Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 827156, 15 pages
doi:10.1155/2012/827156

Research Article

Algorithms for Solving the Variational Inequality Problem over the Triple Hierarchical Problem

Thanyarat Jitpeera and Poom Kumam

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Thung Khru, Bangkok 10140, Thailand

Correspondence should be addressed to Poom Kumam, poom.kum@kmutt.ac.th

Received 29 February 2012; Revised 10 May 2012; Accepted 5 June 2012

Academic Editor: D. Anderson

This paper discusses the monotone variational inequality over the solution set of the variational inequality problem and the fixed point set of a nonexpansive mapping. The strong convergence theorem for the proposed algorithm to the solution is guaranteed under some suitable assumptions.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively.

A mapping $A : H \to H$ is said to be *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$, $\forall x, y \in H$. A is said to be *α-strongly monotone* if there exists $\alpha > 0$ such that $\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2$, $\forall x, y \in H$. A is said to be *β-inverse-strongly monotone* if there exists $\beta > 0$ such that $\langle Ax - Ay, x - y \rangle \geq \beta \|Ax - Ay\|^2$, $\forall x, y \in H$. A is said to be *L-Lipschitz continuous* if there exists $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|, \forall x, y \in H$. A linear bounded operator A is said to be *strongly positive* on H if there exists $\overline{\gamma} > 0$ with the property $\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2$, $\forall x \in H$.

Let $f : C \to C$ be a *ρ-contraction* if there exists $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Let $T : C \to C$ be *nonexpansive* such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [1]). Let A be a nonlinear mapping. The *Hartmann-Stampacchia variational inequality* [2] is to finding $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{VI}(C, A)$. The variational inequality has been extensively studied in the literature [3, 4].

We discuss the following variational inequality problem over the fixed point set of a nonexpansive mapping (see [5–12]), which is called the *hierarchical problem*. Let a monotone, continuous mapping $A : H \to H$ and a nonexpansive mapping $T : H \to H$.

$$\text{Find } x \in \mathrm{VI}(F(T), A) = \{x \in F(T) : \langle Ax, y - x \rangle \geq 0, \ \forall y \in F(T)\}, \quad F(T) \neq \emptyset. \tag{1.4}$$

This solution set is denoted by Ξ.

We introduce the following variational inequality problem over solution set of variational inequality problem and the fixed point set of a nonexpansive mapping (see [13–16]), which is called *the triple hierarchical problem* (or *the triple hierarchical constrained optimization problem* (see also [13])). Let an inverse-strongly monotone $A : H \to H$, a strongly monotone and Lipschitz continuous $B : H \to H$, and a nonexpansive mapping $T : H \to H$.

$$\text{Find } x \in \mathrm{VI}(\Xi, B) = \{x \in \Xi : \langle Bx, y - x \rangle \geq 0, \ \forall y \in \Xi\}, \tag{1.5}$$

where $\Xi := \mathrm{VI}(F(T), A) \neq \emptyset$.

In 2009, Iiduka [13] introduced an iterative algorithm for the following *triple hierarchical constrained optimization problem*, the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_1 \in H$ is chosen arbitrarily,

$$\begin{aligned} y_n &= T(x_n - \lambda_n A_1 x_n), \\ x_{n+1} &= y_n - \mu \alpha_n A_2 y_n, \quad \forall n \geq 0, \end{aligned} \tag{1.6}$$

where $\alpha_n \in (0, 1]$ and $\lambda_n \in (0, 2\alpha]$ satisfies certain conditions. Let $A_1 : H \to H$ be an inverse-strongly monotone, $A_2 : H \to H$ be a strongly monotone and Lipschitz continuous, and $T : H \to H$ be a nonexpansive mapping, then the sequence converges to strong analysis on (1.6).

In 2011, Ceng et al. [17] studied the new following algorithms. For $x_0 \in C$ is chosen arbitrarily, they defined a sequence $\{x_n\}$ iterative by

$$x_{n+1} = P_C[\lambda_n \gamma(\alpha_n f(x_n) + (1 - \alpha_n) S x_n) + (I - \lambda_n \mu F) T x_n], \quad \forall n \geq 0, \tag{1.7}$$

where the mapping S, T are nonexpansive mappings with $F(T) \neq \emptyset$. Let $F : C \to H$ be a Lipschitzian and strongly monotone operator and $f : C \to H$ be a contraction mapping satisfied some conditions. They proved that the proposed algorithms strongly converge to the minimum norm fixed point of T.

Very recently, Yao et al. [18] studied the following algorithms. For $x_0 \in C$ is chosen arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T P_C [I - \alpha_n (A - \gamma f)] x_n, \quad \forall n \geq 0, \tag{1.8}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$. Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality as follows. Find a point $x^* \in F(T)$ such that

$$\langle (A - \gamma f) x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{1.9}$$

where $A : C \to H$ is a strongly positive linear bounded operator, $f : C \to H$ is a ρ-contraction, and $T : C \to C$ is a nonexpansive mapping satisfied some suitable conditions. The solution (1.9) is denoted by $\Upsilon := \mathrm{VI}(F(T), A - \gamma f) := \{x^* \in F(T) : \langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \forall x \in F(T)\}$.

In this paper, we introduce a new iterative algorithm for solving the triple hierarchical problem, which contain algorithms (1.6) and (1.8) as follows:

$$\begin{gathered} y_n = T P_C [I - \delta_n (A - \gamma f)] x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + [(1-\beta_n) I - \alpha_n \mu F] y_n, \quad \forall n \geq 0. \end{gathered} \tag{1.10}$$

The strong convergence for the proposed algorithms to the solution is solved under some assumptions. Our results generalize and improve the results of Ceng et al. [17], Iiduka [13], Yao et al. [18], and some authors.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C). \tag{2.1}$$

The following properties of projection are useful and pertinent to our purposes.

Lemma 2.1. *Given $x \in H$ and $z \in C$,*

(a) $u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0$, $\forall v \in C$,

(b) $u = P_C z \Leftrightarrow \|z - u\|^2 \leq \|z - v\|^2 - \|v - u\|^2$, $\forall v \in C$,

(c) *P_C is a firmly nonexpansive mapping of H onto C and satisfies*

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{2.2}$$

Consequently, P_C is nonexpansive and monotone.

Lemma 2.2. *There holds the following inequality in an inner product space H*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.3}$$

Lemma 2.3 (see [19]). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \qquad x_n - Tx_n \longrightarrow 0 \tag{2.4}$$

implies $x = Tx$.

Lemma 2.4 (see [20]). *Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.5}$$

hold for each $y \in H$ with $y \ne x$.

Lemma 2.5 (see [21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.6 (see [10]). *Let $B : H \to H$ be β-strongly monotone and L-Lipschitz continuous and let $\mu \in (0, 2\beta/L^2)$. For $\lambda \in [0,1]$, define $T_\lambda : H \to H$ by $T_\lambda(x) := x - \lambda\mu B(x)$ for all $x \in H$. Then, for all $x, y \in H$,*

$$\|T_\lambda(x) - T_\lambda(y)\| \le (1 - \lambda\tau)\|x - y\| \tag{2.6}$$

hold, where $\tau := 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0,1]$.

Lemma 2.7 (see [22]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n, \quad \forall n \ge 0, \tag{2.7}$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathcal{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Remark 2.8. If $A : C \to H$ is a strongly positive linear bounded operator and $f : C \to H$ is a ρ-contraction, then for $0 < \gamma < \tilde{\gamma}/\rho$, the mapping $A - \gamma f$ is strongly monotone. In fact, we have

$$\begin{aligned} \langle (A-\gamma f)x - (A-\gamma f)y, x-y \rangle &= \langle A(x-y), x-y \rangle - \gamma \langle f(x) - f(y), x-y \rangle \\ &\ge \tilde{\gamma}\|x-y\|^2 - \gamma\gamma\rho\|x-y\|^2 \\ &\ge 0. \end{aligned} \tag{2.8}$$

3. Main Results

In this section, we introduce a new iterative algorithm for solving monotone variational inequality problem (where $A : C \to H$ is a strongly positive linear bounded operator, $f : C \to H$ is a ρ-contraction) over solution set of variational inequality problem over the fixed point set of a nonexpansive mapping.

Theorem 3.1. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a strongly positive linear bounded operator,* $f : C \to H$ *be a* ρ*-contraction, and* γ *be a positive real number such that* $(\overline{\gamma} - 1)/\rho < \gamma < \overline{\gamma}/\rho$. *Let* $F : C \to C$ *be* κ*-Lipschitzian and* η*-strongly monotone operators with constant* κ *and* $\eta > 0$, *respectively. Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$. *Let* $0 < \mu < 2\eta/\kappa^2$ *and* $0 < \gamma < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. *Assume that* $VI(\Upsilon, F) \neq \emptyset$, *where* $\Upsilon := VI(F(T), A - \gamma f)$. *Suppose* $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily and*

$$\begin{gathered} y_n = TP_C[I - \delta_n(A - \gamma f)]x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + [(1-\beta_n)I - \alpha_n \mu F]y_n, \quad \forall n \ge 0, \end{gathered} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subset (0,1)$ *satisfy the following conditions:*

(C1) $\alpha_n \le \kappa\delta_n$ *and* $\beta_n < \delta_n$;

(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C3) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(C4) $\lim_{n\to\infty}\delta_n = 0$, $\sum_{n=0}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Upsilon$, *which is the unique solution of another variational inequality*

$$\langle (I - \mu F)x^*, x - x^* \rangle \ge 0, \quad \forall x \in \Upsilon. \tag{3.2}$$

Proof. We will divide the proof into four steps.

Step 1. We will show $\{x_n\}$ is bounded. For any $x^* \in F(T)$, we have

$$\begin{aligned}
\|y_n - x^*\| &= \|TP_C[I - \delta_n(A - \gamma f)]x_n - TP_C x^*\| \\
&\le \|[I - \delta_n(A - \gamma f)]x_n - x^*\| \\
&\le \delta_n\|\gamma f(x_n) - \gamma f(x^*)\| + \delta_n\|\gamma f(x^*) - Ax^*\| + \|I - \delta_n A\|\|x_n - x^*\| \\
&\le \delta_n\gamma\rho\|x_n - x^*\| + \delta_n\|\gamma f(x^*) - Ax^*\| + (1 - \delta_n\overline{\gamma})\|x_n - x^*\| \\
&= [1 - (\overline{\gamma} - \gamma\rho)\delta_n]\|x_n - x^*\| + \delta_n\|\gamma f(x^*) - Ax^*\|.
\end{aligned} \tag{3.3}$$

From (3.1), we deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + [(1 - \beta_n)\ I - \alpha_n\mu F]y_n - x^*\| \\
&\le \alpha_n\|u - \mu F x^*\| + \beta_n\|x_n - x^*\| + [(1 - \beta_n)I - \alpha_n\mu F]\|y_n - x^*\| \\
&\le \alpha_n\|u - \mu F x^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\tau)\|y_n - x^*\|.
\end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad\le \alpha_n\|u - \mu F x^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\tau) \\
&\qquad\times\{[1 - (\overline{\gamma} - \gamma\rho)\delta_n]\|x_n - x^*\| + \delta_n\|\gamma f(x^*) - Ax^*\|\} \\
&\quad= \alpha_n\|u - \mu F x^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\tau)[1 - (\overline{\gamma} - \gamma\rho)\delta_n]\|x_n - x^*\| \\
&\qquad+ (1 - \beta_n - \alpha_n\tau)\delta_n\|\gamma f(x^*) - Ax^*\| \\
&\quad\le \alpha_n\|u - \mu F x^*\| + [1 - (1 - \beta_n - \alpha_n\tau)\delta_n(\overline{\gamma} - \gamma\rho)]\|x_n - x^*\| \\
&\qquad+ (1 - \beta_n - \alpha_n\tau)\delta_n\|\gamma f(q) - Aq\| \\
&\quad\le \kappa\delta_n\|u - \mu F x^*\| + [1 - (1 - \beta_n - \alpha_n\tau)\delta_n(\overline{\gamma} - \gamma\rho)]\|x_n - x^*\| \\
&\qquad+ (1 - \beta_n - \alpha_n\tau)\delta_n\|\gamma f(x^*) - Ax^*\|.
\end{aligned} \tag{3.5}$$

By induction, it follows that

$$\begin{aligned}
&\|x_n - x^*\| \\
&\quad\le \max\left\{\|x_0 - x^*\| + \frac{1}{\overline{\gamma} - \gamma\rho}\|\gamma f(x^*) - Ax^*\| + \frac{1}{(1 - \beta_n - \alpha_n\tau)(\overline{\gamma} - \gamma\rho)}\kappa\|u - \mu F x^*\|\right\}, \quad n \ge 0.
\end{aligned} \tag{3.6}$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}, \{Ax_n\}, \{f(x_n)\}$, and $\{F(x_n)\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\|y_n-x_n\|=0$, and $\lim_{n\to\infty}\|y_n-Ty_n\|=0$. From (3.1), we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|TP_C[I-\delta_{n+1}(A-\gamma f)]x_{n+1}-TP_C[I-\delta_n(A-\gamma f)]x_n\| \\
&\le \|P_C[I-\delta_{n+1}(A-\gamma f)]x_{n+1}-P_C[I-\delta_n(A-\gamma f)]x_n\| \\
&\le \|[I-\delta_{n+1}(A-\gamma f)]x_{n+1}-[I-\delta_n(A-\gamma f)]x_n\| \\
&= \|\delta_{n+1}(\gamma f(x_{n+1})-\gamma f(x_n))+(\delta_{n+1}-\delta_n)\gamma f(x_n) \\
&\qquad +(I-\delta_{n+1}A)(x_{n+1}-x_n)+(\delta_n-\delta_{n+1})Ax_n\| \\
&\le \delta_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|+(1-\delta_{n+1}\overline{\gamma})\|x_{n+1}-x_n\| \\
&\quad +|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\|+\|Ax_n\|) \\
&\le \delta_{n+1}\gamma\rho\|x_{n+1}-x_n\|+(1-\delta_{n+1}\overline{\gamma})\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\|+\|Ax_n\|) \\
&= [1-(\overline{\gamma}-\gamma\rho)\delta_{n+1}]\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\|+\|Ax_n\|).
\end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}u+\beta_{n+1}x_{n+1}+[(1-\beta_{n+1})I-\alpha_{n+1}\mu F]y_{n+1} \\
&\qquad -\alpha_n u-\beta_n x_n-[(1-\beta_n)I-\alpha_n\mu F]y_n\| \\
&\le |\alpha_{n+1}-\alpha_n|\|u\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\| \\
&\quad +\|[(1-\beta_{n+1})I-\alpha_{n+1}\mu F]y_{n+1}-[(1-\beta_{n+1})I-\alpha_{n+1}\mu F]y_n\| \\
&\quad +\|[(1-\beta_{n+1})I-\alpha_{n+1}\mu F]y_n-[(1-\beta_n)I-\alpha_n\mu F]y_n\| \\
&\le |\alpha_{n+1}-\alpha_n|\|u\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\| \\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\tau)\|y_{n+1}-y_n\|+(1-\beta_{n+1}-\alpha_{n+1}\mu F-1+\beta_n+\alpha_n\mu F)\|y_n\| \\
&\le |\alpha_{n+1}-\alpha_n|\|u\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\| \\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\tau)\|y_{n+1}-y_n\|+|\beta_{n+1}-\beta_n|\|y_n\|+|\alpha_{n+1}-\alpha_n|\tau\|y_n\| \\
&= |\alpha_{n+1}-\alpha_n|(\|u\|+\tau\|y_n\|)+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|(\|x_n\|+\|y_n\|) \\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\tau)\|y_{n+1}-y_n\| \\
&\le |\alpha_{n+1}-\alpha_n|(\|u\|+\tau\|y_n\|)+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|(\|x_n\|+\|y_n\|) \\
&\quad +(1-\beta_{n+1}-\alpha_{n+1}\tau)\{[1-(\overline{\gamma}-\gamma\rho)\delta_{n+1}]\|x_{n+1}-x_n\| \\
&\quad +|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\|+\|Ax_n\|)\}
\end{aligned}$$

$$
\begin{aligned}
&\le |\alpha_{n+1}-\alpha_n|\left(\|u\|+\tau\|y_n\|\right)+\beta_{n+1}\|x_{n+1}-x_n\|+\left|\beta_{n+1}-\beta_n\right|\left(\|x_n\|+\|y_n\|\right)\\
&\quad+\left(1-\beta_{n+1}-\alpha_{n+1}\tau\right)\left[1-(\overline{\gamma}-\gamma\rho)\delta_{n+1}\right]\|x_{n+1}-x_n\|\\
&\quad+|\delta_{n+1}-\delta_n|\left(\|\gamma f(x_n)\|+\|Ax_n\|\right)\\
&\le \left[1-\left(1-\beta_{n+1}-\alpha_{n+1}\tau\right)\left(\overline{\gamma}-\gamma\rho\right)\delta_{n+1}\right]\|x_{n+1}-x_n\|\\
&\quad+|\alpha_{n+1}-\alpha_n|\left(\|u\|+\tau\|y_n\|\right)+\left|\beta_{n+1}-\beta_n\right|\left(\|x_n\|+\|y_n\|\right)\\
&\quad+|\delta_{n+1}-\delta_n|\left(\|\gamma f(x_n)\|+\|Ax_n\|\right)\\
&\le \left[1-\left(1-\beta_{n+1}-\alpha_{n+1}\tau\right)\left(\overline{\gamma}-\gamma\rho\right)\delta_{n+1}\right]\|x_{n+1}-x_n\|\\
&\quad+\left(|\alpha_{n+1}-\alpha_n|+\left|\beta_{n+1}-\beta_n\right|+|\delta_{n+1}-\delta_n|\right)M_3,
\end{aligned}
\tag{3.8}
$$

where M_3 is a constant such that

$$\sup_{n\ge 0}\left\{\left(\|u\|+\tau\|y_n\|\right),\left(\|x_n\|+\|y_n\|\right),\left(\|\gamma f(x_n)\|+\|Ax_n\|\right)\right\}\le M_3. \tag{3.9}$$

By the conditions $(C2)$–$(C4)$ allow us to apply Lemma 2.7, we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.10}$$

On the other hand, we note that

$$
\begin{aligned}
\|y_n-Tx_n\| &= \left\|TP_C\left[I-\delta_n(A-\gamma f)\right]x_n-Tx_n\right\|\\
&= \left\|TP_C\left[I-\delta_n(A-\gamma f)\right]x_n-TP_Cx_n\right\|\\
&\le \left\|\left[I-\delta_n(A-\gamma f)\right]x_n-x_n\right\|\\
&\le \delta_n\left\|(A-\gamma f)x_n\right\|,
\end{aligned}
\tag{3.11}
$$

by $(C4)$, it follows that

$$\lim_{n\to\infty}\|y_n-Tx_n\|=0. \tag{3.12}$$

From (3.7), we observe that

$$\|y_{n+1}-y_n\|\le\left[1-(\overline{\gamma}-\gamma\rho)\delta_{n+1}\right]\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\left(\|\gamma f(x_n)\|+\|Ax_n\|\right). \tag{3.13}$$

It follows that

$$\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\le(\overline{\gamma}-\gamma\rho)\delta_{n+1}\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\left(\|\gamma f(x_n)\|+\|Ax_n\|\right). \tag{3.14}$$

From the conditions $(C1)$–$(C4)$ and the boundedness of $\{x_n\}, \{f(x_n)\}$, and $\{Ax_n\}$, which implies that

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{3.15}$$

Hence, by Lemma 2.5, we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.16}$$

From (3.12) and (3.16), we obtain

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.17}$$

Step 3. We will show that $\limsup_{n\to\infty}\langle u_n - x^*, \gamma f(x^*) - Ax^*\rangle \le 0$ is proven. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle x_n - x^*, \gamma f(x^*) - Ax^*\right\rangle = \lim_{i\to\infty}\left\langle x_{n_i} - x^*, \gamma f(x^*) - Ax^*\right\rangle. \tag{3.18}$$

The boundedness of $\{x_{n_i}\}$ implies the existences of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $\{x_{n_{i_j}}\}$ converges weakly to $\hat{x}$. We may assume without loss of generality that $\lim_{i\to\infty}\langle x_{n_i}, w\rangle = \langle \hat{x}, w\rangle, w \in H$. Assume $\hat{x} \ne T(\hat{x})$. Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ with $F(T) \ne \emptyset$ guarantee that

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - \hat{x}\| &< \liminf_{i\to\infty}\|x_{n_i} - T(\hat{x})\| \\
&= \liminf_{i\to\infty}\|x_{n_i} - T(x_{n_i}) + T(x_{n_i}) - T(\hat{x})\| \\
&= \liminf_{i\to\infty}\|T(x_{n_i}) - T(\hat{x})\| \\
&\le \liminf_{i\to\infty}\|x_{n_i} - \hat{x}\|,
\end{aligned} \tag{3.19}$$

which has a contradiction. Therefore, $\hat{x} \in F(T)$. Since $x^* \in \mathrm{VI}(\Upsilon, F)$, then $x^* \in \Upsilon := \mathrm{VI}(F(T), A - \gamma f)$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle x_n - x^*, \gamma f(x^*) - Ax^*\right\rangle &= \lim_{i\to\infty}\left\langle x_{n_i} - x^*, \gamma f(x^*) - Ax^*\right\rangle \\
&= \left\langle \hat{x} - x^*, \gamma f(x^*) - Ax^*\right\rangle \\
&\le 0.
\end{aligned} \tag{3.20}$$

Setting $u_n = [I - \delta_n(A - \gamma f)]x_n$ and by $(C4)$, we notice that

$$\|u_n - x_n\| \le \delta_n\left\|(A - \gamma f)\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

Hence, we get

$$\limsup_{n\to\infty}\langle u_n - x^*, \gamma f(x^*) - Ax^*\rangle \le 0. \tag{3.22}$$

Next we will show that $\limsup_{n\to\infty}\langle x_{n+1}-x^*, x^*-\mu F x^*\rangle \le 0$ is proven. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, x^* - \mu F x^*\rangle = \lim_{k\to\infty}\langle x_{n_{k+1}} - x^*, x^* - \mu F x^*\rangle. \tag{3.23}$$

The boundedness of $\{x_{n_k}\}$ implies the existences of a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ and a point $\overline{x} \in H$ such that $\{x_{n_{k_l}}\}$ converges weakly to $\overline{x}$. We may assume without loss of generality that $\lim_{k\to\infty}\langle x_{n_k}, w\rangle = \langle \overline{x}, w\rangle, w \in H$. Assume $\overline{x} \neq T(\overline{x})$. By $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ with $F(T) \neq \emptyset$ guarantee that

$$\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k} - \overline{x}\| &< \liminf_{k\to\infty}\|x_{n_k} - T(\overline{x})\| \\
&= \liminf_{k\to\infty}\|x_{n_k} - T(x_{n_k}) + T(x_{n_k}) - T(\overline{x})\| \\
&= \liminf_{k\to\infty}\|T(x_{n_k}) - T(\overline{x})\| \\
&\le \liminf_{k\to\infty}\|x_{n_k} - \overline{x}\|,
\end{aligned} \tag{3.24}$$

which has a contradiction. Therefore, $\overline{x} \in F(T)$. From $x^* \in \mathrm{VI}(\Upsilon, F) := \mathrm{VI}(\mathrm{VI}(F(T), A-\gamma f), F)$, we compute

$$\begin{aligned}
\limsup_{n\to\infty}\langle x_n - x^*, \gamma x^* - \mu F x^*\rangle &= \lim_{k\to\infty}\langle x_{n_k} - x^*, x^* - \mu F x^*\rangle \\
&= \langle \overline{x} - x^*, x^* - \mu F x^*\rangle \\
&\le 0.
\end{aligned} \tag{3.25}$$

Using (3.10), we get

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, x^* - \mu F x^*\rangle \le 0. \tag{3.26}$$

Step 4. Finally, we prove $x_{n+1} \to x^*$. We observe that

$$\|u_n - x^*\| \le \|x_n - x^*\| + \delta_n\left\|(A - \gamma f)x_n\right\|. \tag{3.27}$$

From (3.1), we compute

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|\alpha_n u+\beta_n x_n+\left[(1-\beta_n)I-\alpha_n\mu F\right]y_n-x^*\right\|^2 \\
&= \left\|\alpha_n(u-x^*)+\alpha_n(x^*-\mu Fx^*)+\beta_n(x_n-x^*)+\left[(1-\beta_n)I-\alpha_n\mu F\right](y_n-x^*)\right\|^2 \\
&\le \left\|\alpha_n(u-x^*)+\beta_n(x_n-x^*)+\alpha_n(x^*-\mu Fx^*)\right\|^2+\left[1-\beta_n-\alpha_n\tau\right]\left\|y_n-x^*\right\|^2 \\
&\le \alpha_n^2\|u-x^*\|^2+\beta_n^2\|x_n-x^*\|^2+2\alpha_n\left\langle x^*-\mu Fx^*, x_{n+1}-x^*\right\rangle \\
&\quad+\left[1-\beta_n-\alpha_n\tau\right]\|u_n-x^*\|^2 \\
&\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\alpha_n\left\langle x^*-\mu Fx^*, x_{n+1}-x^*\right\rangle \\
&\quad+\left[1-\beta_n-\alpha_n\tau\right]\left\|\delta_n(\gamma f(x_n)-Ax^*)+(I-\delta_n A)(x_n-x^*)\right\|^2 \\
&\le \kappa\delta_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\kappa\delta_n\left\langle x^*-\mu Fx^*, x_{n+1}-x^*\right\rangle \\
&\quad+\left[1-\beta_n-\alpha_n\tau\right]\left[(1-\delta_n\widetilde{\gamma})^2\|x_n-x^*\|^2+2\delta_n\left\langle\gamma f(x_n)-Ax^*, u_n-x^*\right\rangle\right] \\
&\le \kappa\delta_n\|u-x^*\|^2+2\kappa\delta_n\left\langle x^*-\mu Fx^*, x_{n+1}-x^*\right\rangle \\
&\quad+\left[1-2\delta_n\widetilde{\gamma}(1-\beta_n-\alpha_n\tau)\right]\|x_n-x^*\|^2+\delta_n^2\widetilde{\gamma}^2(1-\beta_n-\alpha_n\tau)\|x_n-x^*\|^2 \\
&\quad-\alpha_n\tau\|x_n-x^*\|^2+2\delta_n\left\langle\gamma f(x_n)-\gamma fx^*, u_n-x^*\right\rangle+2\delta_n\left\langle\gamma fx^*-Ax^*, u_n-x^*\right\rangle \\
&\le \kappa\delta_n\|u-x^*\|^2+2\kappa\delta_n\left\langle x^*-\mu Fx^*, x_{n+1}-x^*\right\rangle \\
&\quad+\left[1-2\delta_n\widetilde{\gamma}\right]\|x_n-x^*\|^2+\delta_n^2\widetilde{\gamma}^2(1-\beta_n-\alpha_n\tau)\|x_n-x^*\|^2 \\
&\quad+2\delta_n\gamma\rho\|x_n-x^*\|\|u_n-x^*\|+2\delta_n\left\langle\gamma fx^*-Ax^*, u_n-x^*\right\rangle \\
&\le \left[1-2\delta_n(\widetilde{\gamma}-\gamma\rho)\right]\|x_n-x^*\|^2+\delta_n^2\widetilde{\gamma}^2(1-\beta_n-\alpha_n\tau)\|x_n-x^*\|^2 \\
&\quad+2\delta_n^2\gamma\rho\|x_n-x^*\|\left\|(A-\gamma f)x_n\right\|+2\delta_n\left\langle\gamma fx^*-Ax^*, u_n-x^*\right\rangle+\kappa\delta_n\|u-x^*\|^2 \\
&\quad+2\kappa\delta_n\left\langle x^*-\mu Fx^*, x_{n+1}-x^*\right\rangle.
\end{aligned}
\tag{3.28}
$$

Since $\{x_n\}$, $\{Ax_n\}$, $\{f(x_n)\}$, and $\{Fx_n\}$ are all bounded, we can choose a constant $M_4 > 0$ such that

$$
\sup_{n\ge 0}\frac{1}{\widetilde{\gamma}-\gamma\rho}\left\{\frac{(1-\beta_n-\alpha_n\tau)\widetilde{\gamma}^2}{2}\|x_n-x^*\|+\gamma\rho\|x_n-x^*\|\left\|(A-\gamma f)x_n\right\|\right\}\le M_4. \tag{3.29}
$$

It follows that

$$
\|x_{n+1}-x^*\|^2\le\left[1-2(\widetilde{\gamma}-\gamma\rho)\delta_n\right]\|x_n-x^*\|^2+2(\widetilde{\gamma}-\gamma\rho)\delta_n\xi_n, \tag{3.30}
$$

where

$$\begin{aligned}\xi_n := \delta_n M_4 &+ \frac{1}{\widetilde{\gamma} - \gamma\rho}\langle \gamma f x^* - Ax^*, u_n - x^*\rangle + \frac{\kappa}{\widetilde{\gamma} - \gamma\rho}\|u - x^*\|^2 \\ &+ \frac{\kappa}{\widetilde{\gamma} - \gamma\rho}\langle x^* - \mu F x^*, x_{n+1} - x^*\rangle.\end{aligned} \tag{3.31}$$

By the conditions $(C1)$, $(C4)$, (3.22), and (3.26), we get

$$\limsup_{n\to\infty} \xi_n \le 0. \tag{3.32}$$

Now, applying Lemma 2.7 and (3.30), we conclude that $x_n \to x^*$. This completes the proof. □

Next, the following example shows that all conditions of Theorem 3.1 are satisfied.

Example 3.2. For instance, let $\alpha_n = n/(n^2+1), \beta_n = 1/2n$ and $\delta_n = 1/n$. Then, clearly the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ satisfy the following condition $(C1)$:

$$\frac{n}{n^2+1} < \kappa\frac{1}{n}, \qquad \frac{1}{2n} < \frac{1}{n}. \tag{3.33}$$

We will show that the condition $(C2)$ is achieved. Indeed, we have

$$\begin{aligned}\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| &= \sum_{n=1}^{\infty}\left|\frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}\right| \\ &= \sum_{n=1}^{\infty}\left|\frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}\right| \\ &= \sum_{n=1}^{\infty}\left|\frac{1-n-n^2}{n^4+2n^3+3n^2+2n+2}\right|.\end{aligned} \tag{3.34}$$

The sequence $\{\alpha_n\}$ satisfies the condition $(C2)$ by p-series. Next, we will show that the condition $(C3)$ is achieved. We compute

$$\begin{aligned}\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| &= \sum_{n=1}^{\infty}\left|\frac{1}{2(n+1)} - \frac{1}{2n}\right| \\ &\le \left|\frac{1}{2\cdot 1} - \frac{1}{2\cdot 2}\right| + \left|\frac{1}{2\cdot 2} - \frac{1}{2\cdot 3}\right| + \left|\frac{1}{2\cdot 3} - \frac{1}{2\cdot 4}\right| + \cdots \\ &= \frac{1}{2}.\end{aligned} \tag{3.35}$$

The sequence $\{\beta_n\}$ satisfies the condition $(C3)$. Finally, we will show that the condition $(C4)$ is achieved. We compute

$$\begin{aligned}
\lim_{n\to\infty}\delta_n &= \lim_{n\to\infty}\frac{1}{n} = 0,\\
\sum_{n=1}^{\infty}\delta_n &= \sum_{n=1}^{\infty}\frac{1}{n} = \infty,\\
\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| &= \sum_{n=1}^{\infty}\left|\frac{1}{n+1}-\frac{1}{n}\right|\\
&\leq \left|\frac{1}{1}-\frac{1}{2}\right| + \left|\frac{1}{2}-\frac{1}{3}\right| + \left|\frac{1}{4}-\frac{1}{3}\right| + \cdots\\
&= 1.
\end{aligned} \tag{3.36}$$

The sequence $\{\delta_n\}$ satisfies the condition $(C4)$.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction, and γ be a positive real number such that $(\overline{\gamma}-1)/\rho < \gamma < \overline{\gamma}/\rho$. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Assume that $\Upsilon := VI(F(T), A-\gamma f) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily and*

$$\begin{aligned}
y_n &= TP_C\left[I-\delta_n(A-\gamma f)\right]x_n,\\
x_{n+1} &= \alpha_n u + \beta_n x_n + \left(1-\beta_n-\alpha_n\right)y_n,
\end{aligned} \tag{3.37}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subset (0,1)$ satisfy the following conditions (C1)–(C4). Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is the unique solution of variational inequality

$$\left\langle (A-\gamma f)x^*, x-x^*\right\rangle \geq 0, \quad \forall x \in F(T). \tag{3.38}$$

Proof. Putting $\mu = 2$ and $F \equiv I/2$ in Theorem 3.1, we can obtain desired conclusion immediately. □

Corollary 3.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily and*

$$\begin{aligned}
y_n &= TP_C(1-\delta_n)x_n,\\
x_{n+1} &= \alpha_n u + \beta_n x_n + \left(1-\beta_n-\alpha_n\right)y_n, \quad \forall n \geq 0,
\end{aligned} \tag{3.39}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subset (0,1)$ satisfy the following conditions (C1)–(C4). Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Proof. Putting $f \equiv 0$ and $A \equiv I$ in Corollary 3.3, we can obtain desired conclusion immediately. □

Remark 3.5. Our results generalize and improve the recent results of Iiduka [13] and Yao et al. [18].

Acknowledgment

The authors were supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (under Project NRU-CSEC no. 55000613) for financial support during the preparation of this paper.

References

[1] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, pp. 1004–1006, 1965.
[2] P. Hartman and G. Stampacchia, "On some non-linear elliptic differential-functional equations," *Acta Mathematica*, vol. 115, pp. 271–310, 1966.
[3] F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, "On a two-step algorithm for hierarchical fixed point problems and variational inequalities," *Journal of Inequalities and Applications*, vol. 2009, Article ID 208692, 13 pages, 2009.
[4] J.-C. Yao and O. Chadli, "Pseudomonotone complementarity problems and variational inequalities," in *Handbook of Generalized Convexity and Generalized Monotonicity*, vol. 76, pp. 501–558, Springer, New York, NY, USA, 2005.
[5] P. L. Combettes, "A block-iterative surrogate constraint splitting method for quadratic signal recovery," *IEEE Transactions on Signal Processing*, vol. 51, no. 7, pp. 1771–1782, 2003.
[6] S. A. Hirstoaga, "Iterative selection methods for common fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1020–1035, 2006.
[7] H. Iiduka and I. Yamada, "A subgradient-type method for the equilibrium problem over the fixed point set and its applications," *Optimization*, vol. 58, no. 2, pp. 251–261, 2009.
[8] K. Slavakis and I. Yamada, "Robust wideband beamforming by the hybrid steepest descent method," *IEEE Transactions on Signal Processing*, vol. 55, no. 9, pp. 4511–4522, 2007.
[9] K. Slavakis, I. Yamada, and K. Sakaniwa, "Computation of symmetric positive definite Toeplitz matrices by the hybrid steepest descent method," *Signal Processing*, vol. 83, no. 5, pp. 1135–1140, 2003.
[10] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, vol. 8, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.
[11] I. Yamada, N. Ogura, and N. Shirakawa, "A numerically robust hybrid steepest descent method for the convexly constrained generalized inverse problems," in *Inverse Problems, Image Analysis, and Medical Imaging*, vol. 313, pp. 269–305, American Mathematical Society, Providence, RI, USA, 2002.
[12] I. Yamada and N. Ogura, "Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 25, no. 7-8, pp. 619–655, 2004.
[13] H. Iiduka, "Strong convergence for an iterative method for the triple-hierarchical constrained optimization problem," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 12, pp. e1292–e1297, 2009.
[14] H. Iiduka, "Iterative algorithm for solving triple-hierarchical constrained optimization problem," *Journal of Optimization Theory and Applications*, vol. 148, no. 3, pp. 580–592, 2011.
[15] T. Jitpeera and P. Kumam, "A new explicit triple hierarchical problem over the set of fixed points and generalized mixed equilibrium problems," *Journal of Inequalities and Applications*, vol. 2012, article 82, 2012.
[16] N. Wairojjana and P. Kumam, "General iterative algorithms for hierarchical fixed points approach to variational inequalities," *Journal of Applied Mathematics*. Volume 2012, Article ID 174318, 20 pages.

[17] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, “Iterative methods for triple hierarchical variational inequalities in Hilbert spaces,” *Journal of Optimization Theory and Applications*, vol. 151, no. 3, pp. 489–512, 2011.

[18] Y. Yao, Y.-C. Liou, and S. M. Kang, “Algorithms construction for variational inequalities,” *Fixed Point Theory and Applications*, vol. 2011, Article ID 794203, 12 pages, 2011.

[19] F. E. Browder, “Nonlinear operators and nonlinear equations of evolution in Banach spaces,” *Proceedings of Symposia in Pure Mathematics*, vol. 18, pp. 78–81, 1976.

[20] Z. Opial, “Weak convergence of the sequence of successive approximations for nonexpansive mappings,” *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[21] T. Suzuki, “Strong convergence of Krasnoselskii and Mann’s type sequences for one-parameter non-expansive semigroups without Bochner integrals,” *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[22] H.-K. Xu, “Iterative algorithms for nonlinear operators,” *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.