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Research Article

Solutions of Sign-Changing Fractional Differential Equation with the Fractional Derivatives

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We study the singular fractional-order boundary-value problem with a sign-changing nonlinear term $-\mathfrak{D}^{\alpha}x(t) = p(t)f(t, x(t), \mathfrak{D}^{\mu_1}x(t), \mathfrak{D}^{\mu_2}x(t), \dots, \mathfrak{D}^{\mu_{n-1}}x(t))$, $0 < t < 1$, $\mathfrak{D}^{\mu_i}x(0) = 0$, $1 \le i \le n-1$, $\mathfrak{D}^{\mu_{n-1}+1}x(0) = 0$, $\mathfrak{D}^{\mu_{n-1}}x(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}^{\mu_{n-1}}x(\xi_j)$, where $n-1 < \alpha \le n$, $n \in \mathbb{N}$ and $n \ge 3$ with $0 < \mu_1 < \mu_2 < \cdots < \mu_{n-2} < \mu_{n-1}$ and $n-3 < \mu_{n-1} < \alpha - 2$, $a_j \in \mathbb{R}$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$ satisfying $0 < \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1} < 1$, $\mathfrak{D}^{\alpha}$ is the standard Riemann-Liouville derivative, $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a sign-changing continuous function and may be unbounded from below with respect to x_i, and $p : (0,1) \to [0,\infty)$ is continuous. Some new results on the existence of nontrivial solutions for the above problem are obtained by computing the topological degree of a completely continuous field.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines, and particularly in the mathematical modeling of systems and processes in physics, chemistry, aerodynamics, electrodynamics of complex medium, and polymer rheology [1–6]. Fractional-order models have proved to be more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Hence fractional differential equations have attracted great research interest in recent years, and for more details we refer the reader to [7–16] and the references cited therein.

In this paper, we consider the existence of nontrivial solutions for the following singular fractional-order boundary-value problem with a sign-changing nonlinear term and fractional derivatives:

$$-\mathfrak{D}^{\alpha}x(t) = p(t)f\left(t, x(t), \mathfrak{D}^{\mu_1}x(t), \mathfrak{D}^{\mu_2}x(t), \dots, \mathfrak{D}^{\mu_{n-1}}x(t)\right), \quad 0 < t < 1,$$
$$\mathfrak{D}^{\mu_i}x(0) = 0, \quad 1 \le i \le n-1, \qquad \mathfrak{D}^{\mu_{n-1}+1}x(0) = 0, \qquad \mathfrak{D}^{\mu_{n-1}}x(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}^{\mu_{n-1}}x(\xi_j), \tag{1.1}$$

where $n-1 < \alpha \le n$, $n \in \mathbb{N}$ and $n \ge 3$ with $0 < \mu_1 < \mu_2 < \cdots < \mu_{n-2} < \mu_{n-1}$ and $n-3 < \mu_{n-1} < \alpha - 2$, $a_j \in \mathbb{R}$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$ satisfying $0 < \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1} < 1$, $\mathfrak{D}^{\alpha}$ is the standard Riemann-Liouville derivative, $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a sign-changing continuous function and may be unbounded from below with respect to x_i, and $p : (0,1) \to [0,\infty)$ is continuous.

In this paper, we assume that $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$, which implies that the problem (1.1) is changing sign (or semipositone particularly). Differential equations with changing-sign arguments are found to be important mathematical tools for the better understanding of several real-world problems in physics, chemistry, mechanics, engineering, and economics [17–19]. In general, the cone theory is difficult to handle this type of problems since the operator generated by f is not a cone mapping. So to find a new method to solve changing-sign problems is an interesting, important, and difficult work. An effective approach to this problem was recently suggested by Sun [20] based on the topological degree of a completely continuous field. Then, Han and Wu [21, 22] obtained a new Leray-Schauder degree theorem by improving the results of Sun [20]. In [22], Han et al. also investigated a kind of singular two-point boundary-value problems with sign-changing nonlinear terms by applying the new Leray-Schauder degree theorem obtained in [22].

To our knowledge, very few results have been established when f is changing sign [20–24]. In [20, 21, 23], f permits sign changing but required to be bounded from below. In [22, Theorem 1.1], f may be a sign-changing and unbounded function, but the Green function must be symmetric and f is controlled by a special function $h(u) = -b - c|u|^{\mu}$, where $b > 0, c > 0$ and $\mu \in (0,1)$. Recently, by improving and generalizing the main results of Sun [20] and Han et al. [21, 22], Liu et al. [24] established a generalized Leray-Schauder degree theorem of a completely continuous field for solving m-point boundary-value problems for singular second-order differential equations.

Motivated by [20–24], we established some new results on the existence of nontrivial solutions for the problem (1.1) by computing the topological degree of a completely continuous field. The conditions used in the present paper are weaker than the conditions given in previous works [20–24], and particularly we drop the assumption of even function in [24]. The new features of this paper mainly include the following aspects. Firstly, the nonlinear term $f(t, x_1, x_2, \dots, x_n)$ in the BVP (1.1) is allowed to be sign changing and unbounded from below with respect to x_i. Secondly, the nonlinear term f involves fractional derivatives of unknown functions. Thirdly, the boundary conditions involve fractional derivatives of unknown functions which is a more general case, and include the two-point, three-point, multipoint, and some nonlocal problems as special cases of (1.1).

2. Preliminaries and Lemmas

In this section, we give some preliminaries and lemmas.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following two conditions:

(1) $x \in P, \sigma > 0$ implies $\sigma x \in P$;

(2) $x \in P, -x \in P$ implies $x = \theta$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Let E be a real Banach space, E^* the dual space of E, P a total cone in E, that is, $E = \overline{P - P}$, and P^* the dual cone of P.

Lemma 2.3 (Deimling [25]). *Let $L : E \to E$ be a continuous linear operator, P a total cone, and $L(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a positive constant c such that $cL(\psi) \geq \psi$, then the spectral radius $r(L) \neq 0$ and has a positive eigenfunction corresponding to its first eigenvalue $\lambda = r(L)^{-1}$.*

Lemma 2.4 (see [25]). *Let P be a cone of the real Banach space E, and Ω a bounded open subsets of E. Suppose that $T : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $x_0 \in P \setminus \{\theta\}$ such that $x - Tx \neq \mu x_0, \forall x \in \partial\Omega \cap P, \mu \geq 0$, then the fixed-point index $i(T, \Omega \cap P, P) = 0$.*

Let $L : E \to E$ be a completely continuous linear positive operator with the spectral radius $r_1 \neq 0$. On account of Lemma 2.3, there exist $\varphi_1 \in P \setminus \{\theta\}$ and $g_1 \in P^* \setminus \{\theta\}$ such that

$$\begin{aligned} L\varphi_1 &= r_1\varphi_1, \\ r_1 L^* g_1 &= g_1, \end{aligned} \tag{2.1}$$

where L^* is the dual operator of L. Choose a number $\delta > 0$ and let

$$P(g_1, \delta) = \{u \in P : g_1(u) \geq \delta\|u\|\}, \tag{2.2}$$

then $P(g_1, \delta)$ is a cone in E.

Lemma 2.5 (see [24]). *Suppose that the following conditions are satisfied.*

(A1) *$T : E \to P$ is a continuous operator satisfying*

$$\lim_{\|u\| \to +\infty} \frac{\|Tu\|}{\|u\|} = 0; \tag{2.3}$$

(A2) *$F : E \to E$ is a bounded continuous operator and there exists $u_0 \in E$ such that $Fu + u_0 + Tu \in P$, for all $u \in E$;*

(A3) *$r_1 > 0$ and there exist $v_0 \in E$ and $\eta > 0$ such that*

$$LFu \geq r_1(1 + \eta)Lu - LTu - v_0, \quad \forall u \in E. \tag{2.4}$$

Let $A = LF$, then there exists $R > 0$ such that

$$\deg(I - A, B_R, \theta) = 0, \tag{2.5}$$

where $B_R = \{u \in E : \|u\| < R\}$ is the open ball of radius R in E.

Remark 2.6. If the operator T which satisfies the conditions of Lemma 2.5 is a null operator, then Lemma 2.5 turns into Theorem 1 in [20]. On the other hand, if the operator T in Lemma 2.5 is such that there exist constants $\alpha \in (0,1)$ and $N > 0$ satisfying $\|Tu\| \leq N\|u\|^{\alpha}$ for all $u \in E$, then Lemma 2.5 turns into Theorem 2.1 in [22] or Theorem 1 in [21]. So Lemma 2.5 is an improvement of the results of paper [20–22].

Now we present the necessary definitions from fractional calculus theory. These definitions can be found in some recent literatures, for example, [26, 27].

Definition 2.7 (see [26, 27]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0,+\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds \tag{2.6}$$

provided that the right-hand side is pointwisely defined on $(0,+\infty)$.

Definition 2.8 (see [26, 27]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0,+\infty) \to \mathbb{R}$ is given by

$$\mathfrak{D}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}x(s)ds, \tag{2.7}$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwisely defined on $(0,+\infty)$.

Remark 2.9. If $x, y : (0,+\infty) \to \mathbb{R}$ with order $\alpha > 0$, then

$$\mathfrak{D}^{\alpha}\big(x(t) + y(t)\big) = \mathfrak{D}^{\alpha}x(t) + \mathfrak{D}^{\alpha}y(t). \tag{2.8}$$

Lemma 2.10 (see [27]). *(1) If $x \in L^1(0,1)$, $\rho > \sigma > 0$, then*

$$I^{\rho}I^{\sigma}x(t) = I^{\rho+\sigma}x(t), \qquad \mathfrak{D}^{\sigma}I^{\rho}x(t) = I^{\rho-\sigma}x(t), \qquad \mathfrak{D}^{\sigma}I^{\sigma}x(t) = x(t). \tag{2.9}$$

(2) If $\rho > 0$, $\sigma > 0$, then

$$\mathfrak{D}^{\rho}t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\rho)}t^{\sigma-\rho-1}. \tag{2.10}$$

Lemma 2.11 (see [27]). *Assume that $x \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I^{\alpha}\mathfrak{D}^{\alpha}x(t) = x(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}, \tag{2.11}$$

where $c_i \in \mathbb{R}$ $(i = 1,2,\ldots,n)$, n is the smallest integer greater than or equal to α.

Noticing that $2 < \alpha - \mu_{n-1} \leq n - \mu_{n-1} < 3$, let

$$k(t,s) = \begin{cases} \dfrac{(t(1-s))^{\alpha-\mu_{n-1}-1} - (t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \leq s \leq t \leq 1, \\ \dfrac{(t(1-s))^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.12}$$

by [28], for $t, s \in [0,1]$, one has

$$\frac{t^{\alpha-\mu_{n-1}-1}(1-t)s(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} \leq k(t,s) \leq \frac{s(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}. \tag{2.13}$$

Lemma 2.12. *If* $2 < \alpha - \mu_{n-1} < 3$ *and* $\rho \in L^1[0,1]$, *then the boundary-value problem*

$$\begin{gathered} \mathfrak{D}^{\alpha-\mu_{n-1}} w(t) + \rho(t) = 0, \\ w(0) = w'(0) = 0, \qquad w(1) = \sum_{j=1}^{p-2} a_j w(\xi_j), \end{gathered} \tag{2.14}$$

has the unique solution

$$w(t) = \int_0^1 K(t,s)\rho(s)ds, \tag{2.15}$$

where

$$K(t,s) = k(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}} \sum_{j=1}^{p-2} a_j k(\xi_j, s), \tag{2.16}$$

is the Green function of the boundary-value problem (2.14).

Proof. By applying Lemma 2.11, we may reduce (2.14) to an equivalent integral equation:

$$w(t) = -I^{\alpha-\mu_{n-1}}\rho(t) + c_1 t^{\alpha-\mu_{n-1}-1} + c_2 t^{\alpha-\mu_{n-1}-2} + c_3 t^{\alpha-\mu_{n-1}-3}, \quad c_1, c_2, c_3 \in \mathbb{R}. \tag{2.17}$$

Note that $w(0) = w'(0) = 0$ and (2.17), we have $c_2 = c_3 = 0$. Consequently the general solution of (2.14) is

$$w(t) = -I^{\alpha-\mu_{n-1}}\rho(t) + c_1 t^{\alpha-\mu_{n-1}-1}. \tag{2.18}$$

By (2.18) and Lemma 2.10, we have

$$\begin{aligned} w(t) &= -I^{\alpha-\mu_{n-1}}\rho(t) + c_1 t^{\alpha-\mu_{n-1}-1} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds + c_1 t^{\alpha-\mu_{n-1}-1}. \end{aligned} \tag{2.19}$$

So,

$$w(1) = -\int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds + c_1, \tag{2.20}$$

and for $j = 1, 2, \ldots, p-2$,

$$w(\xi_j) = -\int_0^{\xi_j} \frac{(\xi_j - s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds + c_1\xi_j^{\alpha-\mu_{n-1}-1}. \tag{2.21}$$

By $w(1) = \sum_{j=1}^{p-2} a_j w(\xi_j)$, combining with (2.20) and (2.21), we obtain

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha-\mu_{n-1}-1}\rho(s)ds - \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_{n-1}-1}\rho(s)ds}{\Gamma(\alpha-\mu_{n-1})\left(1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}\right)}. \tag{2.22}$$

So, the unique solution of problem (2.14) is

$$\begin{aligned}
w(t) &= -\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds \\
&\quad + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j\xi_j^{\alpha-\mu_{n-1}-1}}\left\{\int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds - \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds\right\} \\
&= -\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds + \int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}t^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds \\
&\quad + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j\xi_j^{\alpha-\mu_{n-1}-1}}\sum_{j=1}^{p-2} a_j \int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}\xi_j^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds \\
&\quad - \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j\xi_j^{\alpha-\mu_{n-1}-1}}\sum_{j=1}^{p-2} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\rho(s)ds \\
&= \int_0^1 \left(k(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j\xi_j^{\alpha-\mu_{n-1}-1}}\sum_{j=1}^{p-2} a_j k(\xi_j, s)\right)\rho(s)ds \\
&= \int_0^1 K(t,s)\rho(s)ds.
\end{aligned} \tag{2.23}$$

The proof is completed. □

Lemma 2.13. *The function $K(t,s)$ has the following properties:*

(1) $K(t,s) > 0$, $for\ t,s \in (0,1)$;

(2) $K(t,s) \le Ms(1-s)^{\alpha-\mu_{n-1}-1}$, $for\ t,s \in [0,1]$, where

$$M = \frac{1}{\Gamma(\alpha-\mu_{n-1})}\left(1 + \frac{\sum_{j=1}^{p-2} a_j}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}}\right). \tag{2.24}$$

Proof. It is obvious that (1) holds. In the following, we will prove (2). In fact, by (2.13), we have

$$\begin{aligned} K(t,s) &= k(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}} \sum_{j=1}^{p-2} a_j k(\xi_j, s) \\ &\le \frac{s(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} + \frac{\sum_{j=1}^{p-2} a_j}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}} \frac{s(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} \\ &\le \left(1 + \frac{\sum_{j=1}^{p-2} a_j}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}}\right) \frac{s(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}. \end{aligned} \tag{2.25}$$

This completes the proof. □

Now let us consider the following modified problems of the BVP (1.1):

$$\begin{gathered} -\mathfrak{D}^{\alpha-\mu_{n-1}} u(t) = p(t) f\left(t, I^{\mu_{n-1}} u(t), I^{\mu_{n-1}-\mu_1} u(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}} u(t), u(t)\right), \\ u(0) = 0, \qquad u'(0) = 0, \qquad \mathfrak{D}^{\mu-\mu_{n-1}} u(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}^{\mu-\mu_{n-1}} u(\xi_j). \end{gathered} \tag{2.26}$$

Lemma 2.14. *Let $x(t) = I^{\mu_{n-1}} u(t), u(t) \in C[0,1]$, Then* (1.1) *can be transformed into* (2.26). *Moreover, if $u \in C([0,1],[0,+\infty)$ is a solution of problem* (2.26)*, then, the function $x(t) = I^{\mu_{n-1}} u(t)$ is a positive solution of problem* (1.1)*.*

Proof. Substituting $x(t) = I^{\mu_{n-1}} u(t)$ into (1.1), by the definition of the Riemann-Liouville fractional derivative and Lemmas 2.10 and 2.11, we obtain that

$$\begin{aligned} \mathfrak{D}^{\alpha} x(t) &= \frac{d^n}{dt^n} I^{n-\alpha} x(t) = \frac{d^n}{dt^n} I^{n-\alpha} I^{\mu_{n-1}} u(t) \\ &= \frac{d^n}{dt^n} I^{n-\alpha+\mu_{n-1}} u(t) = \mathfrak{D}^{\alpha-\mu_{n-1}} u(t), \\ \mathfrak{D}^{\mu_1} x(t) &= \mathfrak{D}^{\mu_1} I^{\mu_{n-1}} u(t) = I^{\mu_{n-1}-\mu_1} u(t), \\ \mathfrak{D}^{\mu_2} x(t) &= \mathfrak{D}^{\mu_2} I^{\mu_{n-1}} u(t) = I^{\mu_{n-1}-\mu_2} u(t), \\ &\ \ \vdots \qquad\qquad\qquad\qquad \vdots \\ \mathfrak{D}^{\mu_{n-2}} x(t) &= \mathfrak{D}^{\mu_{n-2}} I^{\mu_{n-1}} u(t) = I^{\mu_{n-1}-\mu_{n-2}} u(t), \\ \mathfrak{D}^{\mu_{n-1}} x(t) &= \mathfrak{D}^{\mu_{n-1}} I^{\mu_{n-1}} u(t) = u(t), \\ \mathfrak{D}^{\mu_{n-1}+1} x(t) &= \left[\mathfrak{D}^{\mu_{n-1}} I^{\mu_{n-1}} u(t)\right]' = u'(t). \end{aligned} \tag{2.27}$$

Also, we have $\mathfrak{D}^{\mu_{n-1}}x(0) = u(0) = 0, \mathfrak{D}^{\mu_{n-1}+1}x(0) = u'(0) = 0$, and it follows from $\mathfrak{D}^{\mu_{n-1}}x(t) = u(t)$ that $u(1) = \sum_{j=1}^{p-2} a_j u(\xi_j)$. Hence, by $x(t) = I^{\mu_{n-1}}u(t), u \in C[0,1]$, (1.1) is transformed into (2.14).

Now, let $u \in C([0,1,0,+\infty))$ be a solution of problem (2.26). Then, by Lemma 2.10, (2.26), and (2.27), one has

$$\begin{aligned} -\mathfrak{D}^{\alpha}x(t) &= -\frac{d^n}{dt^n}I^{n-\alpha}x(t) = -\frac{d^n}{dt^n}I^{n-\alpha}I^{\mu_{n-1}}u(t) \\ &= -\frac{d^n}{dt^n}I^{n-\alpha+\mu_{n-1}}u(t) = -\mathfrak{D}^{\alpha-\mu_{n-1}}u(t) \\ &= p(t)f\left(t, I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right) \\ &= p(t)f(t, x(t), \mathfrak{D}^{\mu_1}x(t), \mathfrak{D}^{\mu_2}x(t), \ldots, \mathfrak{D}^{\mu_{n-1}}x(t)), \quad 0 < t < 1. \end{aligned} \tag{2.28}$$

Noticing that

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \tag{2.29}$$

which implies that $I^{\alpha}u(0) = 0$, from (2.27), for $i = 1, 2, \ldots, n-1$, we have

$$\mathfrak{D}^{\mu_i}x(0) = 0, \qquad \mathfrak{D}^{\mu_{n-1}+1}x(0) = 0, \qquad \mathfrak{D}^{\mu_{n-1}}x(1) = \sum_{j=1}^{p-2}a_j\mathfrak{D}^{\mu_{n-1}}x(\xi_j). \tag{2.30}$$

Moreover, it follows from the monotonicity and property of $I^{\mu_{n-1}}$ that

$$I^{\mu_{n-1}}u \in C([0,1],[0,+\infty)). \tag{2.31}$$

Consequently, $x(t) = I^{\mu_{n-1}}u(t)$ is a positive solution of problem (1.1). □

In the following let us list some assumptions to be used in the rest of this paper.

(H1) $p : (0,1) \to [0,+\infty)$ is continuous, $p(t) \not\equiv 0$ on any subinterval of (0,1), and

$$\int_0^1 p(s)ds < +\infty. \tag{2.32}$$

(H2) $f : [0,1] \times \mathbb{R}^n \to (-\infty,+\infty)$ is continuous.

In order to use Lemma 2.5, let $E = C[0,1]$ be our real Banach space with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$ and $P = \{u \in C[0,1] : u(t) \geq 0, \text{ for all } t \in [0,1]\}$, then P is a total cone in E.

Define two linear operators $L, J : C[0,1] \to C[0,1]$ by

$$(Lu)(t) = \int_0^1 K(t,s)p(s)u(s)ds, \quad t \in [0,1], \ u \in C[0,1], \tag{2.33}$$

$$(Ju)(t) = \int_0^1 K(s,t)p(s)u(s)ds, \quad t \in [0,1], \ u \in C[0,1], \tag{2.34}$$

and define a nonlinear operator $A : C[0,1] \to C[0,1]$ by

$$(Au)(t) = \int_0^1 K(t,s)p(s)f\left(t, I^{\mu_{n-1}}u(s), I^{\mu_{n-1}-\mu_1}u(s), \ldots, I^{\mu_{n-1}-\mu_{n-2}}u(s), u(s)\right)ds, \quad t \in [0,1]. \tag{2.35}$$

Lemma 2.15. *Assume (H1) holds. Then*

(i) $L, J : C[0,1] \to C[0,1]$ *are completely continuous positive linear operators with the first eigenvalue* $r > 0$ *and* $\lambda > 0$*, respectively.*

(ii) L *satisfies* $L(P) \subset P(g_1, \delta)$*.*

Proof. (i) By using the similar method of paper [22], it is easy to know that $L, J : C[0,1] \to C[0,1]$ are completely continuous positive linear operators. In the following, by using the Krein-Rutmann's theorem, we prove that L, J have the first eigenvalue $r > 0$ and $\lambda > 0$, respectively.

In fact, it is obvious that there is $t_1 \in (0,1)$ such that $K(t_1,t_1)p(t_1) > 0$. Thus there exists $[a,b] \subset (0,1)$ such that $t_1 \in (a,b)$ and $K(t,s)p(s) > 0$ for all $t, s \in [a,b]$. Choose $\psi \in P$ such that $\psi(t_1) > 0$ and $\psi(t) = 0$ for all $t \notin [a,b]$. Then for $t \in [a,b]$,

$$(L\psi)(t) = \int_0^1 K(t,s)p(s)\psi(s)ds \geq \int_a^b K(t,s)p(s)\psi(s)ds > 0. \tag{2.36}$$

So there exists $\nu > 0$ such that $\nu(L\psi)(t) \geq \psi(t)$ for $t \in [0,1]$. It follows from Lemma 2.3 that the spectral radius $r_1 \neq 0$. Thus corresponding to $r = r_1^{-1}$, the first eigenvalue of L, and L has a positive eigenvector φ_1 such that

$$rL\varphi_1 = \varphi_1. \tag{2.37}$$

In the same way, J has a positive first eigenvalue λ and a positive eigenvector φ_2 corresponding to the first eigenvalue λ, which satisfy

$$\lambda J\varphi_2 = \varphi_2. \tag{2.38}$$

(ii) Notice that $K(t,0) = K(t,1) \equiv 0$ for $t \in [0,1]$, by $\lambda J\varphi_2 = \varphi_2$ and (2.12)–(2.16), we have $\varphi_2(0) = \varphi_2(1) = 0$. This implies that $\varphi_2'(0) > 0$ and $\varphi_2'(1) < 0$ (see [29]). Define a function χ on $[0,1]$ by

$$\chi(s) = \begin{cases} \varphi_2'(0), & s = 0, \\ \dfrac{\varphi_2(s)}{(1-s)s}, & 0 < s < 1, \\ -\varphi_2'(1), & s = 1. \end{cases} \tag{2.39}$$

Then χ is continuous on $[0,1]$ and $\chi(s) > 0$ for all $s \in [0,1]$. So, there exist $\delta_1, \delta_2 > 0$ such that $\delta_1 \leq \chi(s) \leq \delta_2$ for all $s \in [0,1]$. Thus

$$\delta_1(1-s)s \leq \varphi_2(s) \leq \delta_2(1-s)s, \tag{2.40}$$

for all $s \in [0,1]$.

Let L^* be the dual operator of L, we will show that there exists $g_1 \in P^* \setminus \{\theta\}$ such that

$$\lambda L^* g_1 = g_1. \tag{2.41}$$

In fact, let

$$g_1(u) = \int_0^1 p(t)\varphi_2(t)u(t)dt \quad \forall u \in E. \tag{2.42}$$

Then by (H1) and (2.40), we have

$$\int_0^1 p(t)\varphi_2(t)u(t)dt \le \delta_2\|u\| \int_0^1 t(1-t)p(t)dt \le \delta_2\|u\| \int_0^1 p(t)dt < +\infty, \tag{2.43}$$

which implies that g_1 is well defined. We state that g_1 of (2.42) satisfies (2.41). In fact, by (2.40), (2.41), and interchanging the order of integration, for any $s, t \in [0,1]$, we have

$$\begin{aligned}
\lambda^{-1} g_1(u) &= \int_0^1 p(t)\left(\lambda^{-1}\varphi_2(t)\right)u(t)dt = \int_0^1 p(t)(J\varphi_2)(t)u(t)dt \\
&= \int_0^1 p(t)u(t)\int_0^1 K(s,t)p(s)\varphi_2(s)dsdt \\
&= \int_0^1 p(s)\varphi_2(s)\int_0^1 K(s,t)p(t)u(t)dtds \\
&= \int_0^1 p(s)\varphi_2(s)(Lu)(s)ds \\
&= g_1(Lu) = \left(L^* g_1\right)(u) \quad \forall u \in E.
\end{aligned} \tag{2.44}$$

So (2.41) holds.

In the following we prove that $L(P) \subset P(g_1, \delta)$. In fact, by (2.41) and $\alpha - \mu_{n-1} - 1 > 1$, we have

$$\varphi_2(s) \ge \delta_1(1-s)s \ge \delta_1(1-s)^{\alpha-\mu_{n-1}-1}s \ge \delta_1 M^{-1}K(t,s), \quad t, s \in [0,1]. \tag{2.45}$$

Take $\delta = \delta_1 M^{-1}\lambda^{-1} > 0$ in (2.2). For any $u \in P$, by (2.44), (2.45), we have

$$\begin{aligned}
g_1(Lu) &= \lambda^{-1} g_1(u) = \lambda^{-1}\int_0^1 p(s)\varphi_2(s)u(s)ds \\
&\ge \delta_1 M^{-1}\lambda^{-1}\int_0^1 K(t,s)p(s)u(s)ds = \delta(Lu)(t) \quad \forall t \in [0,1].
\end{aligned} \tag{2.46}$$

Hence, $g_1(Lu) \ge \delta\|Lu\|$, that is, $L(P) \subset P(g_1, \delta)$. The proof is completed. □

3. Main Result

Theorem 3.1. *Assume that (H1)(H2) hold, and the following conditions are satisfied.*

(H3) There exist nonnegative continuous functions $b, c : [0,1] \to (0,+\infty)$ and a nondecreasing continuous function $h : \mathbb{R}^n \to [0,+\infty)$ satisfying

$$\lim_{\sum_{i=1}^{n}|x_i| \to +\infty} \frac{h(x_1, x_2, \ldots, x_n)}{\sum_{i=1}^{n}|x_i|} = 0,$$
$$f(t, x_1, x_2, \ldots, x_n) \geq -b(t) - c(t)h(x_1, x_2, \ldots, x_n), \quad \forall x_i \in \mathbb{R}. \tag{3.1}$$

(H4) f also satisfies

$$\liminf_{\substack{\sum_{i=1}^{n} x_i \to +\infty \\ \sum_{i=1}^{n-1} x_i \geq 0}} \frac{f(t, x_1, x_2, \ldots, x_n)}{\sum_{i=1}^{n} x_i} > \lambda \qquad \limsup_{\sum_{i=1}^{n}|x_i| \to 0} \frac{|f(t, x_1, x_2, \ldots, x_n)|}{\sum_{i=1}^{n}|x_i|} < \lambda, \tag{3.2}$$

uniformly on $t \in [0,1]$, where λ is the first eigenvalue of the operator J defined by (2.34)*.*

Then the singular fractional-order boundary-value problem (1.1) *has at least one nontrivial solution.*

Proof. According to Lemma 2.15, L satisfies $L(P) \subset P(g_1, \delta)$. Let

$$(Tu)(t) = ch\left(I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right) \quad \text{for } u \in E, \tag{3.3}$$

where $c = \max_{t\in[0,1]} c(t)$. It follows from (H3) that $T : E \to P$ is a continuous operator. Note that

$$I^{\mu_{n-1}}u(t) = \int_0^t \frac{(t-s)^{\mu_{n-1}-1}u(s)}{\Gamma(\mu_{n-1})} ds \leq \frac{\|u\|}{\Gamma(\mu_{n-1})},$$
$$I^{\mu_{n-1}-\mu_i}u(t) = \int_0^t \frac{(t-s)^{\mu_{n-1}-\mu_i-1}u(s)}{\Gamma(\mu_{n-1}-\mu_i)} ds \leq \frac{\|u\|}{\Gamma(\mu_{n-1}-\mu_i)}, \quad i = 1, 2, \ldots, n-2. \tag{3.4}$$

Thus from the monotone assumption of h on x_i, we have

$$h\left(I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right)$$
$$\leq h\left(\frac{\|u\|}{\Gamma(\mu_{n-1})}, \frac{\|u\|}{\Gamma(\mu_{n-1}-\mu_1)}, \ldots, \frac{\|u\|}{\Gamma(\mu_{n-1}-\mu_{n-2})}, \|u\|\right), \quad \text{for any } u \in E, t \in [0,1], \tag{3.5}$$

which implies that

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]}\left\{ch\left(I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \dots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right)\right\} \\ &\le ch\left(\frac{\|u\|}{\Gamma(\mu_{n-1})}, \frac{\|u\|}{\Gamma(\mu_{n-1}-\mu_1)}, \dots, \frac{\|u\|}{\Gamma(\mu_{n-1}-\mu_{n-2})}, \|u\|\right) \quad \text{for any } u \in E.\end{aligned} \tag{3.6}$$

Let

$$\tau = \frac{1}{\Gamma(\mu_{n-1})} + \frac{1}{\Gamma(\mu_{n-1}-\mu_1)} + \cdots + \frac{1}{\Gamma(\mu_{n-1}-\mu_{n-2})} + 1, \tag{3.7}$$

then

$$\begin{aligned}\lim_{\|u\|\to+\infty}\frac{\|Tu\|}{\|u\|} &\le \lim_{\|u\|\to+\infty}\frac{ch\left(\|u\|/\Gamma(\mu_{n-1}), \|u\|/\Gamma(\mu_{n-1}-\mu_1), \dots, \|u\|/\Gamma(\mu_{n-1}-\mu_{n-2}), \|u\|\right)}{\|u\|} \\ &= \lim_{\|u\|\to+\infty}\frac{c\tau h\left(\|u\|/\Gamma(\mu_{n-1}), \|u\|/\Gamma(\mu_{n-1}-\mu_1), \dots, \|u\|/\Gamma(\mu_{n-1}-\mu_{n-2}), \|u\|\right)}{\tau\|u\|} \\ &\le \lim_{\|u\|\to+\infty}\frac{c\tau h\left(\|u\|/\Gamma(\mu_{n-1}), \|u\|/\Gamma(\mu_{n-1}-\mu_1), \dots, \|u\|/\Gamma(\mu_{n-1}-\mu_{n-2}), \|u\|\right)}{\|u\|/\Gamma(\mu_{n-1}) + \|u\|/\Gamma(\mu_{n-1}-\mu_1) + \cdots + \|u\|/\Gamma(\mu_{n-1}-\mu_{n-2}) + \|u\|} \\ &= c\tau\lim_{\sum_{i=1}^{n}|x_i|\to+\infty}\frac{h(x_1, x_2, \dots, x_n)}{\sum_{i=1}^{n}|x_i|} = 0,\end{aligned} \tag{3.8}$$

that is,

$$\lim_{\|u\|\to+\infty}\frac{\|Tu\|}{\|u\|} = 0. \tag{3.9}$$

Hence T satisfies condition (A1) in Lemma 2.5.

Next take $u_0(t) \equiv b(t)$, and $(Fu)(t) = f(t, I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \dots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t))$ for $u \in E$. Then it follows from (H3) that

$$\begin{aligned}Fu + u_0 + Tu &= f\left(t, I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \dots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right) \\ &\quad + b(t) + ch\left(I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \dots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right) \ge 0,\end{aligned} \tag{3.10}$$

that yields

$$Fu + u_0 + Tu \in P, \quad \forall u \in E, \tag{3.11}$$

namely, condition (A2) in Lemma 2.5 holds.

From (H4), there exists $\varepsilon > 0$ and a sufficiently large $l_1 > 0$ such that, for any $t \in [0,1]$,

$$\begin{aligned} f(t, x_1, x_2, \ldots, x_n) &\geq \lambda(1+\varepsilon)(x_1 + x_2 + \cdots + x_n) \\ &\geq \lambda(1+\varepsilon)x_n, \qquad x_1 + x_2 + \cdots + x_n > l_1. \end{aligned} \tag{3.12}$$

Combining (H3) with (3.12), there exists $b_1 \geq 0$ such that

$$\begin{aligned} Fu &= f\left(t, I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right) \\ &\geq \lambda(1+\varepsilon)u(t) - b_1 - ch\left(I^{\mu_{n-1}}u(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right), \quad \forall u \in E, \end{aligned} \tag{3.13}$$

that is,

$$\mathrm{F}u \geq \lambda(1+\varepsilon)u - b_1 - Tu \quad \forall u \in E. \tag{3.14}$$

As L is a positive linear operator, it follows from (3.14) that

$$(LFu)(t) \geq \lambda(1+\varepsilon)(Lu)(t) - Lb_1 - (LTu)(t), \quad \forall t \in [0,1]. \tag{3.15}$$

So condition (A3) in Lemma 2.5 holds. According to Lemma 2.5, there exists a sufficiently large number $R > 0$ such that

$$\deg(I - A, B_R, \theta) = 0. \tag{3.16}$$

On the other hand, it follows from (H4) that there exist $0 < \varepsilon < 1$ and $0 < r < R$, for any $t \in [0,1]$, such that

$$\begin{aligned} \left|f(t, x_1, x_2, \ldots, x_n)\right| &\leq \frac{1-\varepsilon}{M\tau \int_0^1 p(s)ds}(|x_1| + |x_2| + \cdots + |x_n|), \\ &\qquad |x_1| + |x_2| + \cdots + |x_n| < r. \end{aligned} \tag{3.17}$$

Thus for any $u \in E$ with $\|u\| \leq r/\tau \leq r \leq R$, we have

$$\begin{aligned} &|I^{\mu_{n-1}}u(t)| + \left|I^{\mu_{n-1}-\mu_1}u(t)\right| + \cdots + \left|I^{\mu_{n-1}-\mu_{n-2}}u(t)\right| + |u(t)| \\ &\quad \leq \frac{\|u\|}{\Gamma(\mu_{n-1})} + \frac{\|u\|}{\Gamma(\mu_{n-1}-\mu_1)} + \cdots + \frac{\|u\|}{\Gamma(\mu_{n-1}-\mu_{n-2})} + \|u\| \\ &\quad \leq \tau\|u\| \leq r. \end{aligned} \tag{3.18}$$

By (3.17), for any $t \in [0,1]$, we have

$$\begin{aligned}&\left|f\left(t, I^{\mu_{n-1}}u(t), I^{\mu_{n-1}-\mu_1}u(t), \dots, I^{\mu_{n-1}-\mu_{n-2}}u(t), u(t)\right)\right| \\ &\quad \leq \frac{1-\varepsilon}{M\tau\int_0^1 p(s)ds}\left(\left|I^{\mu_{n-1}}u(t)\right| + \left|I^{\mu_{n-1}-\mu_1}u(t)\right| + \cdots + \left|I^{\mu_{n-1}-\mu_{n-2}}u(t)\right| + |u(t)|\right).\end{aligned} \tag{3.19}$$

Thus if there exist $u_1 \in \partial B_{r/\tau}$ and $\mu_1 \in [0,1]$ such that $u_1 = \mu_1 Au_1$, then by (3.19), we have

$$\begin{aligned}g_1(\|u_1\|) &= g_1(\mu_1\|Au_1\|) = \mu_1 g_1(\|Au_1\|) \\ &= \mu_1 g_1\left(\max_{0\leq t\leq 1}\left|\int_0^1 K(t,s)p(s)f\left(s, I^{\mu_{n-1}}u_1(s), \dots, I^{\mu_{n-1}-\mu_{n-2}}u_1(s), u_1(s)\right)ds\right|\right) \\ &\leq \frac{1-\varepsilon}{M\tau\int_0^1 p(s)ds}g_1\left(\int_0^1 Mp(s)\left(\left|I^{\mu_{n-1}}u_1(s)\right| + \cdots + \left|I^{\mu_{n-1}-\mu_{n-2}}u_1(s)\right| + |u_1(s)|\right)ds\right) \\ &\leq \frac{1-\varepsilon}{M\tau\int_0^1 p(s)ds}M\tau\int_0^1 p(s)ds\, g_1(\|u_1\|) = (1-\varepsilon)g_1(\|u_1\|).\end{aligned} \tag{3.20}$$

Therefore, $g_1(\|u_1\|) \leq 0$.

But $\varphi_2(t) > 0$ for all $t \in (0,1)$ by the maximum principle, and $u_1(t)$ attains zero on isolated points by the Sturm theorem. Hence, from (2.42),

$$g_1(\|u_1\|) = \int_0^1 p(t)\varphi_2(t)\|u_1\|dt > 0. \tag{3.21}$$

This is a contradiction. Thus

$$u \neq \mu Au, \quad \forall u \in \partial B_r,\ \mu \in [0,1]. \tag{3.22}$$

It follows from the homotopy invariance of the Leray-Shauder degree that

$$\deg(I - A, B_{r/\tau}, \theta) = 1. \tag{3.23}$$

By (3.16), (3.23), and the additivity of Leray-Shauder degree, we obtain

$$\deg(I - A, B_R \setminus B_{r/\tau}, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_{r/\tau}, \theta) = -1. \tag{3.24}$$

As a result, A has at least one fixed point u on $B_R \setminus B_{r/\tau}$, namely, the BVP (1.1) has at least one nontrivial solution $x(t) = I^{\mu_{n-1}}u(t)$. □

Corollary 3.2. *Assume that (H1), (H2), and (H4) hold. If the following condition is satisfied.*

*(H*3) There exists a nonnegative continuous functions* $b : [0,1] \to (0,+\infty)$ *such that*

$$f(t, x_1, x_2, \ldots, x_n) \geq -b(t), \quad \text{for any} \quad t \in [0,1], \ \forall x_i \in \mathbb{R}, \tag{3.25}$$

then the singular higher multipoint boundary-value problems (1.1) *have at least one nontrivial solution.*

Corollary 3.3. *Assume that (H1), (H2), and (H4) hold. If the following condition is satisfied.*

*(H3**) There exist three constants* $b > 0, c > 0$, *and* $\alpha_i \in (0,1)$ *such that*

$$f(t, x_1, x_2, \ldots, x_n) \geq -b - c\sum_{i=1}^{n} |x_i|^{\alpha_i}, \quad \text{for any } t \in [0,1], \ \forall x_i \in \mathbb{R}, \tag{3.26}$$

then the singular higher multipoint boundary-value problems (1.1) *have at least one nontrivial solution.*

Remark 3.4. Noticing that the Green function of the BVP (1.1) is not symmetrical, which implies that the existence of nontrivial solutions of the BVP (1.1) cannot be obtained by Theorem 2.1 in [22] and Theorem 1 in [20]. It is interesting that we construct a new linear operator J instead of K in paper [22] and use its first eigenvalue and its corresponding eigenfunction to seek a linear continuous functional g of P. As a result, we overcome the difficulty caused by the nonsymmetry of the Green function. In [24], the nonlinearity does not contain derivatives and a stronger condition is required, that is, h must be an even function; here we omit this stronger assumption.

Remark 3.5. The results of [20–22] is a special case of the Corollary 3.2 and Corollary 3.3 when α_1 $(i = 1, 2, \ldots, n)$ are integer and the nonlinear term does not involve derivatives of unknown functions.

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