Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 808514, 26 pages
doi:10.1155/2012/808514

Research Article

Adaptive Finite Element Method for Optimal Control Problem Governed by Linear Quasiparabolic Integrodifferential Equations

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Received 21 June 2012; Accepted 10 July 2012

Academic Editor: Xinguang Zhang

The mathematical formulation for a quadratic optimal control problem governed by a linear quasiparabolic integrodifferential equation is studied. The control constrains are given in an integral sense: $U_{ad} = \{u \in X; \int_{\Omega_U} u \geqslant 0, t \in [0,T]\}$. Then the a posteriori error estimates in $L^\infty(0,T;H^1(\Omega))$-norm and $L^2(0,T;L^2(\Omega))$-norm for both the state and the control approximation are given.

1. Introduction

Integrodifferential equations of quasiparabolic and their control of this nature appear in applications such as biology mechanics, nuclear reaction dynamics, heat conduction in materials with memory, and viscoelasticity. All these models express a conservation of a certain quantity in any moment for any subdomain and the historical accumulation feature in the physical models. This in many applications is the most desirable feature of the approximation method when it comes to numerical solution of the corresponding initial boundary value problem. The existence and uniqueness of the solution of the quasiparabolic Integrodifferential equations has been studied in [1]. Finite element methods for quasiparabolic Integrodifferential equations problems with a smooth kernel have been discussed in Cui [2]. Although there is so much work for the finite element approximation of this problem, to our knowledge, there has been a lack of a posteriori error estimates for finite element approximation of any quasiparabolic Integrodifferential optimal control problem.

The finite element approximation of optimal control problems has been an important topic in engineering design works. There have been extensive theoretical and numerical studies for various optimal control problems, see, for instance, [3–11], although it is impossible to give even a very brief review here. And research on finite element approximation of parabolic optimal control problems can be found in, for example, [12, 13].

Among many finite element methods, the adaptive finite element method based on a posteriori error estimates has become a central theme in scientific and engineering computations for its high efficiency. In order to obtain a numerical solution of acceptable accuracy, it is essential for the adaptive finite element method to use a posteriori error estimate indicators to guide the mesh refinement procedure. We only need refine the area where the error indicators are larger, so that a higher density of nodes are distributed over the area where the solution is difficult to approximate. In this sense, adaptive finite element approximation relies very much on the error indicators used, which are often based on a posteriori error estimates of the solutions.

The purpose of this paper is to derive the a posteriori error estimates for the semidiscrete finite element approximation of a quadratic optimal control problem governed by a linear quasiparabolic Integrodifferential equation, which paves a way to derive the a posteriori error estimates for the full discrete finite element approximation for this control problem and thus to develop its adaptive finite element schemes. We extend the existing techniques and results in [14–16] to the optimal control problem governed by the Integrodifferential equation of quasiparabolic type.

The outline of the paper is as follows. In Section 2, we first briefly introduce the optimal control problem and give the optimality conditions, then construct the finite element approximation schemes for the optimal control problem. In Section 3, we give the a posteriori error bounds in $L^\infty(0,T;H^1(\Omega))$-norm for the control problem. And the a posteriori error bounds in $L^2(0,T;L^2(\Omega))$-norm for the control problem are derived in Section 4.

2. Optimal Control Problem and Its Finite Element Approximation

Let Ω and Ω_U be bounded convex polygon domains in R^d with Lipschitz boundary $\partial\Omega$ and $\partial\Omega_U$. In this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$, and seminorm $|\cdot|_{m,q,\Omega}$. We set $W_0^{m,q}(\Omega) = \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$, with norm $\|\cdot\|_{m,\Omega}$, and seminorm $|\cdot|_{m,\Omega}$.

We denote by $L^s(0,T;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,q}(\Omega))} = (\int_0^T \|v\|_{W^{m,q}(\Omega)}^s dt)^{1/s}$ for $s \in [1,\infty)$ and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^1(0,T;W^{m,q}(\Omega))$ and $C^k(0,T;W^{m,q}(\Omega))$. The details can be found in [17]. In addition, c or C denotes a general positive constant independent of the mesh size h.

In the following, we will give semi-discrete finite element approximation schemes for the optimal control problem governed by a linear quasiparabolic Integrodifferential equation.

2.1. Model Problem and Its Weak Formulation

We will take the state space $W = L^2(0,T;V)$ with $V = H_0^1(\Omega)$ and the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega_U)$. Let the observation space $Y = L^2(0,T;H)$ with $H = L^2(\Omega)$ and $U_{ad} \subseteq X$ a convex subset.

We are interested in the following optimal control problem:

$$\min_{u\in U_{ad}\subset X} J(u,y(u)) = \frac{1}{2}\left\{\int_0^T \|y-z_d\|_{0,\Omega}^2 dt + \int_0^T \|u\|_{0,\Omega_U}^2 dt\right\}, \tag{2.1}$$

subject to

$$\begin{gathered} y_t - \operatorname{div}\left(A\nabla y_t + D\nabla y + \int_0^t C(t,\tau)\nabla y(x,\tau)d\tau\right) = f + Bu, \quad \text{in } \Omega\times(0,T], \\ y = 0, \quad \text{on } \partial\Omega\times[0,T], \\ y|_{t=0} = y^0, \quad \text{in } \Omega, \end{gathered} \tag{2.2}$$

where u is control, y is state, z_d is the observation, U_{ad} is a closed convex subset, $f(x,t)\in L^2(0,T;L^2(\Omega))$, and z_d and $y^0\in H^1(\Omega)$ are some suitable functions to be specified later. B is a linear bounded operator from $L^2(\Omega_U)$ to $L^2(\Omega)$ independent of t. And

$$A = A(x) = \left(a_{i,j}(\cdot)\right)_{n\times n} \in \left(C^\infty\left(\overline{\Omega}\right)\right)^{n\times n}, \qquad D = D(x) = \left(d_{i,j}(\cdot)\right)_{n\times n} \in \left(C^\infty\left(\overline{\Omega}\right)\right)^{n\times n}, \tag{2.3}$$

such that there is a constant $c>0$ satisfying that for any vector $X\in R^n$ as follows:

$$X^t AX \ge c\|X\|_{R^n}^2, \qquad X^t DX \ge c\|X\|_{R^n}^2, \tag{2.4}$$

$C = C(x,t,\tau) = (c_{i,j}(x,t,\tau))_{n\times n} \in (C^\infty(0,T;L^2(\overline{\Omega}))^{n\times n})$.

Let

$$\begin{gathered} (f_1,f_2) = \int_\Omega f_1 f_2, \quad \forall (f_1,f_2)\in H\times H, \\ (u,v)_U = \int_{\Omega_U} uv, \quad \forall (u,v)\in U\times U, \\ a(z,\omega) = (A\nabla z,\nabla\omega), \qquad d(z,\omega) = (D\nabla z,\nabla\omega), \\ c(t,\tau;z,\omega) = (C(t,\tau)\nabla z,\nabla\omega), \quad \forall z,w\in V\times V. \end{gathered} \tag{2.5}$$

In the case that $f_1\in V$ and $f_2\in V^*$, the dual pair (f_1,f_2) is understood as $\langle f_1,f_2\rangle_{V\times V^*}$.

Assume that there are constants c and C, such that for all t and τ in $[0,T]$ as follows:

$$\begin{aligned} &(a)\ a(z,z) \geqslant c\|z\|_{1,\Omega}^2, \\ &(\text{b})\ |a(z,w)| \leqslant C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \qquad |d(z,w)| \leqslant C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \\ &(\text{c})\ |c(t,\tau;z,w)| \leqslant C\|z\|_{1,\Omega}\|w\|_{1,\Omega}. \end{aligned} \tag{2.6}$$

for any z and w in V.

Then the weak form of the state equation reads as

$$\begin{gathered}(y_t, w) + a(y_t, w) + d(y, w) + \int_0^t c(t, \tau; y(\tau), w)\, d\tau = (f + Bu, w) \\ \forall w \in V, t \in (0, T], \\ y|_{t=0} = y^0.\end{gathered} \tag{2.7}$$

It is well known (see, e.g., [1]) that the above weak formulation has at least one solution in $y \in W(0,T) = \{w \in L^\infty(0,T;H^1(\Omega)), w_t' \in L^2(0,T;H^1(\Omega))\}$.

Therefore, the weak form of the control problem (2.1) and (2.2) reads as (OCP)

$$\begin{gathered}\min_{u \in U_{ad}} J(u, y(u)), \\ (y_t, w) + a(y_t, w) + d(y, w) + \int_0^t c(t, \tau; y(\tau)\ , w)\, d\tau = (f + Bu, w) \quad \forall w \in V,\ t \in (0, T], \\ y|_{t=0} = y^0.\end{gathered} \tag{2.8}$$

In the following, we first give the existence and uniqueness of the solution of the system (2.8).

Theorem 2.1. *Assume that the condition* (2.6) *(a)–(c) holds. There exists the unique solution* (u, y) *for the minimization problem* (2.8) *such that* $u \in L^2(0,T;L^2(\Omega_U))$, $y \in L^\infty(0,T;H^1(\Omega))$, *and* $y_t \in L^2(0,T;H^1(\Omega))$.

Proof. Let $\{(u^n, y^n)\}_{n=1}^\infty$ be a minimization sequence for the system (2.8), then the sequence $\{u^n\}_{n=1}^\infty$ is bounded in $L^2(0,T;L^2(\Omega_U))$. Thus there is a subsequence of $\{u^n\}_{n=1}^\infty$ (still denote by $\{u^n\}_{n=1}^\infty$) such that u^n converges to u^* weakly in $L^2(0,T;L^2(\Omega_U))$. For the subsequence $\{u^n\}_{n=1}^\infty$, we have

$$\begin{gathered}(y_t^n, w) + a(y_t^n, w) + d(y^n, w) + \int_0^t c(t, \tau; y^n(\tau), w(t))\, d\tau = (f + Bu^n, w) \\ \forall w \in V,\ t \in (0, T].\end{gathered} \tag{2.9}$$

By setting $w = y^n$ and integrating from 0 to t in (2.9), we give

$$\begin{aligned}&\|y^n(t)\|_{1,\Omega}^2 + \int_0^t \|y^n\|_{1,\Omega}^2 d\tau \\ &\quad \leqslant C\left\{\left\|y^0\right\|_{1,\Omega} + C\int_0^t \left(\|f\|_{-1,\Omega}^2 + \|u^n\|_{0,\Omega_U}^2\right)dt + \int_0^t\int_0^\tau \|y(s)\|_{1,\Omega}^2 ds\, d\tau\right\}.\end{aligned} \tag{2.10}$$

Applying Gronwall's inequality to (2.10) yields

$$\|y^n\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|y^n\|_{L^2(0,T;H^1(\Omega))}^2 \leqslant C\left\{\left\|y^0\right\|_{1,\Omega}^2 + \int_0^T \left(\|f\|_{-1,\Omega}^2 + \|u^n\|_{0,\Omega_U}^2\right)\right\}. \tag{2.11}$$

So $\{u^n\}_{n=1}^{\infty}$ is a bounded set in $L^2(0,T;L^2(\Omega_U))$ and $\{y^n\}_{n=1}^{\infty}$ is a bounded set in $L^\infty(0,T;H^1(\Omega))$. Thus

$$\begin{gathered} u^n \longrightarrow u \quad \text{weakly in } L^2\Big(0,T;L^2(\Omega_U)\Big), \\ y^n \longrightarrow y \quad \text{weakly in } L^\infty\Big(0,T;H^1(\Omega)\Big), \\ y^n(T) \longrightarrow y(T) \quad \text{weakly in } H^1(\Omega). \end{gathered} \tag{2.12}$$

Let $W = \{w;\ w \in L^\infty(0,T;H^1(\Omega)),\ w_t' \in L^2(0,T;H^1(\Omega))\}$.

By integrating time from 0 to T in (2.9) and taking limit as $n \to \infty$, we obtain

$$\begin{aligned} &(y(T),w(T)) + a(y(T),w(T)) - \int_0^T \left[(y,w_t') + a(y,w_t') + d(y,w)\right] \\ &\quad + \int_0^T\int_0^t c(t,\tau;y(\tau),w(t))d\tau\, dt \\ &= \left(y^0,w(0)\right) + a\left(y^0,w(0)\right) + \int_0^T (f+Bu,w), \quad \forall w \in W. \end{aligned} \tag{2.13}$$

Then,

$$(y_t,w) + a(y_t,w) + d(y,w) + \int_0^t c(t,\tau;y(\tau),w)d\tau = (f+Bu,w), \quad \forall w \in V,\ t \in (0,T]. \tag{2.14}$$

Furthermore, we have

$$\int_0^T \left[(y_t,y_t) + a(y_t,y_t) + d(y,y_t) + \int_0^t c(t,\tau;y(\tau),y_t(t))d\tau\right]dt = \int_0^T (f+Bu,y_t). \tag{2.15}$$

Then, we get

$$\int_0^T \|y_t\|_{1,\Omega}^2 \leqslant C\int_0^T \left[\|f\|_{-1,\Omega}^2 + \|u\|_{0,\Omega_U}^2 + \|y\|_{1,\Omega}^2 + \int_0^t \|y\|_{1,\Omega}^2 d\tau\right]. \tag{2.16}$$

This means $y_t \in L^2(0,T;H^1(\Omega))$. So (u,y) is one solution of (2.8).

Since $\int_0^T \|y - z_d\|_{0,\Omega}^2$ is a convex function on space $L^2(0,T;L^2(\Omega))$ and $(\alpha/2)\int_0^T \|u\|_{0,\Omega_U}^2$ is a strictly convex function on U, hence $J(u,y(u))$ is a strictly convex function on U, so the minimization problem (2.8) has one unique solution. □

2.2. Optimality Conditions and Their Finite Element Approximation

By the theory of optimal control problem (see [18]), we can similarly deduce the following optimality conditions of the problem (2.8).

Theorem 2.2. *A pair* $(y,u) \in L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;L^2(\Omega_U))$ *is the solution of the optimal control problem* (2.8)*, if and only if there exists a costate* $p \in L^2(0,T;H_0^1(\Omega))$ *such that the triple* (y,p,u) *satisfies the following optimality conditions:*

$$(y_t,w) + a(y_t,w) + d(y,w) + \int_0^t c(t,\tau;y(\tau),w(t))d\tau = (f + Bu, w) \quad \forall w \in V,\ t \in (0,T], \qquad y|_{t=0} = y^0; \tag{2.17}$$

$$-(q,p_t) - a(q,p_t) + d(q,p) + \int_t^T c(\tau,t;q(t),p(\tau))d\tau = (y - z_d, q) \quad \forall q \in V, t \in [0,T), \qquad p|_{t=T} = 0; \tag{2.18}$$

$$\int_0^T (u + B^* p, v - u)_U dt \geqslant 0, \quad \forall v \in U_{ad}, \tag{2.19}$$

where B^* *is the adjoint operator of* B.

Let us consider the semi-discrete finite element approximation of the control problem (2.8). Here, we only consider triangular and conforming elements.

Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$. Let T^h be a partitioning of Ω^h into disjoint regular n-simplices τ, so that $\overline{\Omega}^h = \bigcup_{\tau \in T^h} \overline{\tau}$. Each element has at most one face on $\partial\Omega^h$, and $\overline{\tau}$ and $\overline{\tau}'$ have either only one common vertex or a whole edge or face if $\overline{\tau}$ and $\overline{\tau}' \in T^h$. We further require that $P_i \in \partial\Omega^h \Rightarrow P_i \in \partial\Omega$ where P_i $(i = 1,\dots,J)$ is the vertex set associated with the triangulation T_h. As usual, h denotes the diameter of the triangulation T^h. For simplicity, we assume that Ω is a convex polygon so that $\Omega = \Omega^h$.

Associated with T^h is a finite-dimensional subspace S^h of $C(\overline{\Omega}^h)$, such that $\chi|_\tau$ are polynomials of order m $(m \geq 1)$ for all $\chi \in S^h$ and $\tau \in T^h$. Let $V^h = \{v_h \in S_h : \ v_h(P_i) = 0\ (i = 1,\dots,J)\}, W^h = L^2(0,T;V^h)$. Note that we do not impose a continuity requirement. It is easy to see that $V^h \subset V,\ W^h \subset W$.

Let T_U^h be a partitioning of Ω_U^h into disjoint regular n-simplices τ_U, so that $\overline{\Omega}_U^h = \bigcup_{\tau_U \in T_U^h} \overline{\tau}_U$. $\overline{\tau}_U$ and $\overline{\tau}_U'$ have either only one common vertex or a whole edge or face if $\overline{\tau}_U$ and $\overline{\tau}_U' \in T_U^h$. We further require that $P_i \in \partial\Omega_U^h \Rightarrow P_i \in \partial\Omega_U$, where P_i $(i = 1,\dots,J)$ is the vertex set associated with the triangulation T_U^h. For simplicity, we again assume that Ω_U is a convex polygon so that $\Omega_U = \Omega_U^h$.

Associated with T_U^h is another finite-dimensional subspace U^h of $L^2(\Omega_U^h)$, such that $\chi|_{\tau_U}$ are polynomials of order m $(m \geqslant 0)$ for all $\chi \in U^h$ and $\tau_U \in T_U^h$. Here there is no requirement for the continuity. Let $X^h = L^2(0,T;U^h)$. It is easy to see that $X^h \subset X$. Let $h_\tau(h_{\tau_U})$ denote the maximum diameter of the element $\tau(\tau_U)$ in $T^h(T_U^h)$.

Due to the limited regularity of the optimal control u in general, there will be no advantage in considering higher-order finite element spaces than the piecewise constant space for the control. We therefore only consider the piecewise constant finite element space for the approximation of the control, though higher-order finite element spaces will be used to approximate the state and the co-state. Let $P_0(\Omega)$ denote all the 0-order polynomial over Ω. Therefore, we always take $X^h = \{u \in X : \ u(x,t)|_{x\in\tau_U} \in P_0(\tau_U), \text{ for all } t \in [0,T]\}$. U_{ad}^h is a closed convex set in X^h. For ease of exposition, in this paper, we assume that $U_{ad}^h \subset U_{ad} \cap X^h$.

Then a possible semi-discrete finite element approximation of (OCP) is as follows $(\text{OCP})^h$:

$$\min_{u_h \in U_{ad}^h} J(u_h, y_h) = \frac{1}{2}\left\{\int_0^T \|y_h - z_d\|_{0,\Omega}^2 + \int_0^T \|u_h\|_{0,\Omega_U}^2\right\}, \tag{2.20}$$

such that

$$\begin{gathered}\left(\frac{\partial y_h}{\partial t}, w_h\right) + a\left(\frac{\partial y_h}{\partial t}, w_h\right) + d(y_h, w_h) + \int_0^t c(t,\tau; y_h(\tau), w_h(t))d\tau = (f + Bu_h, w_h), \\ \forall w_h \in V^h, \ t \in (0,T], \\ y_h|_{t=0} = y_h^0,\end{gathered} \tag{2.21}$$

where $y_h \in W^h$ and $y_h^0 \in V^h$ is the approximation of y^0.

In the same way of proving Theorem 2.1, we can easily prove that the problem (2.20)-(2.21) has a unique solution $(y_h, u_h) \in W^h \times U_{ad}^h$.

It is well known (see [18]) that a pair $(y_h, u_h) \in W^h \times U_{ad}^h$ is a solution of (2.20)-(2.21), if and only if there exists a co-state $p_h \in W^h$ such that the triple (y_h, p_h, u_h) satisfies the following optimality conditions:

$$\begin{gathered}\left(\frac{\partial y_h}{\partial t}, w_h\right) + a\left(\frac{\partial y_h}{\partial t}, w_h\right) + d(y_h, w_h) + \int_0^t c(t,\tau; y_h(\tau), w_h(t))d\tau = (f + Bu_h, w_h), \\ \forall w_h \in V^h, \\ y_h|_{t=0} = y_h^0,\end{gathered} \tag{2.22}$$

$$\begin{gathered}-\left(q_h, \frac{\partial p_h}{\partial t}\right) - a\left(q_h, \frac{\partial p_h}{\partial t}\right) + d(q_h, p_h) + \int_t^T c(\tau, t; q_h, p_h(\tau))d\tau = (y_h - z_d, q_h), \quad \forall q_h \in V^h, \\ p_h|_{t=T} = 0,\end{gathered} \tag{2.23}$$

$$\int_0^T (u_h + B^* p_h, v_h - u_h)_U dt \geqslant 0, \quad \forall v_h \in U_{ad}^h. \tag{2.24}$$

The optimality conditions in (2.22)–(2.24) are the semi-discrete approximation to the problem (2.17)–(2.19).

Introduce the local averaging operator π_h given by

$$(\pi_h w)\mid_{\tau_U} := \frac{\int_{\tau_U} w}{\int_{\tau_U} 1}, \quad \forall \tau_U \in T_U^h. \tag{2.25}$$

Then, we have $\int_{\Omega_U} w = \int_{\Omega_U} \pi_h w$ for any $w \in L^2(0,T;L^2(\Omega_U))$, $t \in [0,T]$ and (2.24) is equivalent to

$$\int_0^T \left(u_h + \pi_h(B^* p_h), v_h - u_h\right)_U dt \geqslant 0, \quad \forall v_h \in U_{ad}^h. \tag{2.26}$$

In the following, we derive the a posteriori error estimates for semi-discrete finite element approximation (2.22)–(2.24), allowing different meshes to be used for the state and the control.

The following lemmas are important in deriving the a posteriori error estimates of residual type.

Lemma 2.3 (see [19]). *Let $\widehat{\pi}_h$ be the standard Lagrange interpolation operator. For $m = 0$ or 1, $q > n/2$ and $v \in W^{2,q}(\Omega)$ as*

$$|v - \widehat{\pi}_h v|_{m,q,\Omega} \leqslant Ch^{2-m}|v|_{2,q,\Omega}. \tag{2.27}$$

Lemma 2.4 (see [20]). *Let π_h be the average interpolation operator defined in (2.25). For $m = 0$ or 1, $1 \leqslant q \leqslant \infty$ and for all $v \in W^{1,q}(\Omega^h)$ as*

$$|v - \pi_h v|_{m,q,\tau} \leqslant \sum_{\overline{\tau}' \cap \overline{\tau} \neq \emptyset} Ch_\tau^{1-m}|v|_{1,q,\tau'}. \tag{2.28}$$

Lemma 2.5 (see [21]). *For all $v \in W^{1,q}(\Omega)$, $1 \leqslant q < \infty$ as*

$$\|v\|_{0,q,\partial\tau} \leqslant C\left(h_\tau^{-1/q}\|v\|_{0,q,\tau} + h_\tau^{1-1/q}|v|_{1,q,\tau}\right). \tag{2.29}$$

3. A Posteriori Error Estimates in $L^\infty(0,T;H^1(\Omega))$-Norm

In this paper, the control constraints are given in an integral sense as follows:

$$U_{ad} = \left\{ v \in X;\ \int_{\Omega_U} v \geqslant 0, t \in [0,T] \right\}. \tag{3.1}$$

The following lemma is the first step to derive the a posteriori error estimates of residual type.

Lemma 3.1. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (2.17)–(2.19) *and* (2.22)–(2.24). *Then, we have*

$$\|u - u_h\|^2_{L^2(0,T;L^2(\Omega_U))} \leqslant C\eta_1^2 + C\|p_h - p(u_h)\|^2_{L^2(0,T;L^2(\Omega))}, \tag{3.2}$$

where

$$\eta_1^2 = \int_0^T \left\{ \sum_{\tau_U} \int_{\tau_U} \left(B^* p_h - P_h(B^* p_h)\right)^2 \right\} dt, \tag{3.3}$$

P_h *is the* L^2*-projection from* $L^2(\Omega)$ *to* U^h, *and* $p(u_h)$ *is defined by the following system:*

$$\begin{aligned} &\left(\frac{\partial}{\partial t} y(u_h), \omega\right) + a\left(\frac{\partial}{\partial t} y(u_h), \omega\right) + d\left(y(u_h), \omega\right) + \int_0^t c\left(t, \tau; y(u_h)(\tau), \omega(t)\right) d\tau \\ &\quad = \left(f + Bu_h, \omega\right), \quad \forall\, \omega \in V, \end{aligned} \tag{3.4}$$

$$y(u_h)(0) = y_0^h(x), \quad x \in \Omega,$$

$$\begin{aligned} &-\left(q, \frac{\partial}{\partial t} p(u_h)\right) - a\left(q, \frac{\partial}{\partial t} p(u_h)\right) + d\left(q, p(u_h)\right) + \int_t^T c\left(\tau, t; q(t), p(u_h)(\tau)\right) d\tau \\ &\quad = \left(y(u_h) - z_d, q\right), \quad \forall q \in V. \end{aligned} \tag{3.5}$$

Proof. From (2.19), we have

$$(u, u - u_h)_U \leqslant -\left(B^* p, u - u_h\right)_U. \tag{3.6}$$

Then, by (2.24) and (3.6), we have

$$\begin{aligned} \|u - u_h\|^2_{L^2(0,T;L^2(\Omega_U))} &= \int_0^T \left[(u, u - u_h)_U - (u_h, u - u_h)_U\right] dt = \int_0^T -\left(B^* p + u_h, u - u_h\right)_U dt \\ &= -\int_0^T \left(B^* p_h + u_h, u - v_h\right)_U dt - \int_0^T \left(B^* p_h + u_h, v_h - u_h\right)_U dt \\ &\quad + \int_0^T \left(B^* p_h - B^* p(u_h), u - u_h\right)_U dt + \int_0^T \left(B^* p(u_h) - B^* p, u - u_h\right)_U dt \\ &\leqslant \inf_{v_h \in U^h_{ad}} \int_0^T \left(B^* p_h + u_h, v_h - u\right)_U dt \\ &\quad + \int_0^T \left(B^*\left(p_h - p(u_h)\right), u - u_h\right)_U dt + \int_0^T \left(B^*\left(p(u_h) - p\right), u - u_h\right)_U dt \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.7}$$

Next, we will estimate I_1, I_2, and I_3, respectively.

(1) We first estimate I_1. Let P_h be the L^2-projection from $L^2(\Omega)$ to U^h.
We have

$$\int_{\Omega_U} (P_h v - v)\phi = 0, \quad \forall \phi \in X^h,\ v \in U_{ad},\ t \in (0,T]. \tag{3.8}$$

Since $v \in U_{ad}$, so $\int_{\Omega_U} P_h v \geqslant 0$, then $P_h v \in U_{ad}^h$. So that we can take $v_h = P_h u$ in I_1.

For given $t \in (0,T]$, let

$$u_h = P_h\left(-B^* p_h + \max\left\{0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1}\right\}\right). \tag{3.9}$$

We have $u_h \in X^h$. We will show that u_h is the solution of the variational inequality in (2.24) assuming p_h is known.

Since $\int_{\Omega_U}[P_h(-B^*p_h+\max\{0,\int_{\Omega_U} B^*p_h/\int_{\Omega_U} 1\})-(-B^*p_h+\max\{0,\int_{\Omega_U} B^*p_h/\int_{\Omega_U} 1\})] = 0$, we have

$$\int_{\Omega_U} u_h = -\int_{\Omega_U} B^* p_h + \int_{\Omega_U} \max\left\{0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1}\right\} = \begin{cases} -\int_{\Omega_U} B^* p_h, & \int_{\Omega_U} B^* p_h < 0, \\ 0, & \int_{\Omega_U} B^* p_h \geqslant 0. \end{cases} \tag{3.10}$$

Thus, $\int_{\Omega_U} u_h \geqslant 0$, we have $u_h \in U_{ad}^h$. Note that for all $v_h \in U_{ad}^h$, $t \in (0,T]$, we have

$$\begin{aligned}
&(u_h + B^* p_h, v_h - u_h)_U \\
&= \int_{\Omega_U}\left[P_h\left(-B^* p_h + \max\left\{0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1}\right\}\right)\right. \\
&\qquad \left. -\left(-B^* p_h + \max\left\{0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1}\right\}\right) + \max\left\{0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1}\right\}\right] (v_h - u_h) \\
&= \int_{\Omega_U} \max\left\{0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1}\right\}(v_h - u_h).
\end{aligned} \tag{3.11}$$

If $\int_{\Omega_U} B^* p_h < 0$, then

$$(u_h + B^* p_h, v_h - u_h)_U = \int_{\Omega_U} 0 \cdot (v_h - u_h) = 0. \tag{3.12}$$

If $\int_{\Omega_U} B^* p_h \geqslant 0$, since

$$\int_{\Omega_U} u_h = \int_{\Omega_U}\left(-B^* p_h + \max\left\{0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1}\right\}\right) = 0. \tag{3.13}$$

we have

$$\left(u_h + B^* p_h, v_h - u_h\right)_U = \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1} \int_{\Omega_U} (v_h - u_h) = \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1} \cdot \int_{\Omega_U} v_h \geqslant 0. \tag{3.14}$$

From (3.11)–(3.14), we obtain

$$\left(u_h + B^* p_h, v_h - u_h\right)_U \geqslant 0, \quad \forall v_h \in U_{ad}^h. \tag{3.15}$$

So $u_h = P_h(-B^* p_h + \max\{0, \int_{\Omega_U} B^* p_h / \int_{\Omega_U} 1\})$ is the solution of the variational inequality in (2.24) assuming p_h is known.

Then,

$$\begin{aligned} I_1 &\leqslant \int_0^T \left(B^* p_h + u_h, P_h u - u\right)_U dt \\ &= \int_0^T \left\{ \sum_{\tau_U} \int_{\tau_U} \left[P_h \left(-B^* p_h + \max\left\{ 0, \frac{\int_{\Omega_U} B^* p_h}{\int_{\Omega_U} 1} \right\} \right) + B^* p_h \right] (P_h u - u) \right\} dt. \end{aligned} \tag{3.16}$$

Since $\int_{\tau_U} (P_h u - u) = 0$, we have

$$\begin{aligned} I_1 &\leqslant \int_0^T \left\{ \sum_{\tau_U} \int_{\tau_U} \left(-P_h\left(B^* p_h\right) + B^* p_h\right)(P_h u - u) \right\} dt \\ &= \int_0^T \left\{ \sum_{\tau_U} \int_{\tau_U} \left(-P_h\left(B^* p_h\right) + B^* p_h\right)\left(P_h(u - u_h) - (u - u_h)\right) \right\} dt \\ &\leqslant C(\delta) \int_0^T \left\{ \sum_{\tau_U} \int_{\tau_U} \left(-P_h\left(B^* p_h\right) + B^* p_h\right)^2 \right\} dt + \delta \|u - u_h\|^2_{L^2(0,T;L^2(\Omega_U))} \\ &= C\eta_1^2 + \delta \|u - u_h\|^2_{L^2(0,T;L^2(\Omega_U))}. \end{aligned} \tag{3.17}$$

(2) Consider

$$I_2 = \int_0^T \left(B^*\left(p_h - p(u_h)\right), u - u_h\right)_U dt \leqslant C\left\|p_h - p(u_h)\right\|^2_{L^2(0,T;L^2(\Omega))} + \delta \|u - u_h\|^2_{L^2(0,T;L^2(\Omega_U))}. \tag{3.18}$$

(3) By (3.4) and (2.17), we have for $t \in (0, T]$

$$\begin{aligned} &\left(\frac{\partial}{\partial t}\left(y - y(u_h)\right), \omega\right) + a\left(\frac{\partial}{\partial t}\left(y - y(u_h)\right), \omega\right) + d\left(y - y(u_h), w\right) \\ &\quad + \int_0^t c\left(t, \tau; \left(y - y(u_h)\right)(\tau), \omega(t)\right) d\tau = (B(u - u_h), \omega), \quad \forall \omega \in V, \end{aligned} \tag{3.19}$$

and from (3.5) and (2.18), we have

$$
\begin{aligned}
&-\left(q,\frac{\partial}{\partial t}(p-p(u_h))\right)-a\left(q,\frac{\partial}{\partial t}(p-p(u_h))\right)+d(q,p-p(u_h))\\
&\quad+\int_t^T c(\tau,t;q(t),(p-p(u_h))(\tau))d\tau=(y-y(u_h),q),\quad \forall q\in V.
\end{aligned}
\tag{3.20}
$$

Then, from (3.19), (3.20), and integrating by part we have

$$
\begin{aligned}
I_3&=\int_0^T (B^*(p(u_h)-p),u-u_h)_U dt=\int_0^T (p(u_h)-p,B(u-u_h))_U dt\\
&=\int_0^T\left[\left(\frac{\partial}{\partial t}(y-y(u_h)),p(u_h)-p\right)+a\left(\frac{\partial}{\partial t}(y-y(u_h)),p(u_h)-p\right)\right.\\
&\qquad\left.+d(y-y(u_h),p(u_h)-p)+\int_0^t c(t,\tau;(y-y(u_h))(\tau),(p(u_h)-p)(t))d\tau\right]dt\\
&=\int_0^T\left[-\left(y-y(u_h),\frac{\partial}{\partial t}(p(u_h)-p)\right)-a\left(y-y(u_h),\frac{\partial}{\partial t}(p(u_h)-p)\right)\right.\\
&\qquad\left.+d(y-y(u_h),p(u_h)-p)+\int_t^T c(\tau,t;(y-y(u_h))(t),(p(u_h)-p)(\tau))d\tau\right]dt\\
&=\int_0^T -(y-y(u_h),y-y(u_h))dt\leqslant 0.
\end{aligned}
\tag{3.21}
$$

Following from (3.17)–(3.21), let δ be small enough as

$$
\|u-u_h\|^2_{L^2(0,T;L^2(\Omega_U))}\leqslant C\eta_1^2+C\|p_h-p(u_h)\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.22}
$$

This completes the proof. □

Lemma 3.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of* (2.17)–(2.19), *and* (2.22)–(2.24) *respectively. Then, there hold the a posteriori error estimates as*

$$
\begin{aligned}
&\|y_h-y(u_h)\|^2_{L^\infty(0,T;H^1(\Omega))}+\left\|\frac{\partial}{\partial t}(y_h-y(u_h))\right\|^2_{L^2(0,T;H^1(\Omega))}\\
&\quad+\|p_h-p(u_h)\|^2_{L^\infty(0,T;H^1(\Omega))}+\left\|\frac{\partial}{\partial t}(p_h-p(u_h))\right\|^2_{L^2(0,T;H^1(\Omega))}\leqslant C\sum_{i=2}^{6}\eta_i^2,
\end{aligned}
\tag{3.23}
$$

where

$$\eta_2^2 = \int_0^T \left\{ \sum_\tau h_\tau^2 \int_\tau \left(\frac{\partial p_h}{\partial t} - \operatorname{div}\left(A^* \nabla \frac{\partial p_h}{\partial t} \right) + \operatorname{div}(D^* \nabla p_h) + \int_t^T \operatorname{div}(C^*(\tau,t)\nabla p_h(\tau))d\tau + y_h - z_d \right)^2 d\tau \right\} dt,$$

$$\eta_3^2 = \int_0^T \sum_\tau h_l \int_{\partial\tau} \left[-\left(A^* \nabla \frac{\partial p_h}{\partial t} \right) \cdot n + (D^* \nabla p_h) \cdot n + \int_t^T (C^*(\tau,t)\nabla p_h(\tau)) \cdot n d\tau \right]^2 dl\, dt,$$

$$\eta_4^2 = \int_0^T \left\{ \sum_\tau h_\tau^2 \int_\tau \left(\frac{\partial}{\partial t} y_h - \operatorname{div}\left(A \nabla \frac{\partial y_h}{\partial t} \right) - \operatorname{div}(D \nabla y_h) - \int_0^t \operatorname{div}(C(t,\tau)\nabla y_h(\tau))d\tau - f - Bu_h \right)^2 \right\} dt,$$

$$\eta_5^2 = \int_0^T \sum_\tau h_l \int_{\partial\tau} \left[\left(A \nabla \frac{\partial y_h}{\partial t} \right) \cdot n + (A \nabla y_h) \cdot n + \int_0^t (C(t,\tau)\nabla y_h(\tau)) \cdot n d\tau \right]^2 dl\, dt,$$

$$\eta_6^2 = \left\| y_0^h - y_0 \right\|_{1,\Omega}^2, \tag{3.24}$$

where l is a face of an element τ, h_l is the maximum diameter of l, and $[\nabla p_h \cdot n]$ and $[\nabla y_h \cdot n]$ are the normal derivative jumps over the interior face l defined by

$$\begin{aligned} [\nabla p_h \cdot n]_l &= \left(\nabla p_h h|_{\tau_l^1} - \nabla p_h|_{\tau_l^2} \right) \cdot n, \\ [\nabla y_h \cdot n]_l &= \left(\nabla y_h|_{\tau_l^1} - \nabla y_h|_{\tau_l^2} \right) \cdot n, \end{aligned} \tag{3.25}$$

where n is the unit normal vector on $l = \tau_l^1 \cap \tau_l^2$ outwards τ_l^1. For later convenience, one can define $[\nabla p_h \cdot n]_l = 0$ and $[\nabla y_h \cdot n]_l = 0$ when $l \subset \partial\Omega$.

Proof. Let

$$\begin{aligned} \langle R(u_h), v \rangle = &-\left(v, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) - a\left(v, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) + d(v, p_h - p(u_h)) \\ &+ \int_t^T c(\tau, t; v(t), (p_h - p(u_h))(\tau)) d\tau, \end{aligned} \tag{3.26}$$

and π_h the average interpolation operator defined as in (2.25) and $e = p_h - p(u_h)$. Then, it follows from (2.23) and (3.5) that

$$\begin{aligned} &-\left(q_h, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) - a\left(q_h, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) + d(q_h, p_h - p(u_h)) \\ &+ \int_t^T c(\tau, t; q_h(t), (p_h - p(u_h))(\tau)) d\tau = (y_h - y(u_h), q_h), \quad \forall q_h \in V^h. \end{aligned} \tag{3.27}$$

We have

$$
\begin{aligned}
&\langle R(u_h), v\rangle \\
&= -\left(v-\pi_h v, \frac{\partial}{\partial t}(p_h - p(u_h))\right) - a\left(v-\pi_h v, \frac{\partial}{\partial t}(p_h - p(u_h))\right) + d(v-\pi_h v, p_h - p(u_h)) \\
&\quad + \int_t^T c(\tau, t; (v-\pi_h v)(t), (p_h - p(u_h))(\tau))d\tau - \left(\pi_h v, \frac{\partial}{\partial t}(p_h - p(u_h))\right) \\
&\quad - a\left(\pi_h v, \frac{\partial}{\partial t}(p_h - p(u_h))\right) + d(\pi_h v, p_h - p(u_h)) + \int_t^T c(\tau, t; \pi_h v(t), (p_h - p(u_h))(\tau))d\tau \\
&= -\left(v-\pi_h v, \frac{\partial p_h}{\partial t}\right) - a\left(v-\pi_h v, \frac{\partial p_h}{\partial t}\right) + d(v-\pi_h v, p_h) + \int_t^T c(\tau, t; (v-\pi_h v)(t), p_h(\tau))d\tau \\
&\quad - (y(u_h) - z_d, v-\pi_h v) + (y_h - y(u_h), \pi_h v) \\
&= \sum_\tau \int_\tau \left(-\frac{\partial p_h}{\partial t} + \operatorname{div}\left(A^*\nabla \frac{\partial p_h}{\partial t}\right) - \operatorname{div}(D^*\nabla p_h) - \int_t^T \operatorname{div}(C^*(\tau,t)\nabla p_h(\tau))d\tau - y_h + z_d\right) \\
&\quad \times (v-\pi_h v) + \sum_\tau \int_{\partial\tau} \left[-\left(A^*\nabla \frac{\partial p_h}{\partial t}\right)\cdot n + (D^*\nabla p_h)\cdot n + \int_t^T (C^*(\tau,t)\nabla p_h(\tau))\cdot n d\tau\right] \\
&\quad \times (v-\pi_h v) + (y_h - y(u_h), v) \\
&\leqslant \Bigg\{\sum_\tau \int_\tau h_\tau^2 \Bigg(-\frac{\partial p_h}{\partial t} + \operatorname{div}\left(A^*\nabla \frac{\partial p_h}{\partial t}\right) - \operatorname{div}(D^*\nabla p_h) \\
&\qquad\qquad - \int_t^T \operatorname{div}(C^*(\tau,t)\nabla p_h(\tau))d\tau - y_h + z_d\Bigg)^2 \\
&\qquad + \sum_\tau \int_{\partial\tau} h_l \left[-\left(A^*\nabla \frac{\partial p_h}{\partial t}\right)\cdot n + (D^*\nabla p_h)\cdot n + \int_t^T (C^*(\tau,t)\nabla p_h(\tau))\cdot n d\tau\right]^2\Bigg\}^{1/2} \\
&\quad \times \|v\|_{1,\Omega} + (y_h - y(u_h), v).
\end{aligned}
\tag{3.28}
$$

Taking $v = p_h - p(u_h)$ in (3.28) and from (2.6), we have

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|p_h - p(u_h)\|_{0,\Omega}^2 - \frac{1}{2}\frac{d}{dt}a(p_h - p(u_h), p_h - p(u_h)) + c\|p_h - p(u_h)\|_{1,\Omega}^2 \\
&\quad \leqslant \Bigg\{\sum_\tau \int_\tau h_\tau^2 \Bigg(-\frac{\partial p_h}{\partial t} + \operatorname{div}\left(A^*\nabla \frac{\partial p_h}{\partial t}\right) - \operatorname{div}(D^*\nabla p_h) - \int_t^T \operatorname{div}(C^*(\tau,t)\nabla p_h(\tau))d\tau \\
&\qquad\qquad -y_h + z_d\Bigg)^2 + \sum_\tau \int_{\partial\tau} h_l \left[-\left(A^*\nabla \frac{\partial p_h}{\partial t}\right)\cdot n + (D^*\nabla p_h)\cdot n\right]^2\Bigg\}^{1/2}
\end{aligned}
$$

$$\times\|p_h - p(u_h)\|_{1,\Omega} + (y_h - y(u_h), p_h - p(u_h)) - \int_t^T c(\tau, t; (p_h - p(u_h))(t), (p_h - p(u_h))(\tau))d\tau. \tag{3.29}$$

Integrating time from t to T in (3.29) and by Schwartz inequality, Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
&\frac{1}{2}\|p_h - p(u_h)\|_{0,\Omega}^2 + c\|p_h - p(u_h)\|_{1,\Omega}^2 + c\int_t^T \|p_h - p(u_h)\|_{1,\Omega}^2 d\tau \\
&\leqslant \int_t^T \sum_\tau h_\tau^2 \times \int_\tau \bigg(\frac{\partial p_h}{\partial t} - \operatorname{div}\left(A^*\nabla \frac{\partial p_h}{\partial t}\right) + \operatorname{div}(D^*\nabla p_h) \\
&\qquad + \int_\tau^T \operatorname{div}(C^*(s,\tau)\nabla p_h(s))ds + y_h - z_d\bigg)^2 d\tau \\
&\quad + \int_t^T \sum_\tau h_l \int_{\partial\tau}\left[-\left(A^*\nabla \frac{\partial p_h}{\partial t}\right)\cdot n + (D^*\nabla p_h)\cdot n + \int_\tau^T (C^*(s,\tau)\nabla p_h(s))\cdot n ds\right]^2 d\tau \\
&\quad + \delta\int_t^T \|p_h - p(u_h)\|_{1,\Omega}^2 d\tau + C\int_t^T \|y_h - y(u_h)\|_{0,\Omega}^2 d\tau + C\int_t^T\int_\tau^T \|(p_h - p(u_h))(s)\|_{1,\Omega}^2 ds\, d\tau.
\end{aligned} \tag{3.30}$$

Letting δ be small enough, we have

$$\begin{aligned}
&\int_t^T \|p_h - p(u_h)\|_{1,\Omega}^2 d\tau \\
&\quad\leqslant C\int_t^T \sum_\tau h_\tau^2 \int_\tau \bigg(\frac{\partial p_h}{\partial t} - \operatorname{div}\left(A^*\nabla \frac{\partial p_h}{\partial t}\right) + \operatorname{div}(D^*\nabla p_h) \\
&\qquad\qquad + \int_\tau^T \operatorname{div}(C^*(s,\tau)\nabla p_h(s))ds + y_h - z_d\bigg)^2 d\tau \\
&\qquad + C\int_t^T \sum_\tau h_l \int_{\partial\tau}\bigg[-\left(A^*\nabla \frac{\partial p_h}{\partial t}\right)\cdot n + (D^*\nabla p_h)\cdot n \\
&\qquad\qquad + \int_\tau^T (C^*(s,\tau)\nabla p_h(s))\cdot n ds\bigg]^2 d\tau \\
&\qquad + C\int_t^T \|y_h - y(u_h)\|_{0,\Omega}^2 d\tau + C\int_t^T\int_\tau^T \|(p_h - p(u_h))(s)\|_{1,\Omega}^2 ds\, d\tau.
\end{aligned} \tag{3.31}$$

Then, from Gronwall inequality and (3.28)–(3.31) we have

$$\|p_h - p(u_h)\|_{L^2(0,T;H^1(\Omega))}^2 \leqslant C\eta_2^2 + C\eta_3^2 \quad + C\|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2. \tag{3.32}$$

Similarly,

$$\begin{aligned}\|p_h - p(u_h)\|^2_{L^\infty(0,T;H^1(\Omega))} &\leqslant C\Big(\eta_2^2 + \eta_3^2 + \|y_h - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}\Big) \\ &\quad + C\int_0^T\int_t^T \|(p_h - p(u_h))(\tau)\|^2_{1,\Omega} d\tau\, dt \\ &\leqslant C\eta_2^2 + C\eta_3^2 \quad + C\|y_h - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{3.33}$$

In the same way of getting (3.32),by setting $v = (\partial/\partial t)(p_h - p(u_h))$ in (3.28), we have

$$\left\|\frac{\partial}{\partial t}(p_h - p(u_h))\right\|^2_{L^2(0,T;H^1(\Omega))} \leqslant C\eta_2^2 + C\eta_3^2 + C\|y_h - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.34}$$

Similarly analysis for $\|y_h - y(u_h)\|_{L^\infty(0,T;H^1(\Omega))}$, we let

$$\begin{aligned}\langle Q(u_h), v\rangle = {}& \left(\frac{\partial}{\partial_t}(y_h - y(u_h)), v\right) + a\left(\frac{\partial}{\partial_t}(y_h - y(u_h)), v\right) + d(y_h - y(u_h), v) \\ & + \int_0^t c(t,\tau;(y_h - y(u_h))(\tau), v(t))d\tau.\end{aligned} \tag{3.35}$$

From (2.22) and (3.4), we obtain

$$\begin{aligned}&\left(\omega_h, \frac{\partial}{\partial t}(y_h - y(u_h))\right) + a\left(\frac{\partial}{\partial t}(y_h - y(u_h)), \omega_h\right) + d(y_h - y(u_h), \omega_h) \\ &\quad + \int_0^t c(t,\tau;(y_h - y(u_h))(\tau), \omega_h(t))d\tau = 0, \quad \forall \omega_h \in V^h.\end{aligned} \tag{3.36}$$

We have

$$\begin{aligned}&\langle Q(u_h), v\rangle \\ &= \left(\frac{\partial}{\partial t}(y_h - y(u_h)), v - \pi_h v\right) + a\left(\frac{\partial}{\partial t}(y_h - y(u_h)), v - \pi_h v\right) + d(y_h - y(u_h), v - \pi_h v) \\ &\quad + \int_0^t c(t,\tau;(y_h - y(u_h))(\tau), (v - \pi_h v)(t))d\tau \\ &= \left(\frac{\partial y_h}{\partial t}, v - \pi_h v\right) + a\left(\frac{\partial y_h}{\partial t}, v - \pi_h v\right) + d(y_h, v - \pi_h v) \\ &\quad + \int_0^t c(t,\tau;y_h(\tau), (v - \pi_h v)(t))d\tau - (f + Bu_h, v - \pi_h v) \\ &= \sum_\tau \int_\tau \left(\frac{\partial y_h}{\partial t} - \operatorname{div}\left(A\nabla\frac{\partial y_h}{\partial t}\right) - \operatorname{div}(D\nabla y_h) - \int_0^t \operatorname{div}(C(t,\tau)\nabla y_h)d\tau - f - Bu_h\right)(v - \pi_h v)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\tau}\int_{\partial\tau}\left[\left(A\nabla\frac{\partial y_h}{\partial t}\right)\cdot n+(D\nabla y_h)\cdot n+\int_0^t(C(t,\tau)\nabla y_h)\cdot n d\tau\right](v-\pi_h v)\\
&\leqslant\left\{\sum_{\tau}\int_{\tau}h_\tau^2\left(\frac{\partial}{\partial t}y_h-\operatorname{div}\left(A\nabla\frac{\partial y_h}{\partial t}\right)-\operatorname{div}(D\nabla y_h)-\int_0^t\operatorname{div}(C(t,\tau)\nabla y_h)d\tau-f-Bu_h\right)^2\right.\\
&\qquad\left.+\sum_{\tau}\int_{\partial\tau}h_l\left[\left(A\nabla\frac{\partial y_h}{\partial t}\right)\cdot n+(D\nabla y_h)\cdot n+\int_0^t(C(t,\tau)\nabla y_h)\cdot n d\tau\right]^2\right\}^{1/2}\|v\|_{1,\Omega}.
\end{aligned}
\tag{3.37}
$$

By setting $v=y_h-y(u_h)$ and Swartz inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y_h-y(u_h)\|_{0,\Omega}^2+\frac{1}{2}\frac{d}{dt}a(y_h-y(u_h),y_h-y(u_h))+c\|y_h-y(u_h)\|_{1,\Omega}^2\\
&\leqslant\sum_{\tau}\int_{\tau}h_\tau^2\left(\frac{\partial y_h}{\partial t}-\operatorname{div}\left(A\nabla\frac{\partial y_h}{\partial t}\right)-\operatorname{div}(D\nabla y_h)-\int_0^t\operatorname{div}(C(t,\tau)\nabla y_h)d\tau-f-Bu_h\right)^2\\
&\quad+\sum_{\tau}\int_{\partial\tau}h_l\left[\left(A\nabla\frac{\partial y_h}{\partial t}\right)\cdot n+(D\nabla y_h)\cdot n+\int_0^t(C(t,\tau)\nabla y_h)\cdot n d\tau\right]^2\\
&\quad+\delta\|y_h-y(u_h)\|_{1,\Omega}^2-\int_0^t c(t,\tau;(y_h-y(u_h))(\tau),(y_h-y(u_h))(t))d\tau.
\end{aligned}
\tag{3.38}
$$

Integrating time from 0 to t in (3.38), we obtain

$$
\begin{aligned}
&\|y_h-y(u_h)\|_{1,\Omega}^2+c\int_0^t\|y_h-y(u_h)\|_{1,\Omega}^2d\tau\\
&\leqslant C\left\{\int_0^t\sum_{\tau}h_\tau^2\int_{\tau}\left(\frac{\partial y_h}{\partial t}-\operatorname{div}\left(A\nabla\frac{\partial y_h}{\partial t}\right)-\operatorname{div}(D\nabla y_h)\right.\right.\\
&\qquad\qquad\left.-\int_0^\tau\operatorname{div}(C(\tau,s)\nabla y_h)ds-f-Bu_h\right)^2d\tau\\
&\qquad\left.+\int_0^t\sum_{\tau}h_l\int_{\partial\tau}\left[\left(A\nabla\frac{\partial y_h}{\partial t}\right)\cdot n+(D\nabla y_h)\cdot n+\int_0^\tau(C(\tau,s)\nabla y_h)\cdot n ds\right]^2dt\right\}\\
&\quad+\delta\int_0^t\|y_h-y(u_h)\|_{1,\Omega}^2d\tau+C\int_0^t\int_0^\tau\|y_h-y(u_h)\|_{1,\Omega}^2ds\,d\tau+C\left\|y_0-y_0^h\right\|_{1,\Omega}^2.
\end{aligned}
\tag{3.39}
$$

Since δ is small enough, then from (3.39) and Gronwall inequality, we have

$$
\int_0^t\|y_h-y(u_h)\|_{1,\Omega}^2dt
$$

$$\leqslant C\int_0^t \sum_\tau h_\tau^2 \int_\tau \left(\frac{\partial y_h}{\partial t} - \operatorname{div}\left(A\nabla \frac{\partial y_h}{\partial t} \right) - \operatorname{div}(D\nabla y_h) - \int_0^\tau \operatorname{div}(C(\tau,s)\nabla y_h)ds - f - Bu_h \right)^2 d\tau$$
$$+ C\int_0^t \sum_\tau h_l \int_{\partial\tau} \left[\left(A\nabla \frac{\partial y_h}{\partial t} \right)\cdot n + (D\nabla y_h)\cdot n + \int_0^\tau (C(\tau,s)\nabla y_h)\cdot n ds \right]^2 dt + C\left\| y_0 - y_0^h \right\|_{1,\Omega}^2. \tag{3.40}$$

Then,

$$\begin{aligned} \|y_h - y(u_h)\|_{L^2(0,T;H^1(\Omega))}^2 &\leqslant C\left(\eta_4^2 + \eta_5^2 + \eta_6^2\right), \\ \|y_h - y(u_h)\|_{L^\infty(0,T;H^1(\Omega))}^2 &\leqslant C\left(\eta_4^2 + \eta_5^2 + \eta_6^2\right). \end{aligned} \tag{3.41}$$

In the same way of getting (3.34), we can similarly obtain

$$\left\| \frac{\partial}{\partial t}(y_h - y(u_h)) \right\|_{L^2(0,T;H^1(\Omega))}^2 \leqslant C\left(\eta_4^2 + \eta_5^2 + \eta_6^2\right). \tag{3.42}$$

Then the desired results (3.23) follow from (3.32)–(3.34) and (3.41)-(3.42). □

From Lemmas 3.1 and 3.2, we have the following results.

Theorem 3.3. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.17)–(2.19) and (2.22)–(2.24) respectively. Then, there hold the a posteriori error estimates as*

$$\begin{aligned} &\|u - u_h\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|y - y_h\|_{L^\infty(0,T;H^1(\Omega))}^2 + \left\| \frac{\partial}{\partial t}(y - y_h) \right\|_{L^2(0,T;H^1(\Omega))}^2 \\ &\quad + \|p - p_h\|_{L^\infty(0,T;H^1(\Omega))}^2 + \left\| \frac{\partial}{\partial t}(p - p_h) \right\|_{L^2(0,T;H^1(\Omega))}^2 \leqslant C\sum_{i=1}^6 \eta_i^2, \end{aligned} \tag{3.43}$$

where η_1^2 is defined in Lemma 3.1.

Proof. First, from (3.27) and (3.36), and [2], we have the following stability results:

$$\begin{aligned} \|y - y(u_h)\|_{L^\infty(0,T;H^1(\Omega))}^2 + \left\| \frac{\partial}{\partial t}(y - y(u_h)) \right\|_{L^2(0,T;H^1(\Omega))}^2 &\leqslant C\|u - u_h\|_{L^2(0,T;L^2(\Omega_U))}^2, \\ \|p - p(u_h)\|_{L^\infty(0,T;H^1(\Omega))}^2 + \left\| \frac{\partial}{\partial t}(p - p(u_h)) \right\|_{L^2(0,T;H^1(\Omega))}^2 &\leqslant C\|y - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\leqslant C\|u - u_h\|_{L^2(0,T;L^2(\Omega_U))}^2. \end{aligned} \tag{3.44}$$

Then, the desired results (3.43) follows from triangle inequality, (3.44) and Lemmas 3.1 and 3.2.

This completes the proof. □

4. A Posteriori Error Estimates in $L^2(0,T;L^2(\Omega))$-Norm

In the following, we will derive the a posteriori error estimates in $L^2(0,T;L^2(\Omega))$-norm.

For given $F \in L^2(0,T;L^2(\Omega))$, we have

$$\begin{gathered}\frac{\partial \phi}{\partial t} - \operatorname{div}\left(A\nabla \frac{\partial \phi}{\partial t}\right) - \operatorname{div}(D\nabla\phi) - \int_0^t \operatorname{div}(C(t,\tau)\nabla\phi(\tau))d\tau = F, \quad (x,t) \in \Omega \times (0,T], \\ \frac{\phi}{\partial\Omega} = 0, \quad t \in (0,T], \\ \phi(x,0) = 0, \quad x \in \Omega,\end{gathered} \tag{4.1}$$

and its dual equation

$$\begin{gathered}-\frac{\partial \psi}{\partial t} + \operatorname{div}\left(A^*\nabla \frac{\partial \psi}{\partial t}\right) - \operatorname{div}(D^*\nabla\psi) - \int_t^T \operatorname{div}(C^*(\tau,t)\nabla\psi(\tau))d\tau = F, \quad (x,t) \in \Omega \times (0,T], \\ \frac{\psi}{\partial\Omega} = 0, \quad t \in (0,T], \\ \psi(x,T) = 0, \quad x \in \Omega.\end{gathered} \tag{4.2}$$

From [1, 2], we have the following stability results.

Lemma 4.1. *Assume that Ω is a convex domain. Let ϕ and ψ be the solution of* (4.1) *and* (4.2), *respectively. Then,*

$$\begin{aligned}\|\phi\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))}, \\ \|\nabla\phi\|_{L^2(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))}, \\ \left\|D^2\phi\right\|_{L^2(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))}, \\ \left\|\frac{\partial}{\partial t}\phi\right\|_{L^2(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))}, \\ \|\psi\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))}, \\ \|\nabla\psi\|_{L^2(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))}, \\ \left\|D^2\psi\right\|_{L^2(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))}, \\ \left\|\frac{\partial}{\partial t}\psi\right\|_{L^2(0,T;L^2(\Omega))} &\leqslant C\|F\|_{L^2(0,T;L^2(\Omega))},\end{aligned} \tag{4.3}$$

where $D^2\phi = \partial^2\phi/\partial x_i\partial x_j, 1 \leqslant i,j \leqslant n$, and $D^2\psi$ is defined similarly.

Using Lemmas 3.1 and 4.1, we have the following upperbounds.

Lemma 4.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of* (2.17)–(2.19) *and* (2.22)–(2.24), *respectively. Then, there hold the a posteriori error estimates as*

$$\|y_h - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))} + \|p_h - p(u_h)\|^2_{L^2(0,T;L^2(\Omega))} \leqslant C\left(\sum_{i=2}^{5}\xi_i^2 + \eta_6^2\right), \tag{4.4}$$

where η_6^2 *is defined in Lemma 3.2, and*

$$\begin{aligned}
\xi_2^2 &= \int_0^T \Bigg\{ \sum_\tau h_\tau^4 \int_\tau \left(\frac{\partial p_h}{\partial t} - \operatorname{div}\left(A^*\nabla \frac{\partial p_h}{\partial t}\right) + \operatorname{div}(D^*\nabla p_h) \right.\\
&\qquad \left. + \int_t^T \operatorname{div}(C^*(\tau,t)\nabla p_h(\tau))d\tau + y_h - z_d \right)^2 \Bigg\} dt,\\
\xi_3^2 &= \int_0^T \sum_\tau h_l^3 \int_{\partial\tau} \left[-\left(A^*\nabla\frac{\partial p_h}{\partial t}\right)\cdot n + (D^*\nabla p_h)\cdot n + \int_t^T (C^*(\tau,t)\nabla p_h(\tau))\cdot n d\tau \right]^2 dl\, dt,\\
\xi_4^2 &= \int_0^T \Bigg\{ \sum_\tau h_\tau^4 \int_\tau \left(\frac{\partial y_h}{\partial t} - \operatorname{div}\left(A\nabla \frac{\partial y_h}{\partial t}\right) - \operatorname{div}(D\nabla y_h) \right.\\
&\qquad \left. - \int_0^t \operatorname{div}(C(t,\tau)\nabla y_h(\tau))d\tau - f - Bu_h \right)^2 \Bigg\} dt,\\
\xi_5^2 &= \int_0^T \sum_\tau h_l^3 \int_{\partial\tau} \left[\left(A\nabla\frac{\partial y_h}{\partial t}\right)\cdot n + (D\nabla y_h)\cdot n + \int_0^t (C(t,\tau)\nabla y_h(\tau))\cdot n d\tau \right]^2 dl\, dt.
\end{aligned} \tag{4.5}$$

Proof. We first estimate $\|p_h - p(u_h)\|^2_{L^2(0,T;L^2(\Omega))}$.

Let ϕ be the solution of (4.1) with $F = p_h - p(u_h)$, and $\phi_I = \hat{\pi}_h\phi$ the interpolation of ϕ in Lemma 2.3.

From (4.1), (3.27), and by integrating by parts we obtain

$$\begin{aligned}
&\|p_h - p(u_h)\|^2_{L^2(0,T;L^2(\Omega))}\\
&= \int_0^T (F(t),(p_h - p(u_h))(t))dt\\
&= \int_0^T \Bigg[-\left(\frac{\partial}{\partial t}(p_h - p(u_h)),\phi\right) - a\left(\phi, \frac{\partial}{\partial t}(p_h - p(u_h))\right)\\
&\qquad + d(\phi, p_h - p(u_h)) + \int_0^t c(t,\tau;\phi(\tau),(p_h - p(u_h))(t))d\tau \Bigg] dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \bigg[-\left(\frac{\partial}{\partial t}(p_h - p(u_h)), \phi - \phi_I \right) - a\left(\phi - \phi_I, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) \\
&\qquad + d(\phi - \phi_I, p_h - p(u_h)) + \int_0^t c(t,\tau; (\phi - \phi_I)(\tau), (p_h - p(u_h))(t))\, d\tau \\
&\qquad - \left(\frac{\partial}{\partial t}(p_h - p(u_h)), \phi_I \right) - a\left(\phi_I, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) \\
&\qquad + d(\phi_I, p_h - p(u_h)) + \int_0^t c(t,\tau; \phi_I(\tau), (p_h - p(u_h))(t))\, d\tau \bigg] dt \\
&= \int_0^T \bigg[-\left(\frac{\partial p_h}{\partial t}, \phi - \phi_I \right) - a\left(\phi - \phi_I, \frac{\partial p_h}{\partial t} \right) \\
&\qquad + d(\phi - \phi_I, p_h) + \int_0^t c(t,\tau; (\phi - \phi_I)(\tau), p_h(t))\, d\tau + \left(\frac{\partial p(u_h)}{\partial t}, \phi - \phi_I \right) \\
&\qquad + a\left(\phi - \phi_I, \frac{\partial p(u_h)}{\partial t} \right) - d(\phi - \phi_I, p(u_h)) - \int_0^t c(t,\tau; (\phi - \phi_I)(\tau), p(u_h)(t))\, d\tau \\
&\qquad - \left(\frac{\partial}{\partial t}(p_h - p(u_h)), \phi_I \right) - a\left(\phi_I, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) \\
&\qquad + d(\phi_I, p_h - p(u_h)) + \int_0^t c(t,\tau; \phi_I(\tau), (p_h - p(u_h))(t))\, d\tau \bigg] dt \\
&= \int_0^T \bigg[-\left(\frac{\partial p_h}{\partial t}, \phi - \phi_I \right) - a\left(\phi - \phi_I, \frac{\partial p_h}{\partial t} \right) \\
&\qquad + d(\phi - \phi_I, p_h) + \int_t^T c(\tau, t; (\phi - \phi_I)(t), p_h(\tau))\, d\tau \\
&\qquad + \left(\frac{\partial p(u_h)}{\partial t}, \phi - \phi_I \right) + a\left(\phi - \phi_I, \frac{\partial p(u_h)}{\partial t} \right) \\
&\qquad - d(\phi - \phi_I, p(u_h)) - \int_t^T c(\tau, t; (\phi - \phi_I)(t), p(u_h)(\tau))\, d\tau \\
&\qquad - \left(\frac{\partial}{\partial t}(p_h - p(u_h)), \phi_I \right) - a\left(\phi_I, \frac{\partial}{\partial t}(p_h - p(u_h)) \right) \\
&\qquad + d(\phi_I, p_h - p(u_h)) + \int_t^T c(\tau, t; \phi_I(t), (p_h - p(u_h))(\tau))\, d\tau \bigg] dt \\
&= \int_0^T \bigg[-\left(\frac{\partial p_h}{\partial t}, \phi - \phi_I \right) - a\left(\phi - \phi_I, \frac{\partial p_h}{\partial t} \right) + d(\phi - \phi_I, p_h) + \int_t^T c(\tau, t; (\phi - \phi_I)(t), p_h(\tau))\, d\tau \bigg] dt \\
&\quad - \int_0^T (y(u_h) - z_d, \phi - \phi_I)\, dt + \int_0^T (y_h - y(u_h), \phi_I)\, dt \\
&= \int_0^T \bigg\{ \sum_\tau \int_\tau \bigg(-\frac{\partial p_h}{\partial t} + \operatorname{div}\left(A^* \nabla \frac{\partial p_h}{\partial t} \right) - \operatorname{div}(D^* \nabla p_h)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_t^T \operatorname{div}\left(C^*(\tau,t)\nabla p_h(\tau)\right)d\tau + y_h - z_d\bigg)(\phi - \phi_I)\bigg\}dt \\
&+ \int_0^T \sum_\tau \int_{\partial\tau}\left[-\left(A^*\nabla\frac{\partial p_h}{\partial t}\right)\cdot n + \left(D^*\nabla p_h\right)\cdot n + \int_t^T \left(C^*(\tau,t)\nabla p_h(\tau)\right)\cdot n d\tau\right] \\
&\times (\phi - \phi_I)dl\,dt + \int_0^T (y_h - y(u_h), \phi)dt \\
&= J_1 + J_2 + J_3.
\end{aligned}
\tag{4.6}
$$

It follows from Lemmas 2.3, 2.5, and 4.1 that

$$
\begin{aligned}
J_1 \leqslant{} & C(\delta)\int_0^T \left\{\sum_\tau h_\tau^4 \int_\tau \left(\frac{\partial p_h}{\partial t} - \operatorname{div}\left(A^*\nabla\frac{\partial p_h}{\partial t}\right) + \operatorname{div}\left(D^*\nabla p_h\right)\right.\right. \\
&\qquad \left.\left. + \int_t^T \operatorname{div}\left(C^*(\tau,t)\nabla p_h(\tau)\right)d\tau + y_h - z_d\right)^2\right\}dt \\
& + \delta\int_0^T |\phi|_{2,\Omega}^2 dt \leqslant C(\delta)\xi_2^2 + \delta\|p_h - p(u_h)\|_{L^2(0,T;L^2(\Omega))}^2, \\
J_2 \leqslant{} & C(\delta)\int_0^T \sum_\tau h_l^3 \int_{\partial\tau}\left[-\left(A^*\nabla\frac{\partial p_h}{\partial t}\right)\cdot n + \left(D^*\nabla p_h\right)\cdot n + \int_t^T \left(C^*(\tau,t)\nabla p_h(\tau)\right)\cdot n d\tau\right]^2 dl\,dt \\
& + \delta\int_0^T |\phi|_{0,\Omega}^2 dt \\
\leqslant{} & C(\delta)\xi_3^2 + \delta\|p_h - p(u_h)\|_{L^2(0,T;L^2(\Omega))}^2.
\end{aligned}
\tag{4.7}
$$

By Schwartz inequality, we have

$$
J_3 \leqslant C(\delta)\|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2 + \delta\|p_h - p(u_h)\|_{L^2(0,T;L^2(\Omega))}^2. \tag{4.8}
$$

Letting δ be small enough, it follows from (4.6)–(4.8) that

$$
\|p_h - p(u_h)\|_{L^2(0,T;L^2(\Omega))}^2 \leqslant C\sum_{i=2}^3 \xi_i^2 + C\|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2. \tag{4.9}
$$

Next, we estimate $\|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2$. Similarly let ψ be the solution of (4.2) with $F = y_h - y(u_h)$, and $\psi_I = \widehat{\pi}_h\psi$ the interpolation of ψ in Lemma 2.3.

Then, it follows from (3.36) and integrating by parts that

$$
\begin{aligned}
&\|y_h - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}\\
&= \int_0^T (F(t), (y_h - y(u_h))(t))\,dt\\
&= \int_0^T \Bigg[\left(\frac{\partial}{\partial t}(y_h - y(u_h)), \varphi\right) + a\left(\frac{\partial}{\partial t}(y_h - y(u_h)), \varphi\right) + d(y_h - y(u_h), \varphi)\\
&\qquad + \int_t^T c(\tau, t; (y_h - y(u_h))(t), \varphi(\tau))\,d\tau\Bigg]dt\\
&\quad + a\left(y_0^h - y_0, \varphi(0)\right) + \left(y_0^h - y_0, \varphi(0)\right)\\
&= \int_0^T \Bigg[\left(\frac{\partial}{\partial t}(y_h - y(u_h)), \varphi - \varphi_I\right) + a\left(\frac{\partial}{\partial t}(y_h - y(u_h)), \varphi - \varphi_I\right)\\
&\qquad + d(y_h - y(u_h), \varphi - \varphi_I) + \int_t^T c(\tau, t; (y_h - y(u_h))(t), (\varphi - \varphi_I)(\tau))\,d\tau\Bigg]dt\\
&\quad + \int_0^T \Bigg[\left(\frac{\partial}{\partial t}(y_h - y(u_h)), \varphi_I\right) + a\left(\frac{\partial}{\partial t}(y_h - y(u_h)), \varphi_I\right)\\
&\qquad + d(y_h - y(u_h), \varphi_I) + \int_t^T c(\tau, t; (y_h - y(u_h))(t), \varphi_I(\tau))\,d\tau\Bigg]dt\\
&\quad + a\left(y_0^h - y_0, \varphi(0)\right) + \left(y_0^h - y_0, \varphi(0)\right)\\
&= \int_0^T \Bigg[\left(\frac{\partial y_h}{\partial t}, \varphi - \varphi_I\right) + a\left(\frac{\partial y_h}{\partial t}, \varphi - \varphi_I\right) + d(y_h, \varphi - \varphi_I)\\
&\qquad + \int_0^t c(t, \tau; y_h(\tau), (\varphi - \varphi_I)(t))\,d\tau\Bigg]dt\\
&\quad - \int_0^T (f + Bu_h, \varphi - \varphi_I)\,dt + a\left(y_0^h - y_0, \varphi(0)\right) + \left(y_0^h - y_0, \varphi(0)\right)\\
&= \int_0^T \Bigg\{\sum_\tau \int_\tau \Bigg(\frac{\partial y_h}{\partial t} - \operatorname{div}\left(A\nabla \frac{\partial y_h}{\partial t}\right) - \operatorname{div}(D\nabla y_h)\\
&\qquad\qquad - \int_0^t \operatorname{div}(C(\tau, t)\nabla y_h(\tau))\,d\tau - f - Bu_h\Bigg)(\varphi - \varphi_I)\Bigg\}dt\\
&\quad + \int_0^T \sum_\tau \int_{\partial\tau} \left[\left(A\nabla \frac{\partial y_h}{\partial t}\right)\cdot n + (D\nabla y_h)\cdot n - \int_0^t (C(t, \tau)\nabla y_h(\tau))\cdot n\right]d\tau(\varphi - \varphi_I)\,dl\,dt\\
&\quad + a\left(y_0^h - y_0, \varphi(0)\right) + \left(y_0^h - y_0, \varphi(0)\right) = D_1 + D_2 + D_3 + D_4.
\end{aligned}
\tag{4.10}
$$

Similarly, it follows from Lemmas 2.3, 2.5 and 4.1 that

$$D_1 \leqslant C(\delta)\int_0^T \left\{ \sum_\tau h_\tau^4 \int_\tau \left(\frac{\partial y_h}{\partial t} - \operatorname{div}\left(A\nabla \frac{\partial y_h}{\partial t}\right) - \operatorname{div}(D\nabla y_h) - \int_0^t \operatorname{div}(C(\tau,t)\nabla y_h(\tau))d\tau - f - Bu_h \right)^2 \right\} dt$$

$$+ \delta \int_0^T |\varphi|_{2,\Omega}^2 dt \leqslant C\xi_4^2 + \delta \|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2,$$

$$D_2 \leqslant C(\delta)\int_0^T \sum_\tau h_l^3 \int_{\partial\tau} \left[\left(A\nabla \frac{\partial y_h}{\partial t}\right)\cdot n + (D\nabla y_h)\cdot n - \int_0^t (C(t,\tau)\nabla y_h(\tau))\cdot n \right]^2 dl\, dt$$

$$+ \delta \int_0^T |\varphi|_{2,\Omega}^2 dt \leqslant C\xi_5^2 + \delta \|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2,$$

$$D_3 + D_4 \leqslant C\eta_6^2 + \delta \|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2 \tag{4.11}$$

Letting δ be small enough, then from (4.10)–(4.11), we have

$$\|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}^2 \leqslant C\left(\xi_4^2 + \xi_5^2 + \eta_6^2\right). \tag{4.12}$$

The desired results (4.4) follows from (4.9)–(4.12).This completes the proof. □

Using Lemmas 3.1 and 4.2, we have the following upper bounds.

Theorem 4.3. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (2.17)–(2.19) *and* (2.22)–(2.24), *respectively. Then, there hold the a posteriori error estimates as*

$$\|u - u_h\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|y - y_h\|_{L^2(0,T;L^2(\Omega))}^2 + \|p - p_h\|_{L^2(0,T;L^2(\Omega))}^2 \leqslant C\left(\eta_1^2 + \sum_{i=2}^5 \xi_i^2 + \eta_6^2 \right). \tag{4.13}$$

Proof. By triangle inequality, (3.44) and (3.38), Lemmas 3.1 and 4.2, we can easily prove (4.13) in the same way of getting (3.43).

This completes the proof. □

5. Conclusion

In this paper, we study the semi-discrete adaptive finite element method for optimal control problem governed by a linear quasiparabolic Integrodifferential equation. We extend the existing methods in studying adaptive finite element approximation of optimal control governed by a parabolic Integrodifferential equation to the control governed by a quasiparabolic Integrodifferential equation. After presenting the weak form and the existence and uniqueness of the solution for the optimal control problem, the a posteriori error estimates for semi-discrete finite element approximations in $L^{\infty}(0,T;H^1(\Omega))$-norm and $L^2(0,T;L^2(\Omega))$-norm are derived. The work will pave a way to derive the a posteriori error estimates of full discrete finite element approximations of this optimal control problem

Acknowledgments

This research was supported by Science and Technology Development Planning project of Shandong Province (no. 2011GGH20118), Shandong Province Natural Science Foundation (no. ZR2009AQ004), and National Natural Science Foundation of China (no. 11071141).

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