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Research Article

Asymptotic Stability of Differential Equations with Infinite Delay

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A theorem on asymptotic stability is obtained for a differential equation with an infinite delay in a function space which is suitable for the numerical computation of the solution to the infinite delay equation.

1. Introduction and Preliminaries

In this paper, we study the asymptotic stability of the solutions to the infinite delay differential equation given below:

$$\begin{gathered} x'(t) = ax(t) + \sum_{i=1}^{\infty} b_i x(t-\tau_i), \quad t \geq 0, \\ x(\theta) = \phi(\theta), \quad \theta \in (-\infty, 0], \end{gathered} \tag{1.1}$$

under the following assumptions.

(i) There exists $p > 0$ with $|b_i| \leq p\gamma^{-i}$ for all $i \in \mathbb{N}$.

(ii) $\tau_i \leq i\tau_1$ for all $i \in \mathbb{N}$.

The asymptotic stability of a linear infinite delay equation is studied in [1–5] in the context of abstract phase spaces which includes the space:

$$\left\{\phi \in \mathbf{C}(-\infty,0] : \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\left|\phi(\theta)\right| < \infty, \lim_{\theta\to\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\right\}. \tag{1.2}$$

The asymptotic constancy neutral equations are studied in [6]. Linear time-invariant systems with constant point delays are studied in [7] and in [8]; a Razumikhin approach is used to study exponential stability of delay equations. Asymptotic stability and stabilization of linear delay-differential equations are studied in [9].

In this paper, the phase space $\mathbf{C}_\sigma(-\infty,0]$ for the initial function is chosen as follows. Let $m_i = i\tau_1 > 0$ and $\beta_i = p\gamma^{-i}$. The space $\mathbf{C}_\sigma(-\infty,0]$ is defined as

$$\left\{\phi \in \mathbf{C}(-\infty,0] : \sum_{i=1}^{\infty}\beta_i \sup_{\theta\in[-m_i,0]}\left|\phi(\theta)\right| < \infty\right\}. \tag{1.3}$$

Here $\mathbf{C}(-\infty,0]$ is the set of continuous complex valued functions defined on $(-\infty,0]$.

The motivation to consider the above type of phase space is that for numerical computation of solutions it is enough to know the values of the initial data over a finite domain at every stage of computation. See [10, 11].

The following definitions and results are well known, see for example [5] or [12].

Definition 1.1. The Kuratowski measure of noncompactness $\alpha(V)$ of the subset V of a Banach space X is defined by

$$\alpha(V) = \inf\Big\{d > 0 : \text{there exists a finite number of sets } V_1, V_2, \ldots, V_n, \text{ with diam } V_j \le d \text{ such that } V = \cup_{j=1}^{n} V_j\Big\}. \tag{1.4}$$

For a bounded linear operator $L : X \to Y$, $|L|_\alpha$ is defined as

$$|L|_\alpha = \inf\{k > 0 : \alpha(L(V)) \le k\alpha(V) \text{ for all bounded sets } V\}. \tag{1.5}$$

Proposition 1.2. *Let X, Y, Z be Banach spaces and $M : X \to Y$, $L : Y \to Z$ be bounded linear operators. Then, $|M \circ L|_\alpha \le |M|_\alpha |L|_\alpha$. Further, if $M : X \to Y$ is compact, then $|M|_\alpha = 0$.*

Theorem 1.3. *Let X be a Banach space and let $A : \mathbf{D}(A) \to X$ be the infinitesimal generator of a semigroup of operators $S_t : X \to X$. Then, the growth bound of the semigroup ω_0 defined as*

$$\omega_0 = \lim_{t\to\infty}\frac{1}{t}\ln(\|S_t\|) = \inf\{\omega : \exists M \ge 1 \text{ such that } \|S_t\| \le Me^{\omega t}\}, \tag{1.6}$$

is given by

$$\omega_0 = \max\{s(A), \omega_{\text{ess}}\}, \tag{1.7}$$

where $s(A) = \sup\{\Re(\lambda) : \lambda \in \operatorname{spec}(A)\}$ *and*

$$\omega_{\text{ess}} = \lim_{t \to \infty} \frac{1}{t} \ln(|S_t|_\alpha). \tag{1.8}$$

In Theorem 1.3, spec(A) is the compliment of the resolvent set $\rho(A)$ which is the set of all $\lambda \in \mathbf{C}$ such that the operator $\lambda I - A$ is one-one and onto and $(\lambda I - A)^{-1}$ is a bounded linear map.

For a real number r, $\lfloor r \rfloor = \max\{n \in \mathbf{Z} : n \leq r\}$ and $\lceil r \rceil = \min\{n \in \mathbf{Z} : n \geq r\}$. We will make use of the observation $\lceil r \rceil \leq \lfloor r \rfloor \leq r + 1$ for $r \in \mathbb{R}$.

2. Asymptotic Stability of a PDE

Consider the following simple initial boundary value problem for a PDE:

$$\begin{gathered} \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \theta}, \quad t \geq 0, \ \theta \leq 0, \\ u(t, 0) = 0, \quad t \geq 0, \\ u(0, \theta) = u_0(\theta), \quad \theta \leq 0, \end{gathered} \tag{2.1}$$

where $u_0 \in \mathbf{C}_{\sigma,0}(-\infty, 0] = \{u \in \mathbf{C}_\sigma(-\infty, 0] : u(0) = 0\}$.

Its mild solution is given by the semigroup $T_t : C_{\sigma,0}(-\infty, 0] \to C_{\sigma,0}(-\infty, 0]$ defined as

$$\begin{aligned} T_t u_0(\theta) &= u_0(t + \theta), \quad t + \theta < 0 \\ &= 0, \quad t + \theta \geq 0. \end{aligned} \tag{2.2}$$

Proposition 2.1. *Let* $m_i = i\tau_1$ *and* $\beta_i = p\gamma^{-i}$. *The infinitesimal generator of the semigroup defined by* (2.2) *is given by* $B : \mathbf{D}(B) \to \mathbf{C}_{\sigma,0}(-\infty, 0]$, $B\phi = \phi'$, *where*

$$\mathbf{D}(B) = \left\{\phi \in \mathbf{C}_{\sigma,0}(-\infty, 0] : \phi' \in \mathbf{C}_{\sigma,0}(-\infty, 0]\right\}. \tag{2.3}$$

Further, $\rho(B) = \{\lambda : \Re(\lambda) > -\ln\gamma/\tau_1\}$.

Besides, if $\Re(\lambda) > -\ln\gamma/\tau_1$, then $e_\lambda \in \mathbf{C}_\sigma(-\infty, 0]$ and for every $f \in \mathbf{C}_\sigma(-\infty, 0]$, h defined as $h(\theta) = \int_0^\theta e^{\lambda(\theta - \xi)} f(\xi) d\xi$ and e_λ defined as $e_\lambda(\theta) = e^{\lambda\theta}$ are elements of $\mathbf{C}_\sigma(-\infty, 0]$.

Finally, for the semigroup T_t defined in (2.2), $\omega_0 = -\ln\gamma/\tau_1$.

Proof. Since $\theta \in [-i\tau_1, 0] \Rightarrow t + \theta \in [-i\tau_1, t]$,

$$\sup_{\theta \in [-i\tau_1, 0]} \left|T_t \phi(\theta)\right| \leq \sup_{\theta \in [-i\tau_1, 0]} \left|\phi(\theta)\right|, \tag{2.4}$$

and hence $\|T_t\|_\sigma \leq 1, T_{t+s} = T_t T_s$ is obvious, then

$$\lim_{t \to 0} \left\|T_t \phi - \phi\right\|_\sigma = 0 \tag{2.5}$$

can be proved using Proposition 1.9 of [10]. The proof that B is the infinitesimal generator of T_t is also easy.

Note that $\lambda = 0$ trivially satisfies $\Re(\lambda) > -\ln\gamma/\tau_1$. Let $0 \neq \lambda \in \rho(B)$. Define ϕ, as $\phi(\theta) = \theta$. Since $\sum_{i=1}^{\infty} p\gamma^{-i} < \infty$, $\phi \in \mathbf{C}_{\sigma,0}(-\infty,0]$ and hence there is a unique $\psi \in \mathbf{D}(B)$, such that $\lambda\psi - \psi' = \phi$. Indeed, $\psi = (\lambda I_0 - B)^{-1}\phi$. Here, I_0 is the identity on $\mathbf{C}_{\sigma,0}(-\infty,0]$. Let us note that $\psi(0) = 0$. Now, we find that ψ_1, defined as $\psi_1(\theta) = \theta/\lambda + (1/\lambda^2)(1 - e^{\lambda\theta})$ is the unique continuously differentiable function such that $\lambda\psi_1 - \psi_1' = \phi$ and $\psi_1(0) = 0$. From this we infer that $\psi_1 = (\lambda I_0 - B)^{-1}\phi$ and hence $\psi_1 \in \mathbf{C}_{\sigma,0}(-\infty,0]$. Now, since $\phi \in \mathbf{C}_{\sigma,0}(-\infty,0]$, we obtain $(1 - e_\lambda) \in \mathbf{C}_{\sigma,0}(-\infty,0] \subseteq \mathbf{C}_{\sigma}(-\infty,0]$. Since the constant function 1 is an element of $\mathbf{C}_{\sigma}(-\infty,0]$, $e_\lambda \in \mathbf{C}_{\sigma}(-\infty,0]$. Noting that $-\ln\gamma/\tau_1 = \inf\{\Re(\lambda) : e_\lambda \in \mathbf{C}_{\sigma}(-\infty,0]\}$, we obtain $\Re(\lambda) > -\ln\gamma/\tau_1$. □

Let $t \geq \tau_1$. It is clear that for all $i \leq \lfloor t/\tau_1 \rfloor$, and $\theta \in [-i\tau_1, 0]$, $T_t\phi(\theta) = 0$. For $i > \lfloor t/\tau_1 \rfloor$, and $\theta \in [-i\tau_1, 0]$, we have $t + \theta \geq t - i\tau_1 \geq -(i - \lfloor t/\tau_1 \rfloor)\tau_1$. Thus,

$$\begin{aligned} \sup_{\theta\in[-i\tau_1,0]} \left|T_t\phi(\theta)\right| &\leq \sup_{\theta\in[-i\tau_1,0]} \left|\phi(t+\theta)\right| \\ &\leq \sup_{\theta\in[-i-\lfloor t/\tau_1\rfloor\tau_1,0]} \left|\phi(\theta)\right|. \end{aligned} \tag{2.6}$$

Hence

$$\begin{aligned} \left\|T_t\phi\right\|_\sigma &\leq \sum_{i=1}^{\infty} \left|\beta_i\right| \sup_{\theta\in[-i\tau_1,0]} \left|T_t\phi(\theta)\right| \\ &\leq \sum_{i=\lfloor t/\tau_1\rfloor+1}^{\infty} \left|\beta_i\right| \sup_{\theta\in[-i-\lfloor t/\tau_1\rfloor\tau_1,0]} \left|\phi(\theta)\right| \\ &\leq \sum_{i=\lfloor t/\tau_1\rfloor+1}^{\infty} \left|\beta_{i-\lfloor t/\tau_1\rfloor}\right| \frac{\left|\beta_i\right|}{\left|\beta_{i-\lfloor t/\tau_1\rfloor}\right|} \sup_{\theta\in[-i-\lfloor t/\tau_1\rfloor\tau_1,0]} \left|\phi(\theta)\right| \\ &\leq \sup_{i>\lfloor t/\tau_1\rfloor} \frac{\left|\beta_i\right|}{\left|\beta_{i-\lfloor t/\tau_1\rfloor}\right|} \sum_{i=\lfloor t/\tau_1\rfloor+1}^{\infty} \sup_{\theta\in[-i-\lfloor t/\tau_1\rfloor\tau_1,0]} \left|\beta_{i-\lfloor t/\tau_1\rfloor}\right| \left|\phi(\theta)\right| \\ &\leq \sup_{i>\lfloor t/\tau_1\rfloor} \frac{\left|\beta_i\right|}{\left|\beta_{i-\lfloor t/\tau_1\rfloor}\right|} \|\phi\|_\sigma \\ &\leq \sup_{i>\lfloor t/\tau_1\rfloor} \frac{\gamma^{-i}}{\gamma^{-i+\lfloor t/\tau_1\rfloor}} \|\phi\|_\sigma \\ &\leq \gamma^{-\lfloor t/\tau_1\rfloor} \|\phi\|_\sigma. \end{aligned} \tag{2.7}$$

Hence, the operator norm $\|T_t\|_\sigma \leq \gamma^{-\lfloor t/\tau_1\rfloor}$.

To prove the equality, we construct a function $\eta \in \mathbf{C}_{\sigma,0}(-\infty,0]$ such that $\|T_t\eta\|_\sigma = \gamma^{-\lfloor t/\tau_1\rfloor}\|\eta\|_\sigma$ and the result follows.

Let $\delta = (\lfloor t/\tau_1\rfloor + 1)\tau_1 - t = \tau_1(\lfloor t/\tau_1\rfloor + 1 - t/\tau_1)$. We have, $\delta < \tau_1$. Define,

$$\begin{aligned} \eta(\theta) &= \frac{-\theta}{\delta}, \quad -\delta \leq \theta \leq 0 \\ &= 1, \quad \theta < -\delta. \end{aligned} \tag{2.8}$$

It is clear that $\|\eta\|_\sigma = \sum_{i=1}^{\infty} p\gamma^{-i}$, Now,

$$T_t\eta(\theta) = -\left(\frac{\theta+t}{\delta}\right), \quad (-\delta - t) \leq \theta \leq -t$$
$$= 1, \quad \theta < -\delta - t. \tag{2.9}$$

Thus $\|T_t\eta\|_\sigma = p\sum_{i=\lfloor t/\tau_1\rfloor+1}^{\infty} \gamma^{-i}$.

Hence, $\|T_t\eta\|_\sigma = \gamma^{-\lfloor t/\tau_1\rfloor}\|\eta\|_\sigma$.

Now, $\omega_0 = \lim_{t\to\infty}(1/t)\ln(\|T_t\|_\sigma) = -\ln(\gamma)/\tau_1$.

Let $\Re(\lambda) > -\ln\gamma/\tau_1$. Since

$$\begin{aligned}
\left\|(\lambda I_0 - B)^{-1}g\right\|_\sigma &= \left\|\int_0^\infty e^{-\lambda t}T_t g\,dt\right\|_\sigma \\
&\leq \int_0^\infty e^{-\Re(\lambda)t}\|T_t g\|_\sigma dt \\
&\leq \int_0^\infty e^{-\Re(\lambda)t}e^{\omega_0 t}\|g\|_\sigma dt = \int_0^\infty e^{(\omega_0-\Re(\lambda))t}\|g\|_\sigma dt \\
&\leq \int_0^\infty e^{(-\ln(\gamma)/\tau_1-\Re(\lambda))t}\|g\|_\sigma dt,
\end{aligned} \tag{2.10}$$

we have $\lambda \in \rho(B)$.

Let $f \in \mathbf{C}_\sigma(-\infty,0]$. Define $g(\theta) = f(\theta) - f(0)$. Then $g \in \mathbf{C}_{\sigma,0}(-\infty,0]$.

Let $\psi = (\lambda I_0 - B)^{-1}g$. We have, $\psi(0) = 0$.

Define $\psi_1(\theta) = -\int_0^\theta e^{\lambda(\theta-\xi)}\ g(\xi)d\xi$. Now $\psi_1(0) = 0$ and $\psi_1'(0) = 0$.

By the uniqueness of the solution to the initial value problem of the ODE:

$$\begin{aligned}
\lambda\psi - \psi' &= g, \\
\psi(0) &= 0,
\end{aligned} \tag{2.11}$$

it is now obvious that $\psi_1 = \psi$ and hence $\psi_1 \in \mathbf{C}_{\sigma,0}(-\infty,0]$.

Now,

$$\int_0^\theta e^{\lambda(\theta-\xi)}g(\xi)d\xi = \int_0^\theta e^{\lambda(\theta-\xi)}\left[f(\xi) - f(0)\right]d\xi = \int_0^\theta e^{\lambda(\theta-\xi)}f(\xi)d\xi + \frac{1}{\lambda}\left(1 - e^{\lambda\theta}\right)f(0). \tag{2.12}$$

Since $1 - e_\lambda \in \mathbf{C}_{\sigma,0}(-\infty,0]$, $h \in \mathbf{C}_{\sigma,0}(-\infty,0] \subset \mathbf{C}_\sigma(-\infty,0]$, where h is defined as $h(\theta) = \int_0^\theta e^{\lambda(\theta-\xi)}f(\xi)d\xi$.

3. Stability of the Infinite Delay Equation

The proof of the next theorem assuring the existence of a unique solution to (1.1) is similar to the proof of Theorem 2.2 of [10].

Theorem 3.1. *Let $a \in \mathbb{R}$ and the sequences b_i and β_i be as in Section 1. Assume that $\tau_i \leq i\tau_1$. Then there exists a unique solution $x : \mathbb{R} \to \mathbb{R}$ to* (1.1) *such that its restriction to $[0, \infty)$, denoted by y, is in $\mathbf{C}^1[0, \infty)$. Further, for any $t \in [0, \infty)$, there is a constant $c(t) > 0$ such that*

$$\sup_{s\in[0,t]} |y(s)| \leq c(t)\|\phi\|_\sigma. \tag{3.1}$$

In addition, the family of operators $\{S_t : t \geq 0\}$ defined as

$$\begin{aligned} S_t\phi(\theta) &= x(t+\theta), \quad t+\theta \geq 0 \\ &= \phi(t+\theta), \quad t+\theta < 0 \end{aligned} \tag{3.2}$$

forms a semigroup. Also, the infinitesimal generator of S_t is given by $A : \mathbf{D}(A) \to \mathbf{C}_\sigma(-\infty, 0]$, where

$$\begin{gathered} \mathbf{D}(A) = \left\{\phi \in \mathbf{C}_\sigma(-\infty,0] : \phi' \in \mathbf{C}_\sigma(-\infty,0],\ \phi'(0) = a\phi(0) + \sum_{i=1}^{\infty} b_i\phi(-\tau_i)\right\} \\ A\phi = \phi'. \end{gathered} \tag{3.3}$$

Further, $D(A)$ is dense and A is a closed operator.

Theorem 3.2. *For the semigroup S_t defined by* (3.2)

$$|S_t|_\alpha \leq \gamma^{-\lfloor t/\tau_1 \rfloor}. \tag{3.4}$$

Further, assume that $a + \sum_{i=1}^{\infty} b_i \neq 0$. Then for the generator of the semigroup S_t defined by (3.3) and

$$\operatorname{spec}(A) = \left\{\lambda : \Re(\lambda) \leq -\frac{\ln(\gamma)}{\tau_1}\right\} \cup \left\{\lambda : \Re(\lambda) > -\frac{\ln(\gamma)}{\tau_1} : \lambda = a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i}\right\}. \tag{3.5}$$

Besides, suppose that for any $\lambda \in \mathbf{C}$ with $\lambda = a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i}$, we have $\Re(\lambda) < -\mu_1$ for some $\mu_1 > 0$. Then, the semigroup S_t is asymptotically stable.

Proof. Let T_t be as in Proposition 2.1. Fix $t > 0$. Define $V_t : \mathbf{C}_\sigma(-\infty, 0] \to \mathbf{C}_\sigma(-\infty, 0]$ as

$$\begin{aligned} V_t\phi(\theta) &= 0, \quad t+\theta \geq 0 \\ &= \phi(t+\theta) - \phi(0), \quad t+\theta < 0. \end{aligned} \tag{3.6}$$

Define $K_t : \mathbf{C}[0,t] \to \mathbf{C}_\sigma(-\infty,0]$ as

$$[K_t z](\theta) = z(t+\theta) - z(0), \quad t+\theta \geq 0$$
$$= 0, \quad t+\theta < 0. \tag{3.7}$$

□

It is easy to see that

$$\|K_t z\|_\sigma \leq 2\sum_{i=1}^{\infty} |\beta_i| \left(\sup_{s\in[0,t]} |z(s)| \right). \tag{3.8}$$

Thus, K_t is a bounded linear map.

Define $K_1 : \mathbf{C}_\sigma(-\infty,0] \to \mathbf{C}_\sigma(-\infty,0]$ as $[K_1\phi](\theta) = \phi(0)$ for all $\theta \in (-\infty,0]$. It is clear that K_1 is compact. Define $B_t : \mathbf{C}_\sigma(-\infty,0] \to \mathbf{C}[0,t]$ as $B_t\phi = z$, where z is the restriction of y to $[0,t]$. From (3.1), B_t is a bounded linear map. Let S_t be as in (3.3). Then,

$$S_t = V_t + K_t B_t + K_1. \tag{3.9}$$

Now, if I is the identity on $\mathbf{C}_\sigma(-\infty,0]$ and $J : \mathbf{C}_{\sigma,0}(-\infty,0] \to \mathbf{C}_\sigma(-\infty,0]$ is the inclusion map, then $V_t = JT_t(I-K_1)$, and, finally,

$$S_t = JT_t(I-K_1) + K_t B_t + K_1. \tag{3.10}$$

Next, we show that B_t is, in fact, a compact map. Let x be the solution to (1.1) as in Theorem 3.1:

$$z(s) = e^{as}\phi(0) + e^{as}\int_0^s e^{-a\eta}\sum_{i=1}^{\infty} b_i x(\eta-\tau_i)\,d\eta, \quad s\in[0,t]. \tag{3.11}$$

Thus,

$$z'(s) = az(s) + \sum_{i=1}^{\infty} b_i x(s-\tau_i). \tag{3.12}$$

Consider $n\in\mathbb{N}$ such that $t\in[n\tau_1,(n+1)\tau_1]$. From (3.1) and (3.11), we obtain existence of $c_1(t)\geq 0$ such that

$$\sup_{s\in[0,t]} |z'(s)| \leq c_1(t)\|\phi\|_\sigma. \tag{3.13}$$

Hence by Arzela-Ascoli theorem, B_t is a compact operator.

It is easy to show that $|J|_\alpha \leq \|J\|_\sigma = 1$. By the compactness of K_1 and B_t, $|I-K_1|_\alpha = 1$ and $|K_t B_t|_\alpha = |K_1|_\alpha = 0$. Thus, from the relation

$$S_t = JT_t(I-K_1) + K_t B_t + K_1, \tag{3.14}$$

and Propositions 1.2 and 2.1 of this paper, we obtain

$$|S_t|_\alpha \le |T_t|_\alpha \le \|T_t\|_\sigma \le \gamma^{-\lfloor t/\tau_1 \rfloor}. \tag{3.15}$$

So,

$$\omega_{\text{ess}} = \lim_{t\to\infty} \frac{1}{t} \ln(|S_t|_\alpha) \le -\ln \frac{\gamma}{\tau_1}. \tag{3.16}$$

Let $0 \ne \lambda \in \rho(A)$.

There is a unique $\psi \in D(A)$ such that

$$\begin{aligned} \lambda\psi - \psi' &= -1, \\ \psi'(0) &= a\psi(0) + \sum_{i=1}^{\infty} b_i \psi(-\tau_i). \end{aligned} \tag{3.17}$$

It is clear that there is $c \in \mathbf{C}$ such that $\psi(\theta) = (c - 1/\lambda)e^{\lambda\theta} - 1/\lambda$. Now, we claim that $c \ne 1/\lambda$. If $c = 1/\lambda$, then $\psi(\theta) = -1/\lambda$ for all $\theta \in (-\infty, 0]$. Since $\psi \in D(A)$, we must have $\psi'(0) = a\psi(0) + \sum_{i=1}^{\infty} b_i \psi(-\tau_i)$. But this would imply that $a + \sum_{i=1}^{\infty} b_i = 0$ which is a contradiction, to the hypothesis that $a + \sum_{i=1}^{\infty} b_i \ne 0$. Now, since $c - 1/\lambda \ne 0$, it is obvious that $e_\lambda \in \mathbf{C}_\sigma(-\infty, 0]$. But this implies that $\Re(\lambda) > -\ln(\gamma)/\tau_1$. If $0 \in \rho(A)$, the condition $\Re(\lambda) > -\ln(\gamma)/\tau_1$ is obvious. Thus,

$$\rho(A) \subseteq \left\{ \lambda : \Re(\lambda) > -\frac{\ln(\gamma)}{\tau_1} \right\}. \tag{3.18}$$

We now infer that $\{\lambda : \Re(\lambda) \le -\ln(\gamma)/\tau_1\} \subseteq \operatorname{spec}(A)$. Next, if $\lambda = a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i}$, and $\Re(\lambda) > -\ln(\gamma)/\tau_1$, then $e_\lambda \in \mathbf{C}_\sigma(-\infty, 0]$ and hence $e_\lambda \in \mathbf{D}(A)$ with $\lambda e_\lambda = Ae_\lambda$. Thus, $\lambda \in \operatorname{spec}(A)$. So,

$$\left\{ \lambda : \Re(\lambda) > -\frac{\ln(\gamma)}{\tau_1},\ \lambda = a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i} \right\} \subseteq \operatorname{spec}(A). \tag{3.19}$$

Let us assume that $\Re(\lambda) > -\ln(\gamma)/\tau_1$ and $\lambda \ne a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i}$.

Then, by Proposition 2.1, we have $e_\lambda \in \mathbf{C}_\sigma(-\infty, 0]$ and the function h defined as $h(\theta) = \int_0^\theta e^{\lambda(\theta-\xi)} f(\xi) d\xi$ is in $\mathbf{C}_\sigma(-\infty, 0]$.

Defining $\Lambda : \mathbf{C}_\sigma(-\infty, 0] \to \mathbf{C}$ as $\Lambda(\phi) = a\phi(0) + \sum_{i=1}^{\infty} b_i \phi(-\tau_i)$ and taking $c = (\Lambda(h) - f(0))/(\Lambda(e_\lambda) - \lambda)$, we find that ϕ defined as $\phi(\theta) = \int_0^\theta e^{\lambda(\theta-\xi)} f(\xi) d\xi + ce^{\lambda\theta}$ is $(\lambda I - A)^{-1}(f)$. Thus,

$$\left\{ \lambda : \Re(\lambda) > -\frac{\ln(\gamma)}{\tau_1},\ \lambda \ne a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i} \right\} \subseteq \rho(A). \tag{3.20}$$

From (3.18), (3.19), and (3.20), we finally conclude that

$$\operatorname{spec}(A) = \left\{\lambda : \Re(\lambda) \leq -\frac{\ln(\gamma)}{\tau_1}\right\} \cup \left\{\lambda : \Re(\lambda) > -\frac{\ln(\gamma)}{\tau_1}, \lambda = a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i}\right\}, \tag{3.21}$$

or

$$\rho(A) = \left\{\lambda : \Re(\lambda) > -\frac{\ln(\gamma)}{\tau_1},\ \lambda \neq a + \sum_{i=1}^{\infty} b_i e^{-\lambda\tau_i}\right\}. \tag{3.22}$$

Since $\omega_0 = \max\{s(A), \omega_{\text{ess}}\} \leq \max\{-\mu_1, -\ln(\gamma)/\tau_1\}$, the result follows.

Remark 3.3. Consider the PDE:

$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \theta}, \\ u(0,\theta) = \phi(\theta).\end{gathered} \tag{3.23}$$

Let B be as in Proposition 2.1 and A be as in Theorem 3.1. For $\phi \in D(B)$, $u(t,\theta) = T_t\phi \in \mathbf{C}_{\sigma,0}(-\infty, 0]$ is the solution to the above PDE. For $\phi \in D(A)$, $u(t,\theta) = S_t\phi \in \mathbf{C}_\sigma(-\infty, 0]$ is the solution to the above PDE. For the first solution $u(t+\theta) = 0$, $t+\theta \geq 0$ and for the second solution $u(t+\theta) = x(t+\theta)$, $t+\theta \geq 0$. Here x is the solution to the delay equation.

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