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Research Article

On Certain Classes of Meromorphic Functions Associated with Conic Domains

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Making use of the concept of k-uniformly bounded boundary rotation and Ruscheweyh differential operator, we introduce some new classes of meromorphic functions in the punctured unit disc. Convolution technique and principle of subordination are used to investigate these classes. Inclusion results, generalized Bernardi integral operator, and rate of growth of coefficients are studied. Some interesting consequences are also derived from the main results.

1. Introduction

Let $\sum$ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in punctured unit disc $E^* = \{z : 0 < |z| < 1\} = \mathrm{E} \setminus \{0\}$. At $z = 0$, the function $f(z)$ has a simple pole.

Let, for $0 \le \beta < 1$, $\sum^*(\beta)$ and $\sum_c(\beta)$ be well-known subclasses of $\sum$ consisting of functions meromorphic starlike and meromorphic convex of order β, respectively, see [1].

Let $f, g \in \sum$, $g(z) = 1/z + \sum_{n=0}^{\infty} b_n z^n$ and f be given by (1.1). Then convolution (Hadamard product) $f * g$ of f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.2}$$

Robertson [2] showed that $f * g$ also belongs to $\sum$.

Let $f \in \sum$ and define, for $\alpha > -1$, $z \in E^*$, $D^\alpha : \sum \to \sum$ as

$$D^\alpha f(z) = \frac{1}{z(1-z)^{\alpha+1}} * f(z). \tag{1.3}$$

It can easily be seen that, for $\alpha = n \in N_0 = \{0,1,2,\dots\}$,

$$D^n f(z) = \frac{z^{-1}}{n!}\frac{d^n}{dz^n}\left(z^{n+1}f(z)\right). \tag{1.4}$$

We note that

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= zf'(z) + 2f(z), \end{aligned} \tag{1.5}$$

and so

$$D^1 f(z) - 2D^0 f(z) = zf'(z) = \frac{2z-1}{z(1-z)^2} * f(z). \tag{1.6}$$

Equation (1.6) can be verified as follows.

Since

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n = f(z) * \left(\frac{1}{z(1-z)}\right), \tag{1.7}$$

we have

$$\begin{aligned} -zf'(z) &= \frac{1}{z} - \sum_{n=0}^{\infty} n a_n z^n \\ &= f(z) * \left(\frac{1}{z} - \frac{z}{(1-z)^2}\right), \quad z \in E^*, \\ &= f(z) * \left(\frac{1-2z}{z(1-z)^2}\right). \end{aligned} \tag{1.8}$$

From (1.3), we can readily obtain the following identity for $f \in \sum$ and $\alpha > -1$:

$$z\left(D^\alpha f(z)\right)' = (\alpha+1)D^{\alpha+1}f(z) - (\alpha+2)D^\alpha f(z). \tag{1.9}$$

D^α, $\alpha > 1$ is known as generalized Ruscheweyh derivative for meromorphic functions.

For $k \in [0,1]$, define the domain Ω_k as follows, see [3]:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{1.10}$$

For fixed k, Ω_k represents the conic region bounded successively, by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), and a parabola($k = 1$). Related with Ω_k, the domain $\Omega_{k,\gamma}$ can be defined as below, see [4]:

$$\Omega_{k,\gamma} = (1-\gamma)\Omega_k + \gamma, \qquad (0 \le \gamma < 1). \tag{1.11}$$

The functions $p_{k,\gamma}(z)$ with $p_{k,\gamma}(0) = 1$, $p'_{k,\gamma}(0) > 0$, univalent in $E = \{z : |z| < 1\}$, map E onto $\Omega_{k,\gamma}$ and are given as in the following:

$$p_{k,\gamma}(z) = \begin{cases} \dfrac{1 + (1-2\gamma)z}{(1-z)}, & (k = 0), \\ 1 + \dfrac{2(1-\gamma)}{\pi^2}\left(\log \dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k = 1), \\ 1 + \dfrac{2(1-\gamma)}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\text{arc}\cos k\right)\text{arc}\tanh\sqrt{z}\right], & (0 < k < 1). \end{cases} \tag{1.12}$$

The functions $p_{k,\gamma}(z)$ are continuous as regard to k, have real coefficients for $k \in [0,1]$, and play the part of extremal ones for many problems related to $\Omega_{k,\gamma}$.

Let P be the class of analytic functions with positive real part, and let $P(p_{k,\gamma}) \subset P$ be the class of functions $p(z)$ which are analytic in E, $p(0) = 1$ such that $p(z) \prec p_{k,\gamma}(z)$ for $z \in E$, where "$\prec$" denotes subordination, and $p_{k,\gamma}(z)$ are given by (1.12).

We define the following.

Definition 1.1. Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p(z)$ is said to belong to the class $P_m(p_{k,\gamma})$, for $m \ge 2$, $0 \le \gamma < 1$, $k \ge 0$, if and only if there exist $p_1, p_2 \in P(p_{k,\gamma})$ such that

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z), \quad z \in E. \tag{1.13}$$

We note that

(i) $k = 0$, $P_m(p_{0,\gamma}) = P_m(\gamma)$ and with $m = 2$, $P_2(\gamma)$ coincides with $P(\gamma)$ and $p \in P(\gamma)$ implies $\operatorname{Re} p(z) > \gamma$ in E;

(ii) when $k = 0$, $\gamma = 0$, we have the class P_m introduced in [5].

Definition 1.2. Let $f \in \sum$. Then $f(z)$ is said to belong to the class $k - MR_m(\gamma)$ if and only if $(-zf'/f) \in P_m(p_{k,\gamma})$ in E.

For $k = 0$, $m = 2$, we obtain the class $\sum^*(\gamma)$ of meromorphic starlike functions of order γ.

We can define the class $k - MV_m(\gamma)$ by the following relation:

$$f \in k - MV_m(\gamma) \quad \text{if } -zf' \in k - MR_m(\gamma). \tag{1.14}$$

When $k = 0$, $m = 2$, $\gamma = 0$, we obtain $\sum_c$ of meromorphic convex functions.

Definition 1.3. Let $f \in \sum$. Then $f \in k - MR_m^{\alpha}(\gamma)$ if and only if $D^{\alpha} f \in k - MR_m(\gamma)$ for $z \in E^*$.

Similarly $f \in k - MV_m^{\alpha}(\gamma)$ if and only if $D^{\alpha} f \in k - MV_m(\gamma)$. We note that the classes $k - MR_m^{\alpha}(\gamma)$ and $k - MV_m^{\alpha}(\gamma)$ are related by relation (1.14).

For $k = 0$, $\gamma = 0$, we have $0 - MV_m(0) = MV_m$, the class of meromorphic functions of bounded boundary rotation which was studied in [6]. The functions $f \in MV_m$ have integral representation of the form

$$f'(z) = -\frac{1}{z^2} \exp\left\{ \int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t) \right\}, \tag{1.15}$$

where $\mu(t)$ is a real-valued function of bounded variation on $[0, 2\pi]$ satisfying the conditions

$$\int_0^{2\pi} d\mu(t) = 2, \qquad \int_0^{2\pi} |d\mu(t)| \leq m, \qquad \int_0^{2\pi} e^{-it} d\mu(t) = 0. \tag{1.16}$$

With simple computations, it can easily be seen that the third of conditions (1.16) guarantees that the singularity of $f(z)$ at $z = 0$ is a simple pole with no logarithm term.

Also it is known that $f \in MV_m$ if and only if $f(E)$ is a domain containing infinity with boundary rotation at most $m\pi$, see [6]. The class V_m is wellknown [1] and consists of analytic functions with boundary rotation at most $m\pi$. Noonan [6] established the relation between the classes V_m and MV_m as follows.

A function $f \in MV_m$ if and only if there exists $g \in V_m$ of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ with $b_2 = 0$ such that

$$-\frac{1}{z^2 f'(z)} = g'(z). \tag{1.17}$$

It is also shown [6] that, for $f \in MV_m$, $m > 4$, there exist ϕ_1, $\phi_2 \in \sum^*$ given by $\phi_1(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$, $\phi_2(z) = 1/z + \sum_{n=0}^{\infty} b_n z^n$, such that $a_0 = ((m-2)/(m+2)) b_0$ and

$$f'(z) = -\frac{1}{z^2} \frac{\left[z\phi_1(z)\right]^{(m+2)/4}}{\left[z\phi_2(z)\right]^{(m-2)/4}}. \tag{1.18}$$

We note that $\phi_i \in \sum^*$ and therefore $1/\phi_i \in S^*$ of analytic functions, $i = 1, 2$ in (1.18). This give us

$$\left|z\phi_1(z)\right| \leq 4, \qquad \left[z\phi_2(z)\right]^{-1} \prec (1-z)^2, \tag{1.19}$$

by distortion results and subordination for the class S^*.

We can easily extend the relations (1.17) and (1.18) by noting that $f \in V_m(\gamma)$ implies that there exists $f_1 \in V_m$ such that $f'(z) = (f_1'(z))^{1-\gamma}$, $z \in E$, see [7]. For $f \in MV_m(\gamma)$, $m > ((2/(1-\gamma)) + 2)$, we can write relation (1.18) as

$$f'(z) = -\frac{1}{z^2}\frac{[z\phi_1(z)]^{((m+2)/4)(1-\gamma)}}{[z\phi_2(z)]^{((m-2)/4)(1-\gamma)}}, \quad \phi_1, \phi_2 \in \sum^{*}. \tag{1.20}$$

Throughout this paper, we will assume $k \in [0,1]$, $m \geq 2$, $\gamma \in [0,1)$ and $\alpha > 1$ unless otherwise stated.

We also note that all the results proved in this paper hold for $k \geq 0$ in general.

2. Preliminary Results

The following lemma is a generalized version of a result proved in [3].

Lemma 2.1 (see [4]). *Let $0 \leq k < \infty$ and let β, δ be any complex numbers with $\beta \neq 0$ and $\mathrm{Re}(\beta k/(k+1) + \delta) > \gamma$. If $h(z)$ is analytic in E, $h(0) = 1$ and satisfies*

$$\left\{h(z) + \frac{zh'(z)}{\beta h(z) + \delta}\right\} \prec p_{k,\gamma}(z), \tag{2.1}$$

and $q_{k,\gamma}(z)$ is an analytic solution of

$$\left\{q_{k,\gamma}(z) + \frac{zq'_{k,\gamma}(z)}{\beta q_{k,\gamma}(z) + \delta}\right\} = p_{k,\gamma}(z), \tag{2.2}$$

then $q_{k,\gamma}(z)$ is univalent,

$$h(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z), \tag{2.3}$$

and $q_{k,\gamma}(z)$ is the best dominant of (2.1).

Lemma 2.2 (see [8]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\psi(u, v)$ be complex-valued function satisfying the following conditions:*

(i) *$\psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*

(ii) *$(1, 0) \in D$ and $\psi(1, 0) > 0$,*

(iii) *$\mathrm{Re}\,\psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -(1 + u_2^2)/2$.*

If $h(z) = 1 + \sum_{n=1} c_n z^n$ is a function analytic in E such that $(h(z), zh'(z)) \in D$, and $\mathrm{Re}[\psi(h(z), zh'(z))] > 0$ for $z \in E$, then $\mathrm{Re}(h(z)) > 0$ for $z \in E$.

Lemma 2.3 (see [9]). *Let p and q be analytic in E and $\operatorname{Re} p(z) \geq 0$, $q(0) = 1$ and $\operatorname{Re} q(z) > 0$ for $z \in E$. Further let $A \neq 0$ and B be complex constants such that $A + B \neq 0$. Then*

$$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{A + Bq(z)}\right\} > 0 \quad \text{in } |z| < \rho(A, B), \tag{2.4}$$

where

$$\begin{aligned}\rho(A,B) &= \frac{|A+B|}{\left[|A|^2 + 2 + 4|B| + |B|^2 + \sqrt{R}\right]^{1/2}}, \\ R &= \left[|A|^2 + 2 + 4|B| + |B|^2 - \left|A^2 - B^2\right|^2\right].\end{aligned} \tag{2.5}$$

This result is sharp for A real and nonnegative constant.

3. Main Results

Theorem 3.1. *One has*

$$k - MR_m^{\alpha+1}(\gamma) \subset k - MR_m^{\alpha}, \quad \gamma = \frac{1}{\alpha + 2}. \tag{3.1}$$

This result is best possible and sharpness follows from the best dominant property.

Proof. Let $f \in k - MR_m^{\alpha+1}(\gamma)$, and set

$$f_1(z) = z\left[zD^{\alpha} f(z)\right]^{-1/(1+\alpha)}, \tag{3.2}$$

with

$$r_1 = \sup\{r : f_1(z) \neq 0, 0 < |z| < r < 1\}. \tag{3.3}$$

Then $f_1(z)$ is single-valued in $0 < |z| < r_1$.

Using identity (1.9), it follows that

$$p(z) = \frac{zf_1'(z)}{f_1(z)} = -\left(\frac{D^{\alpha+1} f(z)}{D^{\alpha} f(z)} - 2\right), \tag{3.4}$$

is analytic in $|z| < r_1$ and $p(0) = 1$.

Now, from (1.9) and (3.4), we obtain

$$
\begin{aligned}
-\left(\frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)}-2\right) &= \frac{\alpha+1}{\alpha+2}\left[p(z)+\frac{zp'(z)}{(\alpha+1)(2-p(z))}\right]+\frac{1}{\alpha+2} \\
&= \frac{\alpha+1}{\alpha+2}\left[p(z)+\frac{zp'(z)}{2(\alpha+1)-(\alpha+1)p(z)}+\frac{1}{\alpha+1}\right].
\end{aligned}
\tag{3.5}
$$

With $h(z)=(1-\gamma)p(z)+\gamma$, $\gamma=1/(\alpha+2)$, we can write (3.5) as

$$
-\left(\frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)}-2\right)=h(z)+\frac{zh'(z)}{\beta h(z)+\delta}, \tag{3.6}
$$

where $\beta=-(\alpha+2)$, $\delta=2\alpha+3$.

Since $f\in k-MR_m^{\alpha+1}(\gamma)$, it follows from (3.6) that

$$
\left(h(z)+\frac{(1/\beta)zh'(z)}{h(z)+\delta/\beta}\right)\in P_m(p_{k,\gamma}). \tag{3.7}
$$

Define

$$
\phi_{\delta,\beta}(z)=\frac{1}{z(1-z)^{1/\beta+1}}\left\{\frac{1-(\beta/(\beta+\delta))z}{1-z}\right\}, \tag{3.8}
$$

and let

$$
\begin{aligned}
h(z) &= \left(\frac{m}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)h_2(z), \\
p(z) &= \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z).
\end{aligned}
\tag{3.9}
$$

Then using convolution technique, we have

$$
\begin{aligned}
\left(h(z)*\frac{\phi_{\delta,\beta}(z)}{z}\right) &= h(z)+\frac{zh'(z)}{\beta h(z)+\delta} \\
&= \left(\frac{m}{4}+\frac{1}{2}\right)\left(h_1(z)+\frac{zh_1'(z)}{\beta h_1(z)+\delta}\right)-\left(\frac{m}{4}-\frac{1}{2}\right)\left(h_2(z)+\frac{zh_2'(z)}{\beta h_2(z)+\delta}\right).
\end{aligned}
\tag{3.10}
$$

Thus, from (3.7) and (3.10), we obtain

$$
\left(h_i(z)+\frac{zh'_i(z)}{\beta h_i(z)+\delta}\right)\prec p_{k,\gamma}(z), \quad i=1,2. \tag{3.11}
$$

It can easily be seen that $\operatorname{Re}((\beta k/(k+1))+\delta) > \gamma$, so we apply Lemma 2.1 to have from (3.11)

$$h_i(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z), \quad i = 1,2, \tag{3.12}$$

where $q_{k,\gamma}(z)$ is the best dominant and is given as

$$q_{k,\gamma}(z) = \left\{ \left[\beta \int_0^1 \left(t^{\beta+\delta-1} \exp \int_0^{tz} \frac{p_{k,\gamma}(u) - 1}{u} du \right)^{\beta} dt \right]^{-1} - \frac{\delta}{\beta} \right\}. \tag{3.13}$$

Since $h_i(z) = p_i(z) * p_\gamma(z)$, $p_\gamma(z) = (1 + (1-2\gamma)z)/(1-z)$, $i = 1,2$, we have

$$p_i(z) * p_\gamma(z) \prec p_k(z) * p_\gamma(z), \tag{3.14}$$

and from this it follows that $p_i(z) \prec p_k(z)$ for $i = 1,2$.

Now from (3.4) we have $p \in P_m(p_{k,0})$ in E and the proof is complete. □

As a special case, we have the following.

Corollary 3.2. *Let $k = 0$ in Theorem 3.1. Then $MR_m^{\alpha+1}(\gamma) \subset MR_m^{\alpha}(\gamma_1)$, where $\gamma = 1/(\alpha+2)$ and*

$$\gamma_1 = \frac{6}{\left\{ (2\alpha+9) + \sqrt{(2\alpha+9)^2 - 24} \right\}}. \tag{3.15}$$

Proof. From (3.6), we have

$$-\left(\frac{D^{\alpha+2} f(z)}{D^{\alpha+1} f(z)} - 2 \right) = h(z) + \frac{zh'(z)}{\beta h(z) + \delta}, \quad \beta = -(\alpha+2),\ \delta = (2\alpha+3). \tag{3.16}$$

Proceeding as in Theorem 3.1, it follows that

$$\left(h_i(z) + \frac{zh_i'(z)}{\beta h_i(z) + \delta} \right) \in P(\gamma), \quad i = 1,2,\ z \in E. \tag{3.17}$$

Let $h_i(z) = (1-\gamma_1)H_i(z) + \gamma_1$, $i = 1,2$.

Then

$$\left\{ (\gamma_1 - \gamma) + (1-\gamma_1)H_i(z) + \frac{(1-\gamma_1)zH_i'(z)}{\beta(1-\gamma_1)H_i(z) + \beta\gamma_1 + \delta} \right\} \in P. \tag{3.18}$$

We construct a functional $\psi(u,v)$ by taking $u = H_i(z)$, $v = zH_i'(z)$. Then

$$\psi(u,v) = (\gamma_1 - \gamma) + (1-\gamma_1)u + \frac{(1-\gamma_1)v}{\beta(1-\gamma_1)u + \beta\gamma_1 + \delta}. \tag{3.19}$$

The first two conditions of Lemma 2.2 are easily verified. For condition (iii), we proceed as follows:

$$\operatorname{Re}\psi(iu_2, v_1) = (\gamma_1 - \gamma) + \frac{(1-\gamma_1)(\beta\gamma_1 + \delta)v_1}{(\beta\gamma_1+\delta)^2 + \beta^2(1-\gamma_1)^2 u_2^2}. \tag{3.20}$$

By putting $v_1 \leq -(1+u_2^2)/2$, we have

$$\operatorname{Re}\psi(iu_2, v_1) \leq \frac{A_1 + B_1 u_2^2}{2C} \quad \text{iff } A_1 \leq 0,\ B_1 \leq 0, \tag{3.21}$$

where

$$\begin{aligned} A_1 &= 2(\gamma_1 - \gamma)(\beta\gamma_1+\delta)^2 - (1-\gamma_1)(\beta\gamma_1+\delta), \\ B_1 &= 2(\gamma_1-\gamma)\left[\beta^2(1-\gamma_1)^2\right] - (1-\gamma_1)(\beta\gamma_1+\delta), \\ C &= (\beta\gamma_1+\delta)^2 + \beta^2(1-\gamma_1)^2 u_2^2 > 0. \end{aligned} \tag{3.22}$$

From $A_1 \leq 0$, we obtain γ_1 as given by (3.15) and $B_1 \leq 0$ ensures $0 \leq \gamma_1 < 1$.

Applying Lemma 2.2, we now have $H_i \in P$, $i = 1,2$ and therefore $h_i \in P(\gamma_1)$ in E, consequently $h \in P_m(\gamma_1)$ in E and the proof is complete. □

We note that, for $\alpha = 0$, we have

$$MR_m\left(\frac{1}{2}\right) \subset MR_m\left(\frac{6}{9+\sqrt{57}}\right). \tag{3.23}$$

Also, for $\alpha = 1$, $\gamma = 1/3$ and $\gamma_1 \approx 6/21$.

We will now investigate the rate of growth of coefficients for $f \in k - MV_m^\alpha(\gamma)$ and the corresponding result for the class $k - MR_m^\alpha(\gamma)$ will follow from the relation (1.14).

Theorem 3.3. *Let $f \in k - MV_m^\alpha(\gamma)$ and be given by (1.1). Then, for $m > (2/(1-\sigma)+2)$, $\sigma = (k+\gamma)/(1+k)$, $n \geq 3$, one has*

$$a_n = O(1) n^{\beta_1 - (\alpha+2)}, \quad \beta_1 = (1-\sigma)\left(\frac{m}{2} - 1\right), \tag{3.24}$$

and $O(1)$ depends only on m and σ.

The exponent $\{\beta_1 - (\alpha+2)\}$ in (3.24) is best possible for the class $MV_m^\alpha(\sigma)$ as can be seen from the function $f_0 \in MV_m^\alpha(\sigma)$ given by

$$D^\alpha f_0(z) = -\frac{1}{z^2}\frac{\left(1+z^2-2z((m-2)/(m+2))\right)^{((m+2)/4)(1-\sigma)}}{(1-z)^{\beta_1}}. \tag{3.25}$$

Proof. Since $k - MV_m^\alpha(\gamma) \subset MV_m^\alpha(\sigma)$, $\sigma = (k+\gamma)/(1+k)$ and $f \in k - MV_m^\alpha(\gamma)$ implies $D^\alpha f \in MV_m(\sigma)$, we use (1.20) to write

$$F'(z) = \left(D^\alpha f(z)\right)' = -\frac{1}{z^2}\frac{\left[z\phi_1(z)\right]^{((m+2)/4)(1-\sigma)}}{\left[z\phi_2(z)\right]^{((m-2)/4)(1-\sigma)}}, \quad m > \left(\frac{2}{1-\sigma}+2\right), \tag{3.26}$$

and $\phi_1, \phi_2 \in \sum^*$.

Now, with $F(z) = (1/z) + \sum_{n=0}^{\infty} A_n z^n$ and $z = re^{i\theta}$, $0 < r < 1$, we have

$$nA_n = \frac{1}{2\pi r^{n+1}}\int_0^{2\pi} z^2 F'(z) e^{-i(n+1)\theta} d\theta. \tag{3.27}$$

Pommerenke [10] has shown that

$$\int_0^{2\pi} \frac{1}{|1-z|^\lambda} d\theta \sim c(\lambda)\frac{1}{(1-r)^{\lambda-1}}, \quad (r \longrightarrow 1) \text{ for } \lambda > 1. \tag{3.28}$$

Thus we use (1.19), (3.26), and (3.28) to have from (3.27)

$$|nA_n| \le \frac{1}{2\pi r^{n+1}}\int_0^{2\pi} \left|z^2 F'(z)\right| d\theta \le \frac{c(\sigma, m)}{(1-\gamma)^{(1-\sigma)((m/2)-1)-1}}. \tag{3.29}$$

We take $r = 1 - (1/n)$, $A_n = (\Gamma(n+\alpha+2)/(\Gamma(\alpha+1)(n+1)!))a_n$, Γ denotes gamma function, and have

$$a_n = O(1) n^{\beta_1 - (\alpha+2)}, \tag{3.30}$$

where β_1 is as given in (3.24) and $O(1)$ is a constant depending only on m and σ.

This completes the proof. □

Next we will show that the class $k - MR_m^\alpha(\gamma)$ is preserved under an integral operator.

For $\operatorname{Re} c > 0$, the generalized Bernardi operator for the class $\sum$ is defined in [11] as below.

Let $f \in \Sigma$ and be given by (1.1). Then the integral transform F_c is defined as

$$F(z) = \frac{c}{z^{c+1}}\int_0^z t^c f(t) dt = \left[\frac{1}{z} + \sum_{n=0}^{\infty}\frac{c}{c+n+1} a_n z^n\right] * f(z). \tag{3.31}$$

Also

$$D^\alpha f(z) = \frac{1}{z} + \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)(n+1)!} a_n z^n. \tag{3.32}$$

It easily follows from (3.31) that

$$z(D^{\alpha}F(z))' = cD^{\alpha}f(z) - (1+c)D^{\alpha}F(z), \tag{3.33}$$

and as in (1.9),

$$z(D^{\alpha}F(z))' = (\alpha+1)D^{\alpha+1}F(z) - (\alpha+2)D^{\alpha}F(z). \tag{3.34}$$

From (3.33) and (3.34), we have

$$cD^{\alpha}f(z) = (c-\alpha-1)D^{\alpha}F(z) + (\alpha+1)D^{\alpha+1}F(z). \tag{3.35}$$

We now prove the following.

Theorem 3.4. *Let* $f \in k - MR_m^{\alpha}(\gamma)$*. Then* $F(z)$*, defined by* (3.31)*, also belong to the same class in* E^**.*

Proof. We put

$$F_1(z) = z\left[zD^{\alpha}F(z)\right]^{-1/(\alpha+1)}, \qquad r_1 = \sup\{r : F(z) \neq 0,\ 0 < |z| < 1\}. \tag{3.36}$$

Then $F_1(z)$ is single valued and analytic in $|z| < r_1$ and $H(z)$ defined by

$$H(z) = \left\{\left(\frac{m}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)H_2(z)\right\} = \frac{zF_1'(z)}{F_1(z)} = -\left(\frac{D^{\alpha+1}F(z)}{D^{\alpha}F(z)} - 2\right), \tag{3.37}$$

is analytic in $|z| < r_1, H(0) = 1$.

Form (3.33), (3.34), and (3.37), we obtain

$$-\left(\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} - 2\right) = \left\{H(z) + \frac{zH'(z)}{(c+\alpha+1)-(\alpha+1)H(z)}\right\}, \quad |z| < r_1. \tag{3.38}$$

Since $f \in k - MR_m^{\alpha}(\gamma)$, it follows that

$$\left\{H(z) + \frac{zH'(z)}{\beta H(z) + \delta}\right\} \in P_m(p_{k,\gamma}), \quad \beta = -(\alpha+1),\ \delta = (c+\alpha+1), \tag{3.39}$$

and with the convolution technique used before, we have, for $i = 1, 2$

$$\left\{H_i(z) + \frac{zH_i'(z)}{\beta H_i(z) + \delta}\right\} \prec p_{k,\gamma}(z). \tag{3.40}$$

Since $\operatorname{Re}[\beta k/(k+1) + \delta] > \gamma$, we apply Lemma 2.1 to have $H_i(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z)$, where $q_{k,\gamma}(z)$ is the best dominant. The required result now follows from (3.37). □

Corollary 3.5. *Let $k = 0$, $\gamma = 0$. Then $f \in MR_m^\alpha$ and F, defined by* (3.31)*, belongs to $MR_m^\alpha(\gamma_2)$, where γ_2 is given as*

$$\gamma_2 = \frac{1}{2(c+\alpha+1)}. \tag{3.41}$$

The proof follows on the similar lines of Corollary 3.2.

Remark 3.6. We note that $P(p_{k,\gamma}) \subset P((k+\gamma)/(1+k))$, $(0 \le \gamma < 1)$, and therefore $P_m(p_{k,\gamma}) \subset P_m(\sigma)$, $\sigma = (k+\gamma)/(1+k)$, $(0 \le \sigma < 1)$.

We prove a partial converse of Theorem 3.4 as following.

Theorem 3.7. *Let $F(z)$ be defined by* (3.31) *and let, for* $\operatorname{Re} c > 0$, $F \in MR_m^\alpha(\sigma)$, $\sigma = (k+\gamma)/(1+k)$. *Then $f \in MR_m^\alpha(\sigma)$ for $|z| < \rho$, where*

$$\begin{gathered}
\rho = \rho(A,B) = \frac{|A+B|}{\left[|A|^2 + 2 + 4|B| + |B|^2 + \sqrt{R}\right]^{1/2}}, \\
A = c + (1+\alpha)(1-\sigma), \\
B = -(1+\alpha)(1-\sigma), \\
R = \left[\left(|A|^2 + 2 + 4|B| + |B|^2\right)^2 - \left|A^2 - B^2\right|^2\right].
\end{gathered} \tag{3.42}$$

Proof. We write

$$\begin{aligned}
p(z) &= \frac{1}{1-\sigma}\left[-\left(\frac{D^{\alpha+1}F(z)}{D^\alpha F(z)} - 2\right) - \sigma\right] \\
&= \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z).
\end{aligned} \tag{3.43}$$

Since $F \in MR_m^\alpha(\sigma)$, $p \in P_m$ in E with $p(0) = 1$, and $p_i \in P$, $i = 1, 2$.

Proceeding on the similar lines as before, we obtain form (3.42) and (3.43)

$$\frac{1}{1-\sigma}\left[-\left(\frac{D^{\alpha+1}F(z)}{D^\alpha F(z)} - 2\right) - \sigma\right] = p(z) + \frac{zp'(z)}{\{c + (\alpha+1)(1-\sigma)\}} = (1+\alpha)(1-\sigma)p(z) \tag{3.44}$$

and with convolution technique as previously used, we get from (3.44)

$$\begin{aligned}
&\frac{1}{1-\sigma}\left[-\left(\frac{D^{\alpha+1}F(z)}{D^\alpha F(z)} - 2\right) - \sigma\right] \\
&\quad = \left(\frac{m}{4} + \frac{1}{2}\right)\left[p_1(z) + \frac{zp_1'(z)}{A + Bp_1(z)}\right] - \left(\frac{m}{4} - \frac{1}{2}\right)\left[p_2(z) + \frac{zp_2'(z)}{A + Bp_2(z)}\right],
\end{aligned} \tag{3.45}$$

where $p_1, p_2 \in P$, $A = c + (1+\alpha)(1-\sigma)$, $B = -(1+\alpha)(1-\sigma)$, $(A+B) \neq 0$.

Then, by using, Lemma 2.3, we have

$$\operatorname{Re}\left\{p_i(z) + \frac{zp_i'(z)}{A + Bp_i(z)}\right\} > 0 \quad \text{in } |z| < \rho(A, B), \tag{3.46}$$

where $\rho(A, B)$ is given by (3.42).

Now, from (3.44) and (3.46), we have the required result that $f \in MR_m^{\alpha}(\sigma)$ in $|z| < \rho(A, B)$.

From Lemma 2.3 it follows that this result is sharp for $c > 0$. □

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