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Research Article

Strong Convergence of a Modified Extragradient Method to the Minimum-Norm Solution of Variational Inequalities

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We suggest and analyze a modified extragradient method for solving variational inequalities, which is convergent strongly to the minimum-norm solution of some variational inequality in an infinite-dimensional Hilbert space.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C. \tag{1.1}$$

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$. It is well known that variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems; see [1–36] and the references therein.

It is well known that variational inequalities are equivalent to the fixed point problem. This alternative formulation has been used to study the existence of a solution of the variational inequality as well as to develop several numerical methods. Using this equivalence, one can suggest the following iterative method.

Algorithm 1.1. For a given $u_0 \in C$, calculate the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_C[u_n - \lambda A u_n], \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

It is well known that the convergence of Algorithm 1.1 requires that the operator A must be both strongly monotone and Lipschitz continuous. These restrict conditions rules out its applications in several important problems. To overcome these drawbacks, Korpelevič suggested in [8] an algorithm of the form

$$\begin{aligned} y_n &= P_C[x_n - \lambda A x_n], \\ x_{n+1} &= P_C\left[x_n - \lambda A y_n\right], \quad n \geq 0. \end{aligned} \tag{1.4}$$

Noor [2] further suggested and analyzed the following new iterative methods for solving the variational inequality (1.2).

Algorithm 1.2. For a given $u_0 \in C$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} &= P_C[w_n - \lambda A w_n], \\ w_n &= P_C[u_n - \lambda A u_n], \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.5}$$

which is known as the modified extragradient method. For the convergence analysis of Algorithm 1.2, see Noor [1, 2] and the references therein. We would like to point out that Algorithm 1.2 is quite different from the method of Korpelevič [8]. However, Algorithm 1.2 fails, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces.

In this paper, we suggest and consider a very simple modified extragradient method which is convergent strongly to the minimum-norm solution of variational inequality (1.2) in an infinite-dimensional Hilbert space. This new method includes the method of Noor [2] as a special case.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H. It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}. \tag{2.1}$$

We denote u_0 by $P_C u$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\|P_C x - P_C y\| \le \|x - y\|$ for all $x, y \in H$;

(ii) $\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2$ for every $x, y \in H$;

(iii) $\langle x - P_C x, y - P_C x \rangle \le 0$ for all $x \in H$, $y \in C$.

We need the following lemma for proving our main results.

Lemma 2.1 (see [15]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \delta_n, \tag{2.2}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n \to \infty} \delta_n / \gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n \to \infty} a_n = 0$.

3. Main Result

In this section we will state and prove our main result.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Suppose that $\mathrm{VI}(C, A) \neq \emptyset$. For given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by*

$$\begin{aligned} y_n &= P_C[(1 - \alpha_n)(x_n - \lambda A x_n)], \\ x_{n+1} &= P_C(y_n - \lambda A y_n), \quad n \ge 0, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\lambda \in [a, b] \subset (0, 2\alpha)$ is a constant. Assume the following conditions are satisfied:

(C1) : $\lim_{n \to \infty} \alpha_n = 0$;

(C2) : $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) : $\lim_{n \to \infty} (\alpha_{n+1} / \alpha_n) = 1$.

Then the sequence $\{x_n\}$ generated by (3.1) *converges strongly to $P_{\mathrm{VI}(C,A)}(0)$ which is the minimum-norm element in* $\mathrm{VI}(C, A)$.

We will divide our detailed proofs into several conclusions.

Proof. Take $x^* \in \mathrm{VI}(C, A)$. First we need to use the following facts:

(1) $x^* = P_C(x^* - \lambda A x^*)$ for all $\lambda > 0$; in particular,

$$x^* = P_C[x^* - \lambda(1-\alpha_n)Ax^*] = P_C[\alpha_n x^* + (1-\alpha_n)(x^* - \lambda A x^*)], \quad \forall n \geq 0; \tag{3.2}$$

(2) $I - \lambda A$ is nonexpansive and for all $x, y \in C$

$$\left\|(I-\lambda A)x - (I-\lambda A)y\right\|^2 \leq \left\|x-y\right\|^2 + \lambda(\lambda - 2\alpha)\left\|Ax - Ay\right\|^2. \tag{3.3}$$

From (3.1), we have

$$\begin{aligned}
\left\|y_n - x^*\right\| &= \|P_C[(1-\alpha_n)(x_n - \lambda A x_n)] - P_C[\alpha_n x^* + (1-\alpha_n)(x^* - \lambda A x^*)]\| \\
&\leq \|\alpha_n(-x^*) + (1-\alpha_n)[(x_n - \lambda A x_n) - (x^* - \lambda A x^*)]\| \\
&\leq \alpha_n\|x^*\| + (1-\alpha_n)\|(I-\lambda A)x_n - (I-\lambda A)x^*\| \\
&\leq \alpha_n\|x^*\| + (1-\alpha_n)\|x_n - x^*\|.
\end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|P_C(y_n - \lambda A y_n) - P_C(x^* - \lambda A x^*)\right\| \\
&\leq \left\|(y_n - \lambda A y_n) - (x^* - \lambda A x^*)\right\| \\
&\leq \left\|y_n - x^*\right\| \\
&\leq \alpha_n\|x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\leq \max\{\|x^*\|, \|x_0 - x^*\|\}.
\end{aligned} \tag{3.5}$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{Ax_n\}$, and $\{Ay_n\}$.

From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\|P_C(y_n - \lambda A y_n) - P_C(y_{n-1} - \lambda A y_{n-1})\right\| \\
&\leq \left\|(y_n - \lambda A y_n) - (y_{n-1} - \lambda A y_{n-1})\right\| \\
&\leq \left\|y_n - y_{n-1}\right\| \\
&= \|P_C[(1-\alpha_n)(x_n - \lambda A x_n)] - P_C[(1-\alpha_{n-1})(x_{n-1} - \lambda A x_{n-1})]\| \\
&\leq \|(1-\alpha_n)[(I-\lambda A)x_n - (I-\lambda A)x_{n-1}] - (\alpha_n - \alpha_{n-1})(I-\lambda A)x_{n-1}\| \\
&\leq (1-\alpha_n)\|(I-\lambda A)x_n - (I-\lambda A)x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|(I-\lambda A)x_{n-1}\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M,
\end{aligned} \tag{3.6}$$

where $M > 0$ is a constant such that $\sup_n\{\|(I-\lambda A)x_n\|, \|(I-\lambda A)x_n\|(\|(I-\lambda A)x_n\| + 2\|x_n - x^*\|)\} \leq M$. Hence, by Lemma 2.1, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.7}$$

From (3.4), (3.5) and the convexity of the norm, we deduce

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|\alpha_n(-x^*) + (1-\alpha_n)[(x_n - \lambda A x_n) - (x^* - \lambda A x^*)]\|^2 \\
&\leq \alpha_n\|x^*\|^2 + (1-\alpha_n)\|(I-\lambda A)x_n - (I-\lambda A)x^*\|^2 \\
&\leq \alpha_n\|x^*\|^2 + (1-\alpha_n)\left[\|x_n - x^*\|^2 + \lambda(\lambda - 2\alpha)\|Ax_n - Ax^*\|^2\right] \\
&\leq \alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 + (1-\alpha_n)a(b-2\alpha)\|Ax_n - Ax^*\|^2.
\end{aligned} \tag{3.8}$$

Therefore, we have

$$\begin{aligned}
(1-\alpha_n)a(2\alpha - b)\|Ax_n - Ax^*\|^2 &\leq \alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n\|x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \times \|x_n - x_{n+1}\|.
\end{aligned} \tag{3.9}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain $\|Ax_n - Ax^*\| \to 0$ as $n \to \infty$.

By the property (ii) of the metric projection P_C, we have

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &= \|P_C[(1-\alpha)(x_n - \lambda A x_n)] - P_C(x^* - \lambda A x^*)\|^2 \\
&\leq \left\langle (1-\alpha_n)(x_n - \lambda A x_n) - (x^* - \lambda A x^*), y_n - x^*\right\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - \lambda A x_n) - (x^* - \lambda A x^*) - \alpha_n(I-\lambda A)x_n\|^2 + \left\|y_n - x^*\right\|^2 \\
&\qquad - \left\|(x_n - \lambda A x_n) - (x^* - \lambda A x^*) - \left(y_n - x^*\right) - \alpha_n(I-\lambda A)x_n\right\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|(x_n - \lambda A x_n) - (x^* - \lambda A x^*)\|^2 + \alpha_n M + \left\|y_n - x^*\right\|^2 \\
&\qquad - \left\|(x_n - y_n) - \lambda(Ax_n - Ax^*) - \alpha_n(I-\lambda A)x_n\right\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \alpha_n M + \left\|y_n - x^*\right\|^2 - \left\|x_n - y_n\right\|^2 \\
&\qquad + 2\lambda\left\langle x_n - y_n, Ax_n - Ax^*\right\rangle + 2\alpha_n\left\langle (I-\lambda A)x_n, x_n - y_n\right\rangle \\
&\qquad - \|\lambda(Ax_n - Ax^*) + \alpha_n(I-\lambda A)x_n\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \alpha_n M + \left\|y_n - x^*\right\|^2 - \left\|x_n - y_n\right\|^2 \\
&\qquad + 2\lambda\left\|x_n - y_n\right\|\|Ax_n - Ax^*\| + 2\alpha_n\|(I-\lambda A)x_n\|\left\|x_n - y_n\right\|\Big\}.
\end{aligned} \tag{3.10}$$

It follows that

$$\begin{aligned}\left\|y_n - x^*\right\|^2 \le{}& \|x_n - x^*\|^2 + \alpha_n M - \|x_n - y_n\|^2 \\ &+ 2\lambda\left\|x_n - y_n\right\|\|Ax_n - Ax^*\| + 2\alpha_n\|(I-\lambda A)x_n\|\left\|x_n - y_n\right\|,\end{aligned} \tag{3.11}$$

and hence

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \left\|y_n - x^*\right\|^2 \\ &\le \|x_n - x^*\|^2 + \alpha_n M - \left\|x_n - y_n\right\|^2 + 2\lambda\left\|x_n - y_n\right\|\|Ax_n - Ax^*\| \\ &\quad + 2\alpha_n\|(I-\lambda A)x_n\|\left\|x_n - y_n\right\|\end{aligned} \tag{3.12}$$

which implies that

$$\begin{aligned}\left\|x_n - y_n\right\|^2 \le{}& (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + \alpha_n M + 2\lambda\left\|x_n - y_n\right\|\|Ax_n - Ax^*\| \\ &+ 2\alpha_n\|(I-\lambda A)x_n\|\left\|x_n - y_n\right\|.\end{aligned} \tag{3.13}$$

Since $\alpha_n \to 0, \|x_n - x_{n+1}\| \to 0$, and $\|Ax_n - Ax^*\| \to 0$, we derive $\|x_n - y_n\| \to 0$.

Next we show that

$$\limsup_{n\to\infty}\left\langle z_0, z_0 - y_n\right\rangle \le 0, \tag{3.14}$$

where $z_0 = P_{\mathrm{VI}(C,A)}(0)$. To show it, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\left\langle z_0, z_0 - y_n\right\rangle = \lim_{i\to\infty}\left\langle z_0, z_0 - y_{n_i}\right\rangle. \tag{3.15}$$

As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$ converges weakly to z.

Next we show that $z \in \mathrm{VI}(C, A)$. We define a mapping T by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.16}$$

Then T is maximal monotone (see [16]). Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Av\rangle \ge 0$. On the other hand, from $y_n = P_C[(1-\alpha_n)(x_n - \lambda A x_n)]$, we have

$$\left\langle v - y_n, y_n - (1-\alpha_n)(x_n - \lambda A x_n)\right\rangle \ge 0, \tag{3.17}$$

that is,

$$\left\langle v - y_n, \frac{y_n - x_n}{\lambda} + Ax_n + \frac{\alpha_n}{\lambda}(I - \lambda A)x_n\right\rangle \ge 0. \tag{3.18}$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, w \rangle &\geq \langle v - y_{n_i}, Av \rangle \\
&\geq \langle v - y_{n_i}, Av \rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + Ax_{n_i} + \frac{\alpha_{n_i}}{\lambda}(I - \lambda A)x_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, Av - Ax_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda} - \frac{\alpha_{n_i}}{\lambda}(I - \lambda A)x_{n_i} \right\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i} \rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + \frac{\alpha_{n_i}}{\lambda}(I - \lambda A)x_{n_i} \right\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + \frac{\alpha_{n_i}}{\lambda}(I - \lambda A)x_{n_i} \right\rangle.
\end{aligned} \tag{3.19}$$

Noting that $\alpha_{n_i} \to 0$, $\|y_{n_i} - x_{n_i}\| \to 0$, and A is Lipschitz continuous, we obtain $\langle v - z, w \rangle \geq 0$. Since T is maximal monotone, we have $z \in T^{-1}(0)$, and hence $z \in \mathrm{VI}(C, A)$. Therefore,

$$\limsup_{n \to \infty} \langle z_0, z_0 - y_n \rangle = \lim_{i \to \infty} \langle z_0, z_0 - y_{n_i} \rangle = \langle z_0, z_0 - z \rangle \leq 0. \tag{3.20}$$

Finally, we prove $x_n \to z_0$. By the property (ii) of metric projection P_C, we have

$$\begin{aligned}
\|y_n - z_0\|^2 &= \|P_C[(1 - \alpha_n)(x_n - \lambda Ax_n)] - P_C[\alpha_n z_0 + (1 - \alpha_n)(z_0 - \lambda Az_0)]\|^2 \\
&\leq \langle \alpha_n(-z_0) + (1 - \alpha_n)[(x_n - \lambda Ax_n) - (z_0 - \lambda Az_0)], y_n - z_0 \rangle \\
&\leq \alpha_n \langle z_0, z_0 - y_n \rangle + (1 - \alpha_n)\|(x_n - \lambda Ax_n) - (z_0 - \lambda Az_0)\| \|y_n - z_0\| \\
&\leq \alpha_n \langle z_0, z_0 - y_n \rangle + (1 - \alpha_n)\|x_n - z_0\| \|y_n - z_0\| \\
&\leq \alpha_n \langle z_0, z_0 - y_n \rangle + \frac{1 - \alpha_n}{2}\left(\|x_n - z_0\|^2 + \|y_n - z_0\|^2\right).
\end{aligned} \tag{3.21}$$

Hence,

$$\|y_n - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n \langle z_0, z_0 - y_n \rangle. \tag{3.22}$$

Therefore,

$$\|x_{n+1} - z_0\|^2 \leq \|y_n - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n \langle z_0, z_0 - y_n \rangle. \tag{3.23}$$

We apply Lemma 2.1 to the last inequality to deduce that $x_0 \to z_0$. This completes the proof. □

Remark 3.2. Our Algorithm (3.1) is similar to Noor's modified extragradient method; see [2]. However, our algorithm has strong convergence in the setting of infinite-dimensional Hilbert spaces.

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