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*Research Article*

# A Dynamic Model for Fishery Resource with Reserve Area and Taxation

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The present paper deals with a dynamic reaction model of a fishery. The dynamics of a fishery resource system in an aquatic environment consists of two zones: a free fishing zone and a reserve zone. To protect fish population from over exploitation, a control instrument tax is imposed. The existence of its steady states and their stability are studied. The optimal harvest policy is discussed next with the help of Pontryagin's maximum principle. Our theoretical results are confirmed by numerical simulation.

## 1. Introduction

With the growing need of human for more food and energy, several resources have been increasingly exploited. It has caused wide public concern to protect the ecosystem. A scientific management of commercial exploitation of the biological resource like fisheries and forestry is necessitated. The first attempt of mathematical modelling of resource management problems was made in the article by Hotelling [1], who dealt with the economies of exhaustible resources using the calculus of variations, which was not a very familiar topic for the researchers as Hotelling's article was largely ignored until the 1973 energy crisis which led to a surge of interest in the field [2]. The techniques and issues associated with bioeconomic exploitation of these resources have been discussed in detail by Clark [3, 4]. During the last few decades, several investigations regarding fishing resource have been conducted [5–16]. Dubey et al. [17] proposed and analyzed a mathematical model as follows:

$$\begin{aligned} x'(t) &= rx(t)\left(1-\frac{x(t)}{K}\right)-\sigma_1 x(t)+\sigma_2 y(t)-qEx(t),\\ y'(t) &= sy(t)\left(1-\frac{y(t)}{L}\right)+\sigma_1 x(t)-\sigma_2 y(t), \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ represent biomass densities of the same fish population inside the free fishing zone, and reserve area, respectively, at a time $t$. Let the fish subpopulation of the free fishing zone migrate into reserve area at a rate $\sigma_1$ and the fish subpopulation of the reserve area migrate into free fishing zone at a rate $\sigma_2$. $q$ is the catch ability coefficient of the fish population in free fishing zone, and $E$ denotes the effort devoted to the harvesting. Kar and Misra [18] modified the model proposed in [17] by Dubey et al.

On the other hand, regulation of exploitation of biological resources has become a problem of major concern nowadays in view of the dwindling resource stocks and the deteriorating environment. Exploitation of marine fisheries naturally involves the problems of law enforcement. Several governing instruments are suggested for the choice of a regulatory control variable. These are imposition of taxes and license fees, leasing of property rights, seasonal harvesting, direct control, and so forth. Various issues associated with the choice of an optimal governing instrument and its enforcement in fishery were discussed by Anderson and Lee [19]. Pradhan and Chaudhuri [20] studied a dynamic reaction model of a fishery. A regulatory agency controls exploitation of the fishery by imposing a tax per unit biomass of the landed fish. Ganguli and Chaudhuri [21] studied the bionomic exploitation of single species fishery using taxation as a control variable. Pradhan and Chaudhuri [22] studied a dynamic reaction model of two-species fishery with taxation as a control instrument. It deals with a dynamic reaction model of a fishery consisting of two competing species, each of which obeys the logistic law of growth. A regulatory agency controls exploitation of the fishery by imposing a tax per unit biomass of the landed fish. Kar [23] discussed a problem of selective harvesting in a ratio-dependent predator-prey fishery in which both the predator and prey obey the logistic law of growth. Kar [24] also studied a ratio-dependent prey-predator model with selective harvesting of prey species. A regulatory agency controls exploitation by imposing a tax per unit biomass of the prey species.

In order to keep a sustainable fishing resource. We will take some actions in fishing areas to protect certain fish stocks by restricting the fishermen's fishing action. Such restriction would be implemented in the form of taxation. Following [22], we take $E$ as dynamic (i.e., time-dependent) governed variable. We assume that the fish population in the free fishing zone is subject to a harvesting effort governed by the differential equations. To conserve fish populations, the regulatory agency imposes a tax $\tau > 0$ per unit biomass of the landed fish. ($\tau < 0$ denotes the subsidies given to the fishermen). Keeping these aspects in view, the dynamics of the system may be governed by the following system of equations:

$$\begin{gathered}
x'(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) - \sigma_1 x(t) + \sigma_2 y(t) - qE(t)x(t), \\
y'(t) = sy(t)\left(1 - \frac{y(t)}{L}\right) + \sigma_1 x(t) - \sigma_2 y(t), \\
E'(t) = \left\{\alpha\beta\left[q(p-\tau)x(t) - c\right] - \gamma\right\}E(t), \\
x(0) > 0, \quad y(0) > 0, \quad E(0) > 0, \quad 0 < \alpha \le 1, \quad 0 \le \beta < 1,
\end{gathered} \tag{1.2}$$

where $\gamma$ is constant rate of depreciation of capital. $\alpha$ and $\beta$ are coefficients of proportionality, $\alpha\beta$ is called the stiffness parameter measuring the strength of reaction of effort to the perceived rent; for further details about the biological sense of $\alpha$ and $\beta$, we refer to [23, 24]. $p$ is the constant price per unit biomass of the fish species, and $c$ is the constant cost per unit of harvesting effort.

From [17], we know that there is no migration of fish population from reserve area to free fishing zone (i.e., $\sigma_2 = 0$) and $r - \sigma_1 < 0$, then $\dot{x}(t) < 0$. Similarly, if there is no migration of fish population from free fishing zone to reserve area (i.e., $\sigma_1 = 0$) and $s - \sigma_2 < 0$, then $\dot{y}(t) < 0$. Hence throughout our analysis, we assume that

$$r - \sigma_1 > 0, \qquad s - \sigma_2 > 0. \tag{1.3}$$

The aim of this paper is to find a proper taxation policy which would give the best possible benefit through harvesting to community while preventing the extinction of the fishing species. The structure of this paper is as follows. In the next section, we study the steady-state existence of positive equilibrium. In Section 3, by analyzing the corresponding characteristic equations, we discuss the local stability of equilibria. In Section 4, the global stability of the system is discussed by constructing a suitable Lyapunov function. In Section 5, optimal tax policy is discussed using Pontryagin's maximum principle [25]. In Section 6, we try to interpret our results by numerical simulation. At last, we give some discussions.

## 2. Existence of Equilibria

We find the steady-states of (1.2) by equating the derivatives on the left-hand sides to zero and solving the resulting algebraic equations. This gives three possible steady states, namely: $P_0(0,0,0)$, $P_1(\overline{x},\overline{y},0)$, and $P_2(x^*,y^*,E^*)$. Existence of $P_0(0,0,0)$ is obviously. We will consider the existence of $P_1(\overline{x},\overline{y},0)$ in the following. We know that $\overline{x}$ and $\overline{y}$ are the positive solutions of the following equations:

$$\begin{aligned} \sigma_2 y(t) &= \frac{r x^2(t)}{K} - (r - \sigma_1)x(t), \\ \sigma_1 x(t) &= (\sigma_2 - s)y(t) + \frac{s y^2(t)}{L}. \end{aligned} \tag{2.1}$$

From [17], we get a cubic equation in $x(t)$ as

$$a x^3(t) + b x^2(t) + c x(t) + d = 0, \tag{2.2}$$

where

$$\begin{aligned} a &= \frac{s r^2}{L \sigma_2^2 K^2}, \\ b &= -\frac{2 s r (r - \sigma_1)}{L \sigma_2^2 K}, \\ c &= \frac{s (r - \sigma_1)^2}{L \sigma_2^2} - \frac{(s - \sigma_2) r}{K \sigma_2}, \\ d &= \frac{(s - \sigma_2)}{\sigma_2}(r - \sigma_1) - \sigma_1. \end{aligned} \tag{2.3}$$

Please note that there may be many possibilities that the above equation has positive solutions. We assume that the following inequalities hold:

$$\frac{s(r-\sigma_1)^2}{L\sigma_2} < \frac{(s-\sigma_2)r}{K},$$
$$(s-\sigma_2)(r-\sigma_1) < \sigma_1\sigma_2, \tag{2.4}$$

then above equation has a unique positive solution $x(t) = \overline{x}$.

Knowing the value of $\overline{x}$, the value of $\overline{y}$ can then be computed from (2.1). It may be noted here that for $\overline{y}$ to be positive, we must have

$$\frac{r\overline{x}}{K} > r - \sigma_1. \tag{2.5}$$

Then we have following theorem.

**Theorem 2.1.** *If inequalities* (2.4) *and* (2.5) *are satisfied, then* (1.2) *has a nonnegative equilibrium* $P_1(\overline{x}, \overline{y}, 0)$.

Next, we will consider the existence of the positive equilibrium $P_2(x^*, y^*, E^*)$, where $x^*$, $y^*$, and $E^*$ are positive solutions of

$$\begin{gathered} rx(t)\left(1-\frac{x(t)}{K}\right) - \sigma_1 x(t) + \sigma_2 y(t) - qE(t)x(t) = 0, \\ sy(t)\left(1-\frac{y(t)}{L}\right) + \sigma_1 x(t) - \sigma_2 y(t) = 0, \\ \alpha\beta\{q(p-\tau)x(t) - c\} - \gamma = 0. \end{gathered} \tag{2.6}$$

From (2.6), we obtain

$$x^* = \frac{\alpha\beta c + \gamma}{\alpha\beta(p-\tau)q} > 0 \tag{2.7}$$

for

$$p > \tau, \tag{2.8}$$

$$y^* = \frac{(s-\sigma_2) + \sqrt{(s-\sigma_2)^2 + 4(s/L)\sigma_1 x^*}}{2s/L} > 0. \tag{2.9}$$

Again, if

$$\frac{(r-\sigma_1)r\alpha\beta q(p-\tau)}{K(\alpha\beta c + \gamma)} > 1, \tag{2.10}$$

then

$$E^* = \frac{1}{qx^*}\left\{\left(r - \frac{rx^*}{K} - \sigma_1\right)x^* + \sigma_2 y^*\right\} > 0. \tag{2.11}$$

Thus we have following theorem.

**Theorem 2.2.** *If* (2.8) *and* (2.10) *are satisfied, then* (1.2) *has a unique interior equilibrium point* $P_2(x^*, y^*, E^*)$.

## 3. Local Stability Analysis

We first consider the local stability of equilibria. The variational matrix of the system (1.2) is

$$M(x, y, E) = \begin{pmatrix} r - \dfrac{2rx(t)}{K} - \sigma_1 - qE(t) & \sigma_2 & -qx(t) \\ \sigma_1 & s - \dfrac{2sy(t)}{L} - \sigma_2 & 0 \\ \alpha\beta qE(t)(p-\tau) & 0 & \alpha\beta\{q(p-\tau)x(t) - c\} - \gamma \end{pmatrix}. \tag{3.1}$$

At $P_0(0,0,0)$, the characteristic equation of $M_0(0,0,0)$ is

$$(a_1 - \mu)\left(\mu^2 + a_2\mu + a_3\right) = 0, \tag{3.2}$$

where

$$\begin{aligned} a_1 &= -(\alpha\beta c + \gamma), \\ a_2 &= -(r - \sigma_1 + s - \sigma_2), \\ a_3 &= (r - \sigma_1)(s - \sigma_2) - \sigma_1\sigma_2. \end{aligned} \tag{3.3}$$

In this cubic equation, one root is $a_1 < 0$, the sum of other two roots is $-a_2 > 0$. So (3.2) at least has one positive root. Therefore, $P_0$ is unstable. We get the following theorem.

**Theorem 3.1.** *The equilibrium* $P_0$ *of* (1.2) *is unstable.*

At $P_1(\overline{x}, \overline{y}, 0)$, the characteristic equation of $M_1(\overline{x}, \overline{y}, 0)$ is

$$(d_1 - \mu)\left(\mu^2 + d_2\mu + d_3\right) = 0, \tag{3.4}$$

where

$$d_1 = \alpha\beta q(p-\tau)\overline{x} - (\gamma + \alpha\beta c),$$
$$d_2 = -\left(\frac{\sigma_1\overline{x}}{\overline{y}} + \frac{s\overline{y}}{L} + \frac{\sigma_2\overline{y}}{\overline{x}} + \frac{r\overline{x}}{K}\right) < 0, \tag{3.5}$$
$$d_3 = \frac{s\sigma_2\overline{y}^2}{\overline{x}L} + \frac{\overline{x}^2\sigma_1 r}{K\overline{y}} + \frac{sr\overline{x}\,\overline{y}}{KL} > 0.$$

Similarly, one of the eigenvalues of the variational matrix $M_1(\overline{x}, \overline{y}, 0)$ is $\alpha\beta q(p-\tau)\overline{x} - (\alpha\beta c + \gamma)$. This eigenvalue is positive or negative according to whether $\tau < (1/q\overline{x})(pq\overline{x} - (\alpha\beta c + \gamma)/\alpha\beta)$ or $\tau > (1/q\overline{\overline{x}})(pq\overline{x} - (\alpha\beta c + \gamma)/\alpha\beta)$. The sum of other two roots is $-d_2 > 0$; the product of other two roots is $d_3 > 0$. So (3.4) has two positive roots. Therefore, $P_1$ is unstable. Then we have following theorem.

**Theorem 3.2.** *If inequalities* (2.4) *and* (2.5) *are satisfied, then the equilibrium $P_1$ of* (1.2) *is unstable.*

To determine the local stability character of the interior equilibrium $P_2(x^*, y^*, E^*)$, we compute the variational $M_2(x^*, y^*, E^*)$ about $(x^*, y^*, E^*)$

$$M_2(x^*, y^*, E^*) = \begin{pmatrix} r - \dfrac{2rx^*}{K} - \sigma_1 - qE^* & \sigma_2 & -qx^* \\ \sigma_1 & s - \dfrac{2sy^*}{L} - \sigma_2 & 0 \\ \alpha\beta qE^*(p-\tau) & 0 & 0 \end{pmatrix}. \tag{3.6}$$

The characteristic equation of the variational matrix $M_2(x^*, y^*, E^*)$ is given by

$$\mu^3 + m_1\mu^2 + m_2\mu + m_3 = 0, \tag{3.7}$$

where

$$m_1 = \frac{rx^*}{K} + \sigma_2\frac{y^*}{x^*} + \frac{sy^*}{L} + \sigma_1\frac{x^*}{y^*} > 0,$$
$$m_2 = \left(\frac{rx^*}{K} + \sigma_2\frac{y^*}{x^*}\right)\left(\frac{sy^*}{L} + \sigma_1\frac{x^*}{y^*}\right) + q^2x^*\alpha\beta E^*(p-\tau) - \sigma_1\sigma_2, \tag{3.8}$$
$$m_3 = \left(\frac{sy^*}{L} + \sigma_1\frac{x^*}{y^*}\right)(p-\tau)q^2\alpha\beta x^*E^* > 0.$$

By the Routh-Hurwitz criterion, it follows that all eigenvalues of (3.7) have negative real parts if and only if

$$m_1 > 0, \quad m_3 > 0, \quad m_1m_2 > m_3. \tag{3.9}$$

Here, $m_1 > 0$, $m_3 > 0$, and

$$
\begin{aligned}
m_1 m_2 - m_3 &= \left\{ \left( \frac{r x^*}{K} + \sigma_2 \frac{y^*}{x^*} \right) \left( \frac{s y^*}{L} + \sigma_1 \frac{x^*}{y^*} \right) - \sigma_1 \sigma_2 \right\} \left( \frac{s y^*}{L} + \sigma_1 \frac{x^*}{y^*} + \frac{r x^*}{K} + \sigma_2 \frac{y^*}{x^*} \right) \\
&\quad + \left( \frac{r x^*}{K} + \sigma_2 \frac{y^*}{x^*} \right) q^2 x^* \alpha \beta E^* (p - \tau) \\
&= \left( \frac{r s x^* y^*}{K L} + \sigma_1 \frac{r x^{*2}}{K y^*} + \sigma_2 \frac{s y^{*2}}{L x^*} \right) \left( \frac{s y^*}{L} + \sigma_1 \frac{x^*}{y^*} + \frac{r x^*}{K} + \sigma_2 \frac{y^*}{x^*} \right) \\
&\quad + \left( \frac{r x^*}{K} + \sigma_2 \frac{y^*}{x^*} \right) q^2 x^* \alpha \beta E^* (p - \tau) \\
&> 0.
\end{aligned}
\tag{3.10}
$$

Hence $P_2(x^*, y^*, E^*)$ is locally asymptotically stable. We get the following theorem.

**Theorem 3.3.** *If* (2.8) *and* (2.10) *are satisfied, then the unique interior equilibrium point* $P_2(x^*, y^*, E^*)$ *of* (1.2) *is locally asymptotically stable.*

## 4. Global Stability

In this section, we will consider the global stability of the unique interior equilibrium of system (1.2) by constructing a suitable Lyapunov function. We have the following theorem.

**Theorem 4.1.** *If* (2.8) *and* (2.10) *are satisfied, then the unique interior equilibrium point* $P_2(x^*, y^*, E^*)$ *of* (1.2) *is globally asymptotically stable.*

*Proof.* Define a Lyapunov function

$$
\begin{aligned}
V(x(t), y(t), E(t)) &= \left( (x(t) - x^*) - x^* \ln \frac{x(t)}{x^*} \right) + e_1 \left( (y(t) - y^*) - y^* \ln \frac{y(t)}{y^*} \right) \\
&\quad + e_2 \left( (E(t) - E^*) - E^* \ln \frac{E(t)}{E^*} \right),
\end{aligned}
\tag{4.1}
$$

where $e_1$ and $e_2$ are positive constants to be chosen suitably in the subsequent steps. It can be easily verified that $V(x(t), y(t), E(t))$ is zero at the equilibrium point and positive for all other positive values of $x(t)$, $y(t)$, and $E(t)$.

Differentiating $V$ with respect to $t$ along the solutions of (1.2), a little algebraic manipulation yields

$$
\frac{dV}{dt} = \frac{(x(t) - x^*)}{x(t)} \frac{dx(t)}{dt} + e_1 \frac{(y(t) - y^*)}{y(t)} \frac{dy(t)}{dt} + e_2 \frac{(E(t) - E^*)}{E(t)} \frac{dE(t)}{dt}. \tag{4.2}
$$

Choosing $e_1 = \sigma_2 y^*/\sigma_1 x^*$, $e_2 = 1/\alpha\beta(p-\tau)$, a little algebraic manipulation yields

$$\frac{dV}{dt} = -\frac{r}{K}(x(t)-x^*)^2 - \frac{s\sigma_2 y^*}{L\sigma_1 x^*}(y(t)-y^*)^2 - \frac{\sigma_2}{x^* x(t) y(t)}(x^* y(t) - y^* x(t))^2 < 0. \tag{4.3}$$

So $P_2(x^*, y^*, E^*)$ is globally asymptotically stable. The proof is complete. □

## 5. Optimal Harvest Policy

The objective of the regulatory agency is to maximize the total discounted net revenues that the society drives due to the harvesting activity. Symbolically, this objective amounts to maximizing the present value $J$ of a continuous time stream of revenues given by

$$J = \int_0^\infty e^{-\delta t}(pqx(t) - c)E(t)dt, \tag{5.1}$$

where $\delta$ denotes the instantaneous annual rate of discount.

Our objective is to determine a tax policy $\tau = \tau(t)$ to maximize $J$ subject to the state (1.2) and the control constraint

$$\tau_{\min} \le \tau(t) \le \tau_{\max} \tag{5.2}$$

on the control variable $\tau(t)$.

We apply Pontryagin's maximum principle in Burghes and Graham [26] to obtain the optimal equilibrium solution to this control problem. The Hamiltonian of this control problem is

$$\begin{aligned} H = {} & e^{-\delta t}(pqx(t) - c)E(t) + \lambda_1(t)\left\{rx(t)\left(1 - \frac{x(t)}{K}\right) - \sigma_1 x(t) + \sigma_2 y(t) - qE(t)x(t)\right\} \\ & + \lambda_2(t)\left\{sy(t)\left(1 - \frac{y(t)}{L}\right) + \sigma_1 x(t) - \sigma_2 y(t)\right\} \\ & + \lambda_3(t)\{\alpha\beta[q(p-\tau)x(t) - c]E(t) - \gamma E(t)\}, \end{aligned} \tag{5.3}$$

where $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ are adjoint variables. Hamiltonian $H$ must be maximized for $\tau(t) \in [\tau_{\min}, \tau_{\max}]$. Assuming that the control constraints are not binding (i.e., the optimal solution does not occur at $\tau(t) = \tau_{\min}$ or $\tau_{\max}$), we have singular control [27] given by

$$\frac{\partial H}{\partial \tau} = \lambda_3(t)(-\alpha\beta q x(t) E(t)) = 0 \Longrightarrow \lambda_3(t) = 0. \tag{5.4}$$

Now the adjoint equations are

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial x} = -e^{-\delta t} pqE(t) - \lambda_1(t)\left\{ r - \frac{2rx(t)}{K} - \sigma_1 - qE(t) \right\} - \lambda_2(t)\sigma_1, \quad (5.5)$$

$$\frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial y} = -\lambda_1(t)\sigma_2 - \lambda_2(t)\left\{ s - \frac{2sy(t)}{L} - \sigma_2 \right\}, \quad (5.6)$$

$$\frac{d\lambda_3(t)}{dt} = -\frac{\partial H}{\partial E} = -\left\{ e^{-\delta t}(pqx(t) - c) - \lambda_1(t)qx(t) \right\} = 0. \quad (5.7)$$

From (5.7), we obtain

$$\lambda_1(t) = e^{-\delta t}\left( p - \frac{c}{qx(t)} \right). \quad (5.8)$$

To obtain an optimal equilibrium solution, we shall rewrite (5.6) by considering the interior equilibrium as

$$\frac{d\lambda_2(t)}{dt} = A_2 e^{-\delta t} + A_1 \lambda_2(t), \quad (5.9)$$

where

$$A_1 = \frac{2sy^*}{L} + \sigma_2 - s, \qquad A_2 = \left( \frac{c}{qx^*} - p \right)\sigma_2. \quad (5.10)$$

The solution of this linear equation is

$$\lambda_2(t) = K_0 e^{A_1 t} - \frac{A_2}{A_1 + \delta} e^{-\delta t}. \quad (5.11)$$

The shadow price (the term "shadow price" refers to the fact that the asset's value is not its direct sale value but the value imputed from its future productivity [4]) $\lambda_2(t)e^{\delta t}$ is bounded as $t \to \infty$ if $K_0 = 0$. Then

$$\lambda_2(t) = -\frac{A_2}{A_1 + \delta} e^{-\delta t}. \quad (5.12)$$

Similarly,

$$\lambda_1(t) = -\frac{B_2}{B_1 + \delta} e^{-\delta t}, \quad (5.13)$$

where

$$B_1 = \frac{2rx^*}{K} + \sigma_1 + qE^* - r, \qquad B_2 = \frac{\sigma_1 A_2}{A_1 + \delta} - pqE^*. \quad (5.14)$$

Substituting the value of $\lambda_1(t)$ from (5.8) into (5.13), we get

$$p - \frac{c}{qx^*} = -\frac{B_2}{B_1 + \delta}. \tag{5.15}$$

Now using the values of $x^*$, $y^*$, and $E^*$ from Section 3 into (5.15), we get an equation for $\tau(t)$. Let $\tau_\delta$ be a solution (if it exists) of this equation. Using this value of $\tau(t) = \tau_\delta$, we get the optimal equilibrium point $(x_\delta, y_\delta, E_\delta)$.

The existence of an optimal equilibrium solution has been created, which satisfies the necessary conditions of the maximum principle. As stated by Clark [3], an optimal approach path, which is composed of the combination of bang-bang control and nonequilibrium singular controls, is fairly difficult to find. Faced with the same difficulty, Clark [3] researched a simple model of two ecologically independent fish populations. The present model is far more complex than the model mentioned by Clark. So we only consider an optimal equilibrium.

From the above analysis carried out in this section, we observe the following.

(1) From (5.4), (5.12), and (5.13), we note that $\lambda_i(t)e^{\delta t}(i = 1, 2)$ is independent of time in an optimum equilibrium. Hence they satisfy the transversality condition at $\infty$, that is, they remain bounded as $t \to \infty$.

(2) Considering the interior equilibrium, (5.7) can be written as

$$\lambda_1 qx^* = (pqx^* - c)e^{-\delta t} = e^{-\delta t}\frac{\partial \pi}{\partial E}. \tag{5.16}$$

This implies that the total users cost of harvest per unit effort is equal to the discounted values of the future price at the steady-state effort level.

(3) From (5.15), we get

$$pqx^* - c = -\frac{B_2 qx^*}{B_1 + \delta} \longrightarrow 0 \quad \text{as } \delta \longrightarrow \infty. \tag{5.17}$$

Thus, the net economic revenue $\pi(x_\infty, y_\infty, E_\infty, t) = 0$.

This shows that an infinite discount rate results in the complete dissipation of economic revenue. For zero discount rate, it is indicated that the present value of continuous time stream gains its maximum value.

## 6. Numerical Example

In this section, we use Matlab 7.0 to simulate a numerical example to illustrate our results.

Let $r = 1.5$, $s = 2.7$, $K = 200$, $L = 250$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $q = 0.2$, $c = 10$, $\gamma = 10$, $\alpha = 0.8$, $\beta = 0.4$, $p = 19$, and $\delta = 0.03$, in appropriate units.

Then for the above values of the parameter, optimal tax becomes $\tau_\delta = 15.1$ and corresponding stable optimal equilibrium is (52.88, 223.90, 11.48). The time-paths of the free fishing zone $x(t)$, reserve area $y(t)$, and effort $E(t)$ are shown in Figure 1, and the three dimensional phase space portrait is depicted in Figure 2. From the solution curves, we infer

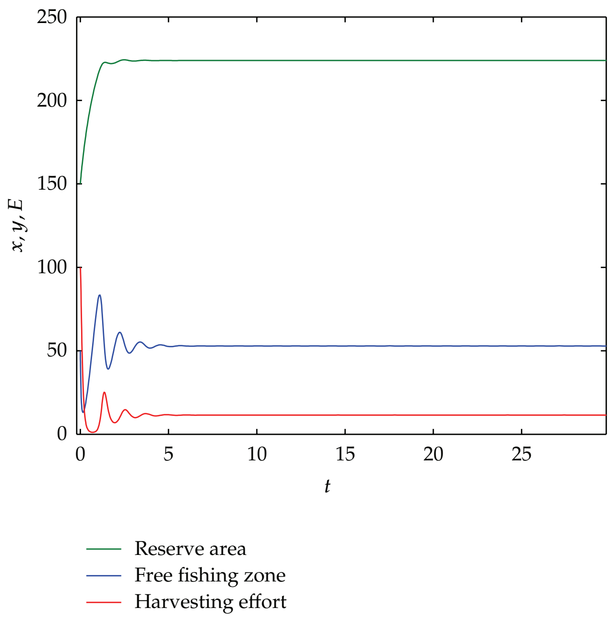


**Figure 1:** Solution curves corresponding to the tax $\tau_\delta = 15.1$, beginning with $x = 50$, $y = 150$, and $E = 100$.

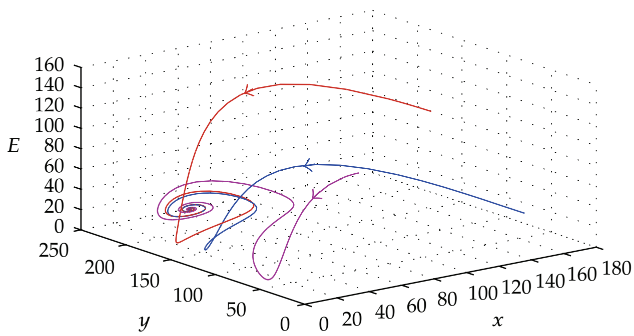


**Figure 2:** Phase space trajectories corresponding to the optimal tax $\tau_\delta = 15.1$, with reference to the different initial levels, we know that the optimal equilibrium (52.88, 223.90, 11.48) is asymptotically stable.

that the system is globally stable about the interior equilibrium point. Biological meaning is that this optimal tax system achieves relatively balance and all the fish population may be maintained at an appropriate equilibrium level.

In Figures 3–5, variation of free fishing zone, reserve area, and harvesting effort against time are plotted for different tax levels. From these plots, we observe that as the rate of tax increases, free fishing zone populations and reserve area populations increase while harvesting effort decreases as expected.

We take $\tau = 0$ and $\tau = 15.1$. Now taking the tax $\tau = 0$ in (2.6), we have $x^* = 10.86$, $y^* = 215.29$, and $E^* = 44.26$, respectively. But for the optimal tax $\tau = 15.1$, we have $x^* = 52.88$, $y^* = 223.90$, and $E^* = 11.48$.

From these results, it is clear that if the fishermen have to pay no tax, then they use a large amount of effort compared to the case when the fishermen have to pay the optimal tax.

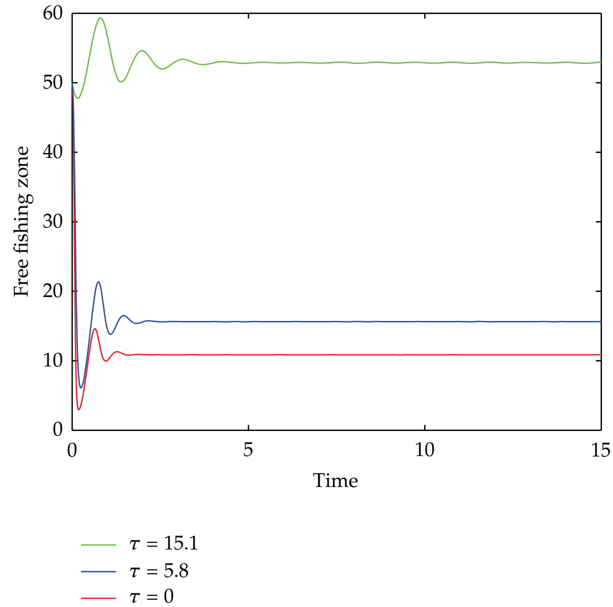


**Figure 3:** Variation of free fishing zone population with time for different tax levels.

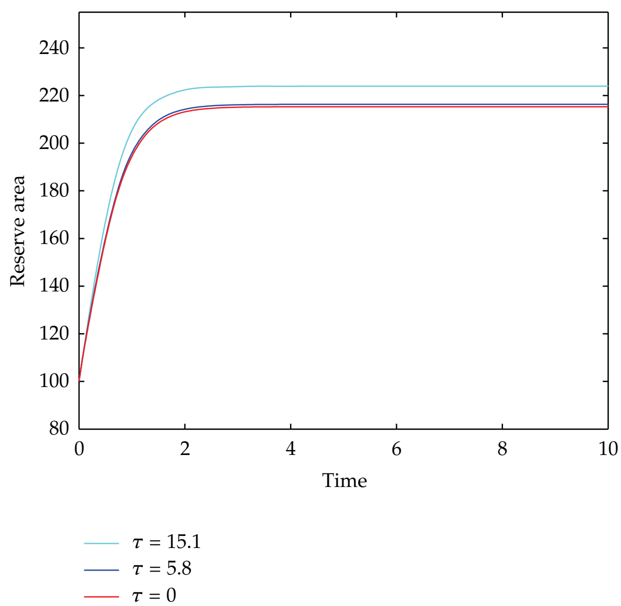


**Figure 4:** Variation of reserve area population with time for different tax levels.

As a result, the steady-state values of the two species for the case of $\tau = 0$ are much less than those for the case of $\tau = 15.1$.

The computer analyzed results for the time course display of the two species $x$ and $y$ and the phase space trajectory for $\tau = 0$ and $\tau = 15.1$ using these parameter values is also shown in Figures 3–5.

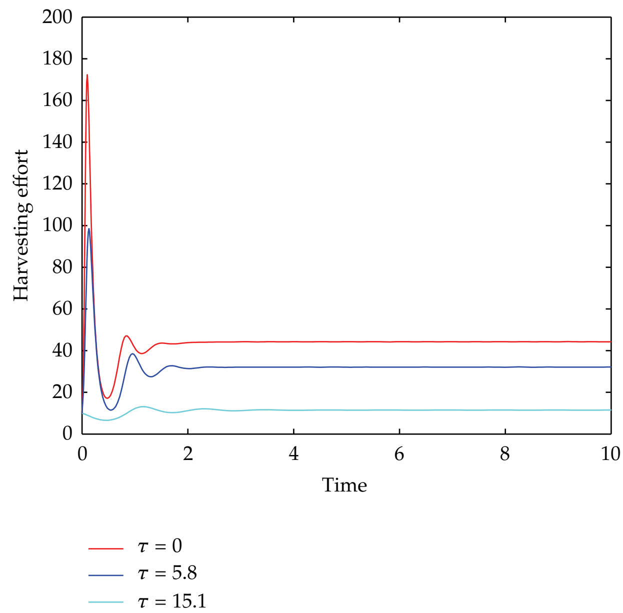


**Figure 5:** Variation of harvesting effort with time for different tax levels.

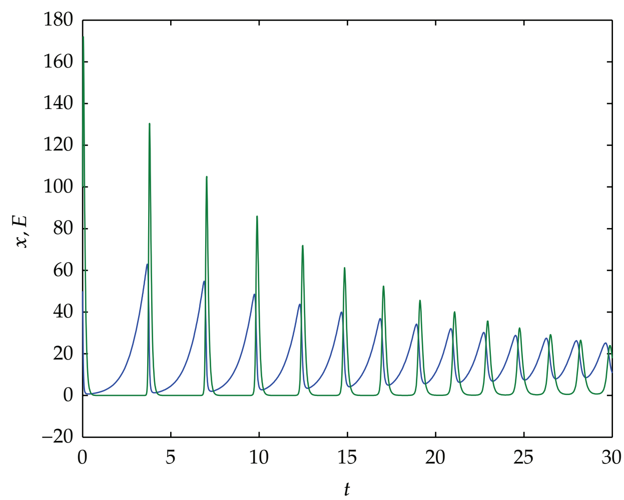


**Figure 6:** The trend of population $x(t)$ and $E(t)$ in the tax $\tau_\delta = 15.1$ rates when the reserved zone not set up.

By comparing Figures 1 and 6, we can see that, in the model without reserve area, there exists oscillation of fish population in free fishing zone for a long time when they tend to positive equilibrium. However, in our model, fish population tend to positive equilibrium quickly in reserved zone, which is beneficial for population protection.

## 7. Discussion

In this paper, we study an optimal harvesting problem for fishery resource with prey dispersal in a two-patch environment: one is a free fishing zone and the other is a reserve zone, focusing attention on the use of taxation as an optimal governing instrument to control exploitation of the fishery. In Sections 3 and 4, we have discussed the local and global stability of the system. It has been observed that, in the case of no taxation, even under continuous harvesting in the free fishing zone, the fish population may be maintained at an appropriate equilibrium level. On the other hand, in the case of taxation, population may be also sustained at an appropriate equilibrium level. But from Figures 3 and 4, we know that as the rate of tax increases free fishing zone populations and reserved area populations increase while harvesting effort decreases. This situation is in accordance with reality. In the model without reserve area, there exists oscillation of fish population in free fishing zone for a long time when they tend to positive equilibrium. However, in our model, fish population tend to positive equilibrium quickly in reserved zone, which is beneficial for population protection.

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