsilon,\eta}|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta} - u_\varepsilon |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \right| dx\, dt \longrightarrow 0.
\end{aligned} \tag{4.22}$$

Proof. Observe that $(u_{\varepsilon,\eta} - u_\varepsilon)/u_{\varepsilon,\eta} \in L^{p(x)}(0,T;W_0^{1,p(x)}(\Omega))$. Multiplying (4.1) by $(u_{\varepsilon,\eta} - u_\varepsilon)/u_{\varepsilon,\eta}$, integrating both sides of the equality over Q_T and integrating by parts, we derive

$$\iint_{Q_T}\left(\frac{\partial u_{\varepsilon,\eta}}{\partial t}\frac{u_{\varepsilon,\eta}-u_\varepsilon}{u_{\varepsilon,\eta}} + \left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}\nabla u_{\varepsilon,\eta}\nabla(u_{\varepsilon,\eta}-u_\varepsilon)\right)dx\,dt = 0. \tag{4.23}$$

By Hölder inequality and Lemma 4.3, we obtain

$$\iint_{Q_T}\frac{\partial u_{\varepsilon,\eta}}{\partial t}\frac{u_{\varepsilon,\eta}-u_\varepsilon}{u_{\varepsilon,\eta}} \longrightarrow 0, \qquad (\eta\longrightarrow 0). \tag{4.24}$$

Hence,

$$\iint_{Q_T}\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}\nabla u_{\varepsilon,\eta}\nabla(u_{\varepsilon,\eta}-u_\varepsilon)dx\,dt \longrightarrow 0. \tag{4.25}$$

We divide the integral in (4.25) in the following way:

$$\begin{aligned}
&\iint_{Q_T}\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}\nabla u_{\varepsilon,\eta}\nabla(u_{\varepsilon,\eta}-u_\varepsilon)dx\,dt\\
&\quad=\iint_{Q_T}\left[\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}\nabla u_{\varepsilon,\eta}-\left(|\nabla u_\varepsilon|^2+\eta\right)^{(p(x)-2)/2}\nabla u_\varepsilon\right]\cdot\nabla(u_{\varepsilon,\eta}-u_\varepsilon)dx\,dt\\
&\qquad+\iint_{Q_T}\left[\left(|\nabla u_\varepsilon|^2+\eta\right)^{(p(x)-2)/2}-|\nabla u_\varepsilon|^{(p(x)-2)}\right]\nabla u_\varepsilon\nabla u_{\varepsilon,\eta}dx\,dt\\
&\qquad+\iint_{Q_T}\left[|\nabla u_\varepsilon|^{(p(x)-2)}-\left(|\nabla u_\varepsilon|^2+\eta\right)^{(p(x)-2)/2}\right]|\nabla u_\varepsilon|^2dx\,dt\\
&\qquad+\iint_{Q_T}|\nabla u_\varepsilon|^{(p(x)-2)}\nabla u_\varepsilon\nabla(u_{\varepsilon,\eta}-u_\varepsilon)dx\,dt\\
&\quad=I_1+I_2+I_3+I_4.
\end{aligned} \tag{4.26}$$

From (4.20), we see that

$$I_4\longrightarrow 0, \qquad (\eta\longrightarrow 0). \tag{4.27}$$

Using Lemma 4.3, we have

$$\begin{aligned}
|I_3| &\le \iint_{Q_T}\left[\left(\nabla u_\varepsilon|^2+\eta\right)^{p(x)/2}-|\nabla u_\varepsilon|^{p(x)}\right]dx\,dt\\
&\le\frac{\eta}{2}\iint_{Q_T}p(x)\left(\nabla u_\varepsilon|^2+\eta\right)^{(p(x)-2)/2}dx\,dt\longrightarrow 0, \qquad (\eta\longrightarrow 0).
\end{aligned} \tag{4.28}$$

Now we estimate I_2. If $p(x) \in (2,3]$, then $(p(x)-1)/2 \in (0,1]$. Using $|a^r - b^r| \leq |a-b|^r$ ($r \in [0,1], a, b \geq 0$) gives

$$|I_2| \leq \iint_{Q_T} \left| \left(|\nabla u_\varepsilon|^2 + \eta\right)^{(p(x)-1)/2} - |\nabla u_\varepsilon|^{p(x)-1} \right| \left|\nabla u_{\varepsilon,\eta}\right| dx\,dt$$
$$\leq \eta^{(p_- -1)/2} \iint_{Q_T} \left|\nabla u_{\varepsilon,\eta}\right| dx\,dt \longrightarrow 0, \qquad (\eta \longrightarrow 0). \tag{4.29}$$

If $p(x) > 3$, we obtain

$$|I_2| \leq \frac{\eta}{2} \iint_{Q_T} (p(x)-1)\left(|\nabla u_\varepsilon|^2 + \eta\right)^{(p-3)/2} \left|\nabla u_{\varepsilon,\eta}\right| dx\,dt \longrightarrow 0, \quad (\eta \longrightarrow 0). \tag{4.30}$$

By (4.25), (4.26), and $I_2, I_3, I_4 \to 0$, we obtain

$$I_1 = \iint_{Q_T} \left[\left(\left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta} - \left(|\nabla u_\varepsilon|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_\varepsilon \right] \cdot \nabla\left(u_{\varepsilon,\eta} - u_\varepsilon\right) dx\,dt \longrightarrow 0,$$
$$(\eta \longrightarrow 0). \tag{4.31}$$

Again by Lemma 2.5, we get

$$I_1 = \iint_{Q_T} \left[\left(\left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta} - \left(|\nabla u_\varepsilon|^2 + \eta\right)^{(p(x)-2)/2} \nabla u_\varepsilon \right] \cdot \nabla\left(u_{\varepsilon,\eta} - u_\varepsilon\right) dx\,dt$$
$$\geq C \iint_{Q_T} \left|\nabla u_{\varepsilon,\eta} - \nabla u_\varepsilon\right|^{p(x)} dx\,dt. \tag{4.32}$$

Letting $\eta \to 0$, we obtain (1). Again noticing that

$$\iint_{Q_T} \left| \left|\nabla u_{\varepsilon,\eta}\right|^{p(x)} - |\nabla u_\varepsilon|^{p(x)} \right| dx\,dt$$
$$\leq \iint_{Q_T} p(x)\left(\left|\nabla u_{\varepsilon,\eta}\right| + |\nabla u_\varepsilon|\right)^{p(x)-1} \left| \left|\nabla u_{\varepsilon,\eta}\right| - |\nabla u_\varepsilon| \right| dx\,dt$$
$$\leq \iint_{Q_T} p(x)\left(\left|\nabla u_{\varepsilon,\eta}\right| + |\nabla u_\varepsilon|\right)^{p-1} \left|\nabla u_{\varepsilon,\eta} - \nabla u_\varepsilon\right| dx\,dt \tag{4.33}$$
$$\leq C p_+ \left| \left|\nabla u_{\varepsilon,\eta}\right| + |\nabla u_\varepsilon| \right|_{p(x)} \left|\nabla u_{\varepsilon,\eta} - \nabla u_\varepsilon\right|_{p(x)}$$
$$\leq C \left|\nabla u_{\varepsilon,\eta} - \nabla u_\varepsilon\right|_{p(x)},$$

by Proposition 2.4, we see that (2) holds. To prove (3), we have

$$\begin{aligned}
&\iint_{Q_T}\left|\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}|\nabla u_{\varepsilon,\eta}|^2-|\nabla u_\varepsilon|^{p(x)}\right|dx\,dt\\
&\quad\le\iint_{Q_T}\left|\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}-|\nabla u_{\varepsilon,\eta}|^{p(x)-2}\right||\nabla u_{\varepsilon,\eta}|^2dx\,dt\\
&\qquad+\iint_{Q_T}\left||\nabla u_{\varepsilon,\eta}|^{p(x)}-|\nabla u_\varepsilon|^{p(x)}\right|dx\,dt\\
&\quad\le\iint_{Q_T}\left|\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{p(x)/2}-|\nabla u_{\varepsilon,\eta}|^{p(x)}\right|dx\,dt+\iint_{Q_T}\left||\nabla u_{\varepsilon,\eta}|^{p(x)}-|\nabla u_\varepsilon|^{p(x)}\right|dx\,dt\\
&\quad\le\frac{\eta}{2}\iint_{Q_T}p(x)\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}dx\,dt+\iint_{Q_T}\left||\nabla u_{\varepsilon,\eta}|^{p(x)}-|\nabla u_\varepsilon|^{p(x)}\right|dx\,dt.
\end{aligned}\tag{4.34}$$

Using Lemma 4.3 and (2), we see that (3) holds.

Finally, we prove (4). We have

$$\begin{aligned}
&\iint_{Q_T}\left|u_{\varepsilon,\eta}\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}\nabla u_{\varepsilon,\eta}-u_\varepsilon|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\right|dx\,dt\\
&\quad\le\iint_{Q_T}|u_{\varepsilon,\eta}-u_\varepsilon|\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}|\nabla u_{\varepsilon,\eta}|dx\,dt\\
&\qquad+\iint_{Q_T}u_\varepsilon\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}|\nabla u_{\varepsilon,\eta}-\nabla u_\varepsilon|dx\,dt\\
&\qquad+\iint_{Q_T}u_\varepsilon\left|\left(|\nabla u_{\varepsilon,\eta}|^2+\eta\right)^{(p(x)-2)/2}-|\nabla u_\varepsilon|^{p(x)-2}\right||\nabla u_\varepsilon|dx\,dt\\
&\quad=I_a+I_b+I_c.
\end{aligned}\tag{4.35}$$

Equation (4.19) implies that

$$I_a\longrightarrow 0,\qquad(\eta\longrightarrow 0).\tag{4.36}$$

To estimate I_b, notice that

$$I_b\le C\left||\nabla u_{\varepsilon,\eta}|^2+\eta\right|_{(p(x)-2)p(x)/2(p(x)-1)}|\nabla u_{\varepsilon,\eta}-\nabla u_\varepsilon|_{p(x)}\longrightarrow 0,\qquad(\eta\longrightarrow 0).\tag{4.37}$$

As $p(x) \in (2,3]$, we have $(p(x)-2)/2 \in (0,1]$,

$$\begin{aligned}
I_c &\le C\iint_{Q_T} \left| \left|\nabla u_{\varepsilon,\eta}\right|^2 - |\nabla u_\varepsilon|^2 + \eta \right|^{(p(x)-2)/2} |\nabla u_\varepsilon| dx\, dt \\
&\le C\iint_{Q_T} 2^{(p(x)-2)/2}\left(\left| \left|\nabla u_{\varepsilon,\eta}\right|^2 - |\nabla u_\varepsilon|^2 \right|^{(p(x)-2)/2} + \eta\right|^{(p-2)/2} \Big) |\nabla u_\varepsilon| dx\, dt \\
&\le C\iint_{Q_T} 2^{(p(x)-2)/2} \left|\nabla u_{\varepsilon,\eta} - \nabla u_\varepsilon\right|^{(p(x)-2)/2} \left(\left|\nabla u_{\varepsilon,\eta}\right| + |\nabla u_\varepsilon|\right)^{p(x)/2} dx\, dt \\
&\quad + C\iint_{Q_T} (2\eta)^{(p(x)-2)/2} |\nabla u_\varepsilon| dx\, dt.
\end{aligned} \tag{4.38}$$

By Hölder inequality, we have

$$I_c \longrightarrow 0, \qquad (\eta \longrightarrow 0). \tag{4.39}$$

If $p(x) > 3$, we have

$$\begin{aligned}
I_c &\le C\iint_{Q_T} (p(x)-2)\left(\left(\left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta\right)^{(p-3)/2} + |\nabla u_\varepsilon|^{p(x)-3} \right) \\
&\qquad \cdot \left| \left(\left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta\right)^{1/2} - |\nabla u_\varepsilon| \right| |\nabla u_\varepsilon| dx\, dt \\
&\le C\iint_{Q_T} (p(x)-2)\left(\left(\left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta\right)^{(p-3)/2} + |\nabla u_\varepsilon|^{p(x)-3} \right) \\
&\qquad \cdot \left| \left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta - |\nabla u_\varepsilon|^2 \right|^{1/2} |\nabla u_\varepsilon|\ dx\, dt \\
&\le C2^{1/2}\iint_{Q_T} (p(x)-2)\left(\left(\left|\nabla u_{\varepsilon,\eta}\right|^2 + \eta\right)^{(p-3)/2} + |\nabla u_\varepsilon|^{p(x)-3} \right) \\
&\qquad \cdot \left(\left|\nabla u_{\varepsilon,\eta}\right| + |\nabla u_\varepsilon| \right)^{3/2} \left|\nabla \left(u_{\varepsilon,\eta} - u_\varepsilon\right)\right| dx\, dt \\
&\quad + C(2\eta)^{1/2}\iint_{Q_T} (p(x)-2)\left(\left(|\nabla u_\varepsilon|^2 + \eta\right)^{(p(x)-3)/2} + |\nabla u_\varepsilon|^{p(x)-3} \right) |\nabla u_\varepsilon| dx\, dt.
\end{aligned} \tag{4.40}$$

Hence,

$$I_c \longrightarrow 0, \qquad (\eta \longrightarrow 0). \tag{4.41}$$

Thus (4) is proved, and the proof of Lemma 4.4 is complete. □

Proposition 4.5. *We obtain that u_ε is a weak solution of the problem*

$$\begin{gathered}
\frac{\partial u}{\partial t} = u \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u \right), \\
u|_{S_T} = \varepsilon, \qquad u|_{t=0} = u_0 + \varepsilon,
\end{gathered} \tag{4.42}$$

then

$$\varepsilon \leq u_\varepsilon \leq |u_0|_\infty + \varepsilon, \qquad u_{\varepsilon_1} \leq u_{\varepsilon_2}, \qquad (\varepsilon_1 \leq \varepsilon_2), \qquad \text{a.e. in } Q_T, \tag{4.43}$$

$$\iint_{Q_T} |\nabla u_\varepsilon|^p dx\,dt \leq C, \qquad \iint_{Q_T} \left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 dx\,dt \leq C, \tag{4.44}$$

where C is independent of ε.

Proof. Obviously, for all $\varepsilon \in (0,1)$, $u_\varepsilon - \varepsilon \in L^{p(x)}(0,T;W^{1,p(x)}(\Omega))$. By Proposition 4.2 and (4.19)–(4.21), we know that (4.43) holds. (4.44) follows from (4.5), (4.19)–(4.21), and Lemma 4.3. To prove that u satisfies the integral equality in Definition 2.2, we multiply (4.1) by $\varphi \in C_0^\infty(Q_T)$, integrate both sides of the equality on Q_T, and integrate by parts to derive

$$\begin{aligned} &\iint_{Q_T} \Big[-u_{\varepsilon,\eta}\varphi_t + u_{\varepsilon,\eta}\Big(\big|\nabla u_{\varepsilon,\eta}\big|^2 + \eta\Big)^{(p(x)-2)/2} \nabla u_{\varepsilon,\eta} \nabla\varphi \\ &\quad + \Big(\big|\nabla u_{\varepsilon,\eta}\big|^2 + \eta\Big)^{(p(x)-2)/2} \big|\nabla u_{\varepsilon,\eta}\big|^2 \varphi\Big] dx\,dt = 0. \end{aligned} \tag{4.45}$$

Letting $\eta = \eta_k \to 0$ to pass to limit and using (4.19) and Lemma 4.4 show that

$$\iint_{Q_T} \Big[-u_\varepsilon\varphi_t + u_\varepsilon|\nabla u_\varepsilon|^{(p(x)-2)} \nabla u_\varepsilon \nabla\varphi + |\nabla u_\varepsilon|^{p(x)}\varphi\Big] dx\,dt = 0. \tag{4.46}$$

Applying Lemma 4.3, we derive

$$\int_\Omega \big|u_{\varepsilon,\eta} - u_0(x) - \varepsilon\big| dx \leq Ct^{1/2}, \tag{4.47}$$

where C is independent of ε and η. Hence,

$$\int_\Omega |u_\varepsilon - u_0(x) - \varepsilon| dx \longrightarrow 0, \qquad (t \longrightarrow 0). \tag{4.48}$$

□

From (4.43), we see that u is bounded and increasing in ε, which implies the existence of a function u, such that, as $\varepsilon \to 0$,

$$u_\varepsilon \longrightarrow u, \qquad \text{a.e. in } Q_T, \tag{4.49}$$

$$\nabla u_\varepsilon \longrightarrow \nabla u, \qquad \text{weakly in } L^{p(x)}(Q_T), \tag{4.50}$$

$$\frac{\partial u_\varepsilon}{\partial t} \longrightarrow \frac{\partial u}{\partial t}, \qquad \text{weakly in } L^2(Q_T). \tag{4.51}$$

Lemma 4.6. *As $\varepsilon \longrightarrow 0$, we have*

$$
\begin{aligned}
&(1) \quad \iint_{Q_T} \left|\nabla u_\varepsilon^2 - \nabla u^2\right|^{p(x)} dx\,dt \longrightarrow 0,\\
&(2) \quad \iint_{Q_c^\varepsilon} |\nabla u_\varepsilon - \nabla u|^{p(x)} dx\,dt \longrightarrow 0, \qquad Q_c^\varepsilon = \{(x,t) \in Q_T, u_\varepsilon \geq c, c > 0\},\\
&(3) \quad \iint_{Q_c} |\nabla u_\varepsilon - \nabla u|^{p(x)} dx\,dt \longrightarrow 0, \qquad Q_c = \{(x,t) \in Q_T, u \geq c, c > 0\}.
\end{aligned}
\tag{4.52}
$$

Proof. We may take $\varphi = u_\varepsilon^{p(x)-2}(u_\varepsilon^2 - \varepsilon^2 - u^2)$, in the integral equality satisfied by u. Then it is easy to see that

$$
\begin{aligned}
&\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\,dt\\
&\quad = -\iint_{Q_T} u_\varepsilon |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla\left[u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right)\right] dx\,dt\\
&\qquad - \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\,dt\\
&\quad = -\iint_{Q_T} (p(x)-1)|\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - u^2\right) dx\,dt\\
&\qquad + \varepsilon^2 \iint_{Q_T} (p(x)-1)|\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{p(x)-2} dx\,dt\\
&\qquad - \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla\left(u_\varepsilon^2 - u^2\right) u_\varepsilon^{p(x)-1} dx\,dt\\
&\qquad - \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon u_\varepsilon^{p(x)-1} \nabla p(x) \ln u_\varepsilon \left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\,dt.
\end{aligned}
\tag{4.53}
$$

Hence, by (4.43), we have

$$
\begin{aligned}
&\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\,dt\\
&\quad \leq \varepsilon^2 \iint_{Q_T} (p(x)-1)|\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{p(x)-2} dx\,dt\\
&\qquad - \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla\left(u_\varepsilon^2 - u^2\right) u_\varepsilon^{p(x)-1} dx\,dt\\
&\qquad - \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon u_\varepsilon^{p(x)-1} \nabla p(x) \ln u_\varepsilon \left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\,dt.
\end{aligned}
\tag{4.54}
$$

Notice that

$$\iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla \left(u_\varepsilon^2 - u^2\right) u_\varepsilon^{p(x)-1} dx\, dt = \iint_{Q_T} \left(\frac{1}{2}\right)^{p(x)-1} \left|\nabla u_\varepsilon^2\right|^{p-2} \nabla u_\varepsilon^2 \nabla \left(u_\varepsilon^2 - u^2\right) dx\, dt. \tag{4.55}$$

So

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\, dt + \iint_{Q_T} \left(\frac{1}{2}\right)^{p(x)-1} \left|\nabla u_\varepsilon^2\right|^{p(x)-2} \nabla u_\varepsilon^2 \nabla \left(u_\varepsilon^2 - u^2\right) dx\, dt \\
&\quad \le \varepsilon^2 \iint_{Q_T} (p(x)-1) |\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{p(x)-2} dx\, dt \\
&\qquad - \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon u_\varepsilon^{p(x)-1} \nabla p(x) \ln u_\varepsilon \left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\, dt.
\end{aligned} \tag{4.56}$$

Hence,

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\, dt \\
&\quad + \iint_{Q_T} \left(\frac{1}{2}\right)^{p(x)-1} \left[\left|\nabla u_\varepsilon^2\right|^{p(x)-2} \nabla u_\varepsilon^2 - \left|\nabla u^2\right|^{p(x)-2} \nabla u^2\right] \nabla \left(u_\varepsilon^2 - u^2\right) dx\, dt \\
&\le -\iint_{Q_T} \left(\frac{1}{2}\right)^{p(x)-1} \left|\nabla u^2\right|^{p(x)-2} \nabla u^2 \nabla \left(u_\varepsilon^2 - u^2\right) dx\, dt \\
&\quad + \varepsilon^2 \iint_{Q_T} (p(x)-1) |\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{p(x)-2} dx\, dt \\
&\quad - \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon u_\varepsilon^{p(x)-1} \nabla p(x) \ln u_\varepsilon \left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\, dt \\
&\le -\iint_{Q_T} \left(\frac{1}{2}\right)^{p(x)-1} \left|\nabla u^2\right|^{p(x)-2} \nabla u^2 \nabla \left(u_\varepsilon^2 - u^2\right) dx\, dt \\
&\quad + \varepsilon^2 \iint_{Q_T} (p(x)-1) |\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{p(x)-2} dx\, dt \\
&\quad + C \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-1} u_\varepsilon^{p(x)} \left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\, dt.
\end{aligned} \tag{4.57}$$

By (4.43), (4.44), and (4.49), we see that as $\varepsilon \to 0$,

$$\begin{gathered}\nabla u_\varepsilon^2 \longrightarrow \nabla u^2, \text{ weakly in } L^{p(x)}(Q_T), \\ \iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p(x)-2}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\, dt \longrightarrow 0, \\ \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-1} u_\varepsilon^{p(x)}\left(u_\varepsilon^2 - \varepsilon^2 - u^2\right) dx\, dt \longrightarrow 0, \quad (\varepsilon \longrightarrow 0).\end{gathered} \tag{4.58}$$

Therefore,

$$\overline{\lim_{\varepsilon\to 0}} \iint_{Q_T} \left[\left|\nabla u_\varepsilon^2\right|^{p(x)-2} \nabla u_\varepsilon^2 - \left|\nabla u^2\right|^{p(x)-2} \nabla u^2\right] \nabla\left(u_\varepsilon^2 - u^2\right) dx\, dt \leq 0. \tag{4.59}$$

By Lemma 2.5,

$$\iint_{Q_T} \left[\left|\nabla u_\varepsilon^2\right|^{p(x)-2} \nabla u_\varepsilon^2 - \left|\nabla u^2\right|^{p(x)-2} \nabla u^2\right] \nabla\left(u_\varepsilon^2 - u^2\right) dx\, dt \geq \iint_{Q_T} \left|\nabla u_\varepsilon^2 - \nabla u^2\right|^{p(x)} dx\, dt. \tag{4.60}$$

Hence,

$$\overline{\lim_{\varepsilon\to 0}} \iint_{Q_T} \left|\nabla u_\varepsilon^2 - \nabla u^2\right|^{p(x)} dx\, dt \leq 0, \tag{4.61}$$

that is,

$$\iint_{Q_T} \left|\nabla u_\varepsilon^2 - \nabla u^2\right|^{p(x)} dx\, dt \longrightarrow 0, \quad (\varepsilon \longrightarrow 0). \tag{4.62}$$

Applying $(a+b)^p \leq 2^p(a^p + b^p)$, $(a, b > 0)$, and $2u_\varepsilon \nabla(u_\varepsilon - u) = \nabla(u_\varepsilon^2 - u^2) - 2(u_\varepsilon - u)\nabla u$, we have

$$\begin{aligned}&\iint_{Q_T} u_\varepsilon^{p(x)} |\nabla u_\varepsilon - \nabla u|^{p(x)} dx\, dt \\ &\quad \leq \iint_{Q_T} \left|\nabla u_\varepsilon^2 - \nabla u^2\right|^{p} dx\, dt + \iint_{Q_T} 2^{p(x)} |\nabla u_\varepsilon - \nabla u|^{p(x)} |\nabla u|^{p(x)} dx\, dt.\end{aligned} \tag{4.63}$$

Equation (4.50) and (1) imply that the right side tends to zero as $\varepsilon \to 0$. Since $u_\varepsilon \geq c$ in Q_c^ε, (2) is proved. (3) is an immediate consequence of (2). □

Lemma 4.7. *For any $\alpha \in [0, 1)$, we have*

$$\iint_{Q_T} |\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{-\alpha} dx\, dt \leq C, \tag{4.64}$$

where C is independent of ε.

Proof. From Lemmas 4.3 and 4.4, it is easily seen that

$$\iint_{Q_T} |\nabla u_{\varepsilon,\eta}|^{p(x)} u_{\varepsilon,\eta}^{-\alpha} dx\,dt \le C, \tag{4.65}$$

$$\iint_{Q_T} \left| |\nabla u_{\varepsilon,\eta}|^{p(x)} - |\nabla u_\varepsilon|^{p(x)} \right| dx\,dt \longrightarrow 0, \qquad (\eta = \eta_k \longrightarrow 0). \tag{4.66}$$

Using $u_{\varepsilon,\eta}, u_\varepsilon \ge \varepsilon$, (4.66), and Proposition 4.5, we have

$$\begin{aligned}
&\iint_{Q_T} \left| |\nabla u_{\varepsilon,\eta}|^{p(x)} u_{\varepsilon,\eta}^{-\alpha} - |\nabla u_\varepsilon|^{p(x)} u_\varepsilon^{-\alpha} \right| dx\,dt \\
&\quad \le \iint_{Q_T} \left| |\nabla u_{\varepsilon,\eta}|^{p(x)} - |\nabla u_\varepsilon|^{p(x)} \right| u_{\varepsilon,\eta}^{-\alpha} dx\,dt + \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)} \left| u_{\varepsilon,\eta}^{-\alpha} - u_\varepsilon^{-\alpha} \right| u_{\varepsilon,\eta}^{-\alpha} u_\varepsilon^{-\alpha} dx\,dt \\
&\quad \le \frac{1}{\varepsilon^\alpha} \iint_{Q_T} \left| |\nabla u_{\varepsilon,\eta}|^{p(x)} - |\nabla u_\varepsilon|^{p(x)} \right| dx\,dt + \frac{1}{\varepsilon^{2\alpha}} \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)} \left| u_{\varepsilon,\eta}^{-\alpha} - u_\varepsilon^{-\alpha} \right| dx\,dt \longrightarrow 0.
\end{aligned} \tag{4.67}$$

The proof of Lemma 4.7 is completed by combining (4.67) with (4.65). □

Lemma 4.8. *As $\varepsilon \to 0$, we have*

$$\begin{aligned}
&(1) \quad \iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)} - |\nabla u|^{p(x)} \right| dx\,dt \longrightarrow 0, \\
&(2) \quad \iint_{Q_T} \left| u_\varepsilon |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon - u |\nabla u|^{p(x)-2} \nabla u \right| dx\,dt \longrightarrow 0.
\end{aligned} \tag{4.68}$$

Proof. Let χ_ρ and $\chi_{\varepsilon\rho}$ be the characteristic functions of $\{(x,t) \in Q_T; u(x,t) < \rho\}$ and $\{(x,t) \in Q_T; u_\varepsilon(x,t) < \rho\}$, respectively. Then

$$\begin{aligned}
&\iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)} - |\nabla u|^{p(x)} \right| dx\,dt \\
&\quad \le \iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)} \chi_{\varepsilon\rho} - |\nabla u|^{p(x)} \chi_\rho \right| dx\,dt + \iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)} (1-\chi_{\varepsilon\rho}) - |\nabla u|^{p(x)} (1-\chi_\rho) \right| dx\,dt \\
&\quad \le \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)} \chi_{\varepsilon\rho} dx dt + \iint_{Q_T} |\nabla u|^{p(x)} \chi_\rho dx\,dt + \iint_{Q_T} |\nabla u|^{p(x)} (\chi_{\varepsilon\rho} - \chi_\rho) dx\,dt \\
&\qquad + \iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)} - |\nabla u|^{p(x)} \right| (1-\chi_{\varepsilon\rho}) dx\,dt \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.69}$$

Taking $\alpha = 1/2$ in Lemma 4.7, we %obtain that $I_1 \le C\rho^{1/2}$. Since $I_2 \to 0$ $(\rho \to 0)$, for any $\delta > 0$, we can choose $\rho > 0$ such that $I_1 + I_2 < \delta/3$. For fixed $\rho > 0$, $\chi_{\varepsilon\rho} \to \chi_\rho$ $(\varepsilon \to 0)$ a.e. in

Q_T, so there exists $\varepsilon_1 > 0$ such that $I_3 < \delta/3$ as $\varepsilon < \varepsilon_1$. By Lemma 4.6, $I_4 \to 0$ $(\varepsilon \to 0)$, so there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that $I_4 < \delta/3$ as $\varepsilon \in (0, \varepsilon_2)$. Summing up, we have

$$\iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)} - |\nabla u|^{p(x)} \right| dx\, dt < \delta, \qquad \forall \varepsilon < \varepsilon_2, \tag{4.70}$$

thus (1) holds. To prove (2), observe that

$$\begin{aligned} &\iint_{Q_T} \left| u_\varepsilon |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon - u |\nabla u|^{p(x)-2} \nabla u \right| dx\, dt \\ &\quad \le \frac{1}{2} \iint_{Q_T} |\nabla u_\varepsilon|^{p(x)-2} \left| \nabla u_\varepsilon^2 - \nabla u^2 \right| dx\, dt + \frac{1}{2} \iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)-2} - |\nabla u|^{p(x)-2} \right| \left| \nabla u^2 \right| dx\, dt \\ &\quad = \frac{1}{2} (I_a + I_b). \end{aligned} \tag{4.71}$$

Using Höder's inequality and Lemma 4.7, we obtain that

$$I_a \le |\nabla u_\varepsilon|_{p(x)(p(x)-2)/(p(x)-1)} \left| \nabla u_\varepsilon^2 - \nabla u^2 \right|_{p(x)} \longrightarrow 0. \tag{4.72}$$

By means of the inequality $|a^r - b^r| \le |a - b|^r (r \in [0,1], a, b \ge 0)$, Höder's inequality and (1), we have

$$\begin{aligned} I_b &= \iint_{Q_T} \left| \left(|\nabla u_\varepsilon|^{p(x)} \right)^{(p(x)-2)/p(x)} - \left(|\nabla u|^{p(x)} \right)^{(p(x)-2)/p(x)} \right| \left| \nabla u^2 \right| dx\, dt \\ &\le C \iint_{Q_T} \left| |\nabla u_\varepsilon|^{p(x)} - |\nabla u|^{p(x)} \right|^{(p(x)-2)/p(x)} \left| \nabla u^2 \right| dx\, dt \\ &\le C \left| |\nabla u_\varepsilon|^{p(x)} - |\nabla u|^{p(x)} \right| \left| \nabla u^2 \right|_{p(x)/2} \longrightarrow 0. \end{aligned} \tag{4.73}$$

Thus the proof of Theorem 4.1 is complete. □

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