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Research Article

The Structure of Disjoint Groups of Continuous Functions

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Let I be an open interval. We describe the general structure of groups of continuous self functions on I which are disjoint, that is, the graphs of any two distinct elements of them do not intersect. Initially the class of all disjoint groups of continuous functions is divided in three subclasses: cyclic groups, groups the limit points of their orbits are Cantor-like sets, and finally those the limit points of their orbits are the whole interval I. We will show that (1) each group of the second type is conjugate, via a specific homeomorphism, to a piecewise linear group of the same type; (2) each group of the third type is a subgroup of a continuous disjoint iteration group. We conclude the Zdun's result on the structure of disjoint iteration groups of continuous functions as special case of our results.

1. Introduction

The problem of characterizing disjoint groups of continuous functions appears in connection with the issues, such as, describing the solution of the simultaneous systems of Abel's functional equations (mainly in [1–3]) and systems of differential equations with several deviations (see [1, 4]). In a recent paper on the simultaneous systems of Abel's equations ([5]) this problem has been paid a new attention. Zdun in [6] has investigated the structure of disjoint iteration groups of homeomorphism on an open interval, that is, disjoint groups of the form $G = \{f^t : t \in \mathbb{R}\}$ such that $f^s \circ f^t = f^{s+t}$ for all reals s and t and each f^t is a homeomorphism of an open interval onto itself. This problem was studies earlier by Domsta ([7, Section 4.2]) but not so general. In the present paper we characterize not only disjoint iteration groups but all the disjoint groups of continuous functions on an open interval in a more general setting.

Throughout this paper I is an open interval in the real line; the topology of it is considered to be the subspace topology it inherits from $\mathbb{R}$. We say that a subset L of I is a Cantor-like set fitted in I if L is a nowhere dense set, $L' = L$, and L has no lower or upper bound in I (here L' is the set of all limit points of L in I). We denote by L^*_-, L^*_+, L^*, and L^{**} to the set of all left sided, right sided, one sided, and two sided limit points of L, respectively. Denote by $B_i(I)$ the group of all increasing bijections from I onto itself, with composition of functions as its binary operation. There exists a natural ordering relation on the set $\mathfrak{F}(I)$ of all self mappings as follows: given two functions f and g in $\mathfrak{F}(I)$ define $f < g$ provided $f \neq g$ and $f(x) \leq g(x)$ for all $x \in I$. Define $f \leq g$ if $f < g$ or $f = g$. With this, $(\mathfrak{F}(I), <)$ is a partial ordering set. A subset of $\mathfrak{F}(I)$ is said to be disjoint if the graphs of any two distinct elements of it do not intersect. If G is a disjoint group of continuous functions in $\mathfrak{F}(I)$, then G is a subgroup of $B_i(I)$ and furthermore, $(G, \circ, <)$ is an Archimedean ordered group (see [8, Proposition 3]). Hence according to Hölder's Theorem there exists an additive subgroup A of $\mathbb{R}$ such that $(G, \circ, <)$ is isomorphic to $(A, +, <)$.

The following two theorems, which have been proved in [5], give an introductory characterization of disjoint subgroups of $B_i(I)$.

Theorem 1.1. *Let G be a disjoint subgroup of $B_i(I)$. Then*

(a) *for all x, y in I one has $G(x)' = G(y)'$ where $G(x) := \{f(x) : f \in G\}$ (we denote this set by $L(G)$);*

(b) *for all $f \in G$, $f(L(G)) = L(G)$;*

(c) $L(G)' = L(G)$.

(d) *the set $L(G)$ is either an empty set or a Cantor-like set fitted in I or $L(G) = I$.*

Theorem 1.2. *Let G be a disjoint subgroup of $B_i(I)$. Then G is a cyclic group if and only if $L(G) = \emptyset$.*

By virtue of these theorems one divides disjoint subgroups of $B_i(I)$ in three classes: The first class consists of the cyclic subgroups; the second one consists of the subgroups G of $B_i(I)$ for which $L(G)$ is a Cantor-like set fitted in I. one names such a group a *spoiled disjoint group*. Finally the third class consists of the subgroups G of $B_i(I)$ for which $L(G) = I$. Such a group is called a *dense disjoint group*. One calls a subgroup G of $B_i(I)$*complete* if $G(x) = I$ for all $x \in I$.

One uses as a tool classes of functions $\phi : I \to \mathbb{R}$ that occur as continuous solutions of simultaneous systems of Abel equations

$$\phi(f(x)) = \phi(x) + \lambda(f), \quad f \in \mathcal{S},\ x \in I, \tag{1.1}$$

where $\mathcal{S}$ is a nonempty subset of $B_i(I)$ the group generated by which is noncyclic and disjoint and $\lambda : \mathcal{S} \to \mathbb{R}$ is a given map. In view of Theorem 6 of [5] either ϕ is a homeomorphism or a Cantor function which lives on $L(G)$ in the following sense.

Definition 1.3. Given a Cantor-like set L fitted in I we say that $\phi : I \to \mathbb{R}$ is a Cantor function which lives on L if

(i) ϕ is monotone;

(ii) $\phi(L) = \mathbb{R}$;

(iii) ϕ is strictly monotone on L^{**}.

Such a function ϕ is constant on the components of $I - L^{**}$, $\phi(L^{**}) \bigcap \phi(I - L^{**}) = \emptyset$ and ϕ is continuous.

Given a function $\phi : I \to \mathbb{R}$, we say that a function $f : I \to I$ is in the *realm* of ϕ if there exists a real number α such that $\phi(f(x)) = \phi(x) + \alpha$ for all $x \in I$. Clearly this α is unique and we call it the *index of* f *with respect to* ϕ and denote it by $\operatorname{ind}_\phi(f)$. Denote by $\operatorname{Realm}(\phi)$ the set of all functions $f : I \to I$ that are in the realm of ϕ. We say that two functions $\phi : I \to \mathbb{R}$ and $\psi : I \to \mathbb{R}$ are *associate* if $\psi = r\phi + s$ for some real constants $r \neq 0$ and s. It is easy to see that the associativity is an equivalence relation on the set of all functions from I into $\mathbb{R}$. Moreover, every two associate functions have the same realms. In particular if ϕ and ψ are associate and ϕ is a Cantor function which lives on L (or a homeomorphism), then ψ is also a Cantor function which lives on L (or a homeomorphism). By these notation the following is an immediate consequence of Theorem 10 in [5].

Theorem 1.4. *Let G be a disjoint subgroup of $B_i(I)$. Then $G \subsetneq \operatorname{Realm}(\phi)$ for some continuous $\phi : I \to \mathbb{R}$. If G is spoiled, then ϕ is a Cantor function which lives on $L(G)$. If G is dense, then ϕ is a homeomorphism.*

Theorem 12 of [5] can be restate as follows.

Theorem 1.5. *Let G be a noncyclic disjoint subgroup of $B_i(I)$. If $\phi_1 : I \to \mathbb{R}$ and $\phi_2 : I \to \mathbb{R}$ are two continuous functions such that $G \subsetneq \operatorname{Realm}(\phi_1)$ and $G \subsetneq \operatorname{Realm}(\phi_2)$, then ϕ_1 and ϕ_2 are associate. In particular $\operatorname{Realm}(\phi_1) = \operatorname{Realm}(\phi_2)$.*

In Section 3 we deal with the properties of the elements of $\operatorname{Realm}(\phi)$ when ϕ is a Cantor function.

2. Complete and Dense Disjoint Groups

Theorems 2.1 and 2.5 below describe the structure of complete disjoint subgroups of $B_i(I)$. While Theorem 2.3 describes dense disjoint groups.

Theorem 2.1. *If $\phi : I \to \mathbb{R}$ is a homeomorphism, then $Realm(\phi)$ is a complete disjoint subgroup of $B_i(I)$. Moreover, if ϕ is increasing, ind_ϕ is an isomorphism of $(\operatorname{Realm}(\phi), \circ, <)$ onto $(\mathbb{R}, +, <)$.*

Conversely, to every complete disjoint subgroup G of $B_i(I)$ there corresponds a homeomorphism $\phi : I \to \mathbb{R}$ such that $G = \operatorname{Realm}(\phi)$.

Proof. For the first part of the theorem (whose proof is straightforward) see [8]. So we prove the converse part. Let G be a complete disjoint subgroup of $B_i(I)$. By Theorem 1.4 one has $G \subsetneq \operatorname{Realm}(\phi)$ for some homeomorphism $\phi : I \to \mathbb{R}$. To show that $\operatorname{Realm}(\phi) \subsetneq G$ let $f \in \operatorname{Realm}(\phi)$. Pick a point a in I. Then $f(a) \in G(a)$ because $G(a) = I$. It follows that $f(a) = g(a)$ for some $g \in G$. But since $G \subsetneq \operatorname{Realm}(\phi)$, one has $g \in \operatorname{Realm}(\phi)$. By the first part of the theorem $\operatorname{Realm}(\phi)$ is disjoint; so that $f = g$. Hence $f \in G$. And we are done. $\square$

Our next goal is to present a description of dense disjoint subgroups of $B_i(I)$. But first a proposition.

Proposition 2.2. *Let G and H be two disjoint subgroups of $B_i(I)$ and $H \subsetneq G$. If H is noncyclic, then $L(H) = L(G)$.*

Proof. The proof we present is extracted from the proof of Theorem 3 of the Zdun's paper [9] with a little modification.

Let $x \in I$. Then $H(x) \subseteq G(x)$. So that $L(H) = H(x)' \subseteq G(x)' = L(G)$. On the other hand by noticing that G is Abelian, for every $f \in G$

$$f(H(x)) = \{f(g(x)) : g \in H\} = \{g(f(x)) : g \in H\} = H(f(x)). \tag{2.1}$$

Since f is a homeomorphism,

$$f(L(H)) = f(H(x)') = (f(H(x)))' = (H(f(x)))' = L(H). \tag{2.2}$$

Fix an a in $L(H)$. The by the preceding relation one has $f(a) \in L(H)$ for all $f \in G$. Thus $G(a) \subseteq L(H)$. So $L(G) = G(a)' \subseteq L(H)$. Therefore, $L(H) = L(G)$ and the proof is complete. □

This proposition yields that if H is a noncyclic subgroup of the disjoint group $G \subseteq B_i(I)$, then either both G and H are spoiled or they are both dense.

Theorem 2.3. *Let G be a noncyclic disjoint subgroup of $B_i(I)$. Then G is a dense group if and only if G is embeddable in a complete disjoint subgroup of $B_i(I)$. Moreover, this complete group is unique.*

Proof. If G is dense, then by Theorem 1.4, $G \subseteq \mathrm{Realm}(\phi)$ for some homeomorphism $\phi : I \to \mathbb{R}$. By Theorem 2.1, $\mathrm{Realm}(\phi)$ is complete. Conversely, suppose that $G \subseteq P$ for some complete disjoint subgroup P of $B_i(I)$. By Proposition 2.2 one has $L(G) = L(P) = I$. Hence, G is dense.

The uniqueness part of the theorem follows from Theorems 1.5 and 2.1. □

The following proposition will be used now and later.

Proposition 2.4. *Let G be a spoiled disjoint subgroup of $B_i(I)$, $L := L(G)$ and $f \in G$. Then $f(L_-^*) = L_-^*$, $f(L_+^*) = L_+^*$ and $f(L^{**}) = L^{**}$. Moreover, G is countable.*

Proof. The first part of the proposition is in fact Proposition 5 of [5]. To prove the countability of G pick a point a in L_-^*. Then $G(a) \subseteq L_-^*$. This implies that $G(a)$ is countable since L_-^* is so. On the other hand, G and $G(a)$ have the same cardinality because G is disjoint. Therefore, G is countable. □

Theorem 2.5. *Let G be a disjoint subgroup of $B_i(I)$. Then G is complete if and only if $(G, \circ, <)$ is isomorphic to $(\mathbb{R}, +, <)$.*

Proof. If G is complete, then by Theorem 2.1 one has $G = \mathrm{Realm}(\phi)$ for some homeomorphism $\phi : I \to \mathbb{R}$. Since associate functions have the same realms, one can assume that ϕ is increasing. Therefore ind_ϕ is an isomorphism of $(G, \circ, <)$ onto $(\mathbb{R}, +, <)$.

Conversely, suppose that $(G, \circ, <)$ is isomorphic to $(\mathbb{R}, +, <)$. Then G is uncountable. Since every spoiled group is countable, G must be dense. Then by Theorem 2.3 there exists a complete disjoint subgroup P of $B_i(I)$ containing G. Since by the first part of the theorem P is isomorphic to the additive group $\mathbb{R}$, G is isomorphic to an additive subgroup of $\mathbb{R}$ via the same isomorphism. But $\mathbb{R}$ cannot be isomorphic to a strict subgroup of itself. Therefore, $G = P$ and G is complete. □

3. The Realm of Cantor Functions

Through this section L is a Cantor-like set fitted in I and $\phi : I \to \mathbb{R}$ is an increasing Cantor function which lives on L.

Proposition 3.1. $(\mathrm{Realm}(\phi), \circ)$ *is a monoid and* ind_ϕ *is a monoid homomorphism from* $(\mathrm{Realm}(\phi), \circ)$ *onto* $(\mathbb{R}, +)$.

If f is an invertible function in $\mathrm{Realm}(\phi)$*, then* $f^{-1} \in \mathrm{Realm}(\phi)$ *and* $\mathrm{ind}_\phi(f^{-1}) = -\mathrm{ind}_\phi(f)$.

If f and g are in $\mathrm{Realm}(\phi)$ *and $f(a) \le g(a)$ for some $a \in I$, then* $\mathrm{ind}_\phi(f) \le \mathrm{ind}_\phi(g)$.

Proof. It is clear that i, the identity functions on I, is in $\mathrm{Realm}(\phi)$ and $\mathrm{ind}_\phi(i) = 0$. Let f and g be in $\mathrm{Realm}(\phi)$. Then for all $x \in I$

$$\phi(f \circ g(x)) = \phi(f(g(x))) = \phi(g(x)) + \mathrm{ind}_\phi(f) = \phi(x) + \mathrm{ind}_\phi(g) + \mathrm{ind}_\phi(f). \tag{3.1}$$

This means that $f \circ g \in \mathrm{Realm}(\phi)$ and $\mathrm{ind}_\phi(f \circ g) = \mathrm{ind}_\phi(f) + \mathrm{ind}_\phi(g)$. That is, $\mathrm{ind}_\phi : \mathrm{Realm}(\phi) \to \mathbb{R}$ is a monoid homomorphism.

To show that $\mathrm{ind}_\phi : \mathrm{Realm}(\phi) \to \mathbb{R}$ is surjective let $\alpha \in \mathbb{R}$. Define $f : I \to I$ by

$$f(x) := \min \phi^{-1}(\{\phi(x) + \alpha\}). \tag{3.2}$$

Note that for each $t \in \mathbb{R}$, the set $\phi^{-1}(\{t\})$ is either a component of $I - L^{**}$ (which is a closed interval) or a singleton $\{x\}$ where $x \in L^{**}$. So the definition of f is meaningful. Since $f(x) \in \phi^{-1}(\{\phi(x) + \alpha\})$ for every $x \in \mathbb{R}$, one has $\phi(f(x)) = \phi(x) + \alpha$. Therefore, $f \in \mathrm{Realm}(\phi)$ and $\mathrm{ind}_\phi(f) = \alpha$. This proves (a).

For every $x \in I$ one has

$$\phi(x) = \phi\Big(f\Big(f^{-1}(x)\Big)\Big) = \phi\Big(f^{-1}(x)\Big) + \mathrm{ind}_\phi(f), \tag{3.3}$$

thus

$$\phi\Big(f^{-1}(x)\Big) = \phi(x) - \mathrm{ind}_\phi(f). \tag{3.4}$$

This means that $f^{-1} \in \mathrm{Realm}(\phi)$ and $\mathrm{ind}_\phi(f^{-1}) = -\mathrm{ind}_\phi(f)$.

Let f and g be in $\mathrm{Realm}(\phi)$ and $f(a) \le g(a)$ for some $a \in I$. Then

$$\phi(a) + \mathrm{ind}_\phi(f) = \phi(f(a)) \le \phi(g(a)) = \phi(a) + \mathrm{ind}_\phi(g), \tag{3.5}$$

since ϕ is increasing. Therefore, $\mathrm{ind}_\phi(f) \le \mathrm{ind}_\phi(g)$. □

For each $x \in I$ put $[x] := \{x\}$ provided $x \in L^{**}$ and let $[x]$ be the component of $I - L^{**}$ containing x provided $x \in I - L^{**}$. Set $\widehat{I} := \{[x] : x \in I\}$, $\mathcal{C} := \{[x] : x \in I - L^{**}\}$. The following relation makes $\widehat{I}$ into a linearly ordered set: given two elements J and K of $\widehat{I}$ define $J < K$ whenever $x < y$ for all $x \in J$, $y \in K$. The topology of $\widehat{I}$ is understood to be the order topology. An elementary analysis shows that $(\mathcal{C}, <)$ is a countable dense linearly ordered structure with no minimum or maximum; in other words $(\mathcal{C},<)$ is of order type η (see [10, Chapter 8, Exercises 17 and 18]). In particular $\mathcal{C}$ is dense in $\widehat{I}$. Since $\mathbb{Q}$ is also of order type η, there exists an order preserving bijection θ of $(\mathcal{C},<)$ onto $(\mathbb{Q},<)$. The set $\mathcal{C}$ is dense in $\widehat{I}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$; therefore θ extends uniquely to an order preserving bijection $\overline{\theta}$ of $\widehat{I}$ onto $\mathbb{R}$. This shows that $\widehat{I}$ is a linear continuum.

By this notation one can restate the properties of a Cantor function as follows: If $\psi : I \to \mathbb{R}$ is an increasing Cantor function which lives on L, then

(i) ψ is constant on every $J \in \widehat{I}$;

(ii) if x and y are in I, then $[x] < [y]$ if and only if $\psi(x) < \psi(y)$.

We define the canonical map $\pi : I \to \widehat{I}$ by $\pi(x) = [x]$. Since π is an increasing surjection and both I and $\widehat{I}$ are linear continuums, π is continuous.

The following illustrates general properties of the elements of $\mathrm{Realm}(\phi)$ in the language of the preceding notation.

Proposition 3.2. *Let $f \in \mathrm{Realm}(\phi)$. Then*

(a) *if x and y are in I and $[x] < [y]$, then $[f(x)] < [f(y)]$;*

(b) *for every $x \in I$ one has $f([x]) \subseteq [f(x)]$;*

(c) *for each $K \in \widehat{I}$ there exists a $J \in \widehat{I}$ such that $f(J) \subseteq K$.*

Proof. Put $\alpha := \mathrm{ind}_\phi(f)$.

(a) From $[x] < [y]$ we conclude that $\phi(x) < \phi(y)$. Then

$$\phi(f(x)) = \phi(x) + \alpha < \phi(y) + \alpha = \phi(f(y)). \tag{3.6}$$

Therefore, $[f(x)] < [f(y)]$.

(b) Let $y \in [x]$. Then $\phi(y) = \phi(x)$. Thus $\phi(f(y)) = \phi(y) + \alpha = \phi(x) + \alpha = \phi(f(x))$. This implies that $f(y) \in [f(x)]$.

(c) Part (c) is equivalent to saying that for every $y \in I$ there exists an $x \in I$ such that $f([x]) \subseteq [y]$. To prove this equivalent statement suppose that $y \in I$. Put $t = \phi(y) - \alpha$. Since $\phi(I) = \mathbb{R}$ there exists an $x \in I$ such that $\phi(x) = t = \phi(y) - \alpha$. thus $\phi(f(x)) = \phi(x) + \alpha = \phi(y)$. So $[f(x)] = [y]$. Now by part (b)

$$f([x]) \subseteq \left[f(x)\right] = \left[y\right]. \tag{3.7}$$

□

A particular subset of $\mathrm{Realm}(\phi)$ is of special importance for us since it is useful for constructing spoiled disjoint groups. To indicate it we need the following.

Proposition 3.3. *Let $f \in Realm(\phi)$. The following are equivalent.*

(a) $\phi(L^*) + \mathrm{ind}_\phi(f) = \phi(L^*)$.

(b) $\phi(L^{**}) + \mathrm{ind}_\phi(f) = \phi(L^{**})$.

(c) $f(L^{**}) = L^{**}$.

(d) f *maps each component of* $I - L^{**}$ *into another one; moreover for every* $K \in \mathcal{C}$ *there exists* $J \in \mathcal{C}$ *such that* $f(J) \subseteq K$.

This proposition suggests some notation:

$$\rho(\phi) := \{f \in \mathrm{Realm}(\phi) : f(L^{**}) = L^{**}\},$$
$$\mathrm{Ind}(\phi) := \{\mathrm{ind}_\phi(f) : f \in \rho(\phi)\} = \mathrm{ind}_\phi(\rho(\phi)). \tag{3.8}$$

The equivalency of parts (a) and (c) of Proposition 3.3 yields the following.

Corollary 3.4. *With* ϕ *and* L *as above one has*

$$\mathrm{Ind}(\phi) = \{\alpha \in \mathbb{R} : \alpha + \phi(L^*) = \phi(L^*)\}. \tag{3.9}$$

Proof. Let $\alpha \in \mathrm{Ind}(\phi)$. Then $\alpha = \mathrm{ind}_\phi(f)$ for some $f \in \rho(\phi)$. By Proposition 3.3 we have

$$\alpha + \phi(L^*) = \mathrm{ind}_\phi(f) + \phi(L^*) = \phi(L^*). \tag{3.10}$$

Conversely, suppose that α is a real number such that $\alpha + \phi(L^*) = \phi(L^*)$. By Proposition 3.1(a) there exists an $f \in \mathrm{Realm}(\phi)$ such that $\alpha = \mathrm{ind}_\phi(f)$. Proposition 3.3 and the definition of $\rho(\phi)$ imply that $f \in \rho(\phi)$. Therefore $\alpha \in \mathrm{Ind}(\phi)$.

This completes the proof. □

In general if G is an Abelian group and E a nonempty subset of G, we denote

$$\mathrm{Las}_G(E) := \{x \in G : xE = E\}. \tag{3.11}$$

By this notation Corollary 3.4 is restated as $\mathrm{Ind}(\phi) = \mathrm{Las}_{\mathbb{R}}(\phi(L^*))$. It turns out that $\mathrm{Las}_G(E)$ is a subgroup of G; we name it the *subgroup laterally generated by* E. In fact we have the following.

Proposition 3.5. *Let* G *be an Abelian group and* E *be a nonempty subset of* G. *Then*

(a) $\mathrm{Las}_G(E)$ *is a subgroup of* G.

(b) E *is a subgroup of* G *if and only if* $\mathrm{Las}_G(E) = E$.

(c) $\mathrm{Las}_G(E) = \mathrm{Las}_G(E^c)$ *(where* $E^c = G - E$*).*

Proof. Clearly $e \in \mathrm{Las}_G(E)$. If x and y are in $\mathrm{Las}_G(E)$, then

$$(xy)E = x(yE) = xE = E. \tag{3.12}$$

Thus, $xy \in \mathrm{Las}_G(E)$. Moreover, from $xE = E$ one concludes $E = x^{-1}E$. Hence $x^{-1} \in \mathrm{Las}_G(E)$. This proves (a).

If $\mathrm{Las}_G(E) = E$, then by (a), E is a subgroup of G. Conversely, suppose that E is a subgroup of G. To show that $\mathrm{Las}_G(E) \subseteq E$ let $x \in \mathrm{Las}_G(E)$. Since $e \in E$ one has $xe \in xE$. So that $x \in E$.To show that $E \subseteq \mathrm{Las}_G(E)$ let $x \in E$. Since E is a group, $xE = E$. Hence $x \in \mathrm{Las}_G(E)$. This proves (b).

First we show that $\mathrm{Las}_G(E) \subseteq \mathrm{Las}_G(E^c)$. Let $a \in \mathrm{Las}_G(E)$. Then $aE = E$. We want to show that $aE^c \subseteq E^c$. To do this let $x \in E^c$. If ax was in E, $a^{-1}(ax)$ would be in $a^{-1}E$. Since $a^{-1} \in \mathrm{Las}_G(E)$, $a^{-1}E = E$. Hence x would be in E which is a contradiction. So we must have $ax \in E^c$. This shows that $aE^c \subseteq E^c$.

To show that $E^c \subseteq aE^c$ let $a \in \mathrm{Las}_G(E)$. Since $a^{-1} \in \mathrm{Las}_G(E)$, one has $a^{-1}E^c \subseteq E^c$ by applying the preceding discussion on a^{-1} in place of a. Multiplying both sides of this relation by a it follows that $E^c \subseteq aE^c$. We have therefore shown that $aE^c = E^c$. Or $a \in \mathrm{Las}_G(E^c)$. This shows that $\mathrm{Las}_G(E) \subseteq \mathrm{Las}_G(E^c)$.

Next we show that $\mathrm{Las}_G(E^c) \subseteq \mathrm{Las}_G(E)$. Plugging E^c in the relation $\mathrm{Las}_G(E) \subseteq \mathrm{Las}_G(E^c)$ in place of E one gets

$$\mathrm{Las}_G(E^c) \subseteq \mathrm{Las}_G\big((E^c)^c\big) = \mathrm{Las}_G(E). \tag{3.13}$$

□

Proof of Proposition 3.3. We proceed as follows: (a)⇒(d)⇒(c)⇒(b)⇒(a).

Put $\alpha := \mathrm{ind}_\phi(f)$.

(a)⇒(d): suppose that J is a component of $I - L^{**}$. Pick x in J. In this case $J = [x]$. By Proposition 3.2, we have $f([x]) \subseteq [f(x)]$. We must show that $[f(x)]$ is a component of $I - L^{**}$. To do this it suffices to show $f(x) \in I - L^{**}$. By (a) one has

$$\phi\big(f(x)\big) = \phi(x) + \alpha \in \phi(L^*) + \alpha = \phi(L^*), \tag{3.14}$$

so $f(x) \in I - L^{**}$.

Now let $K \in \mathcal{C}$. By Proposition 3.2 there exists $J \in \widehat{I}$ such that $f(J) \subseteq K$. We seek to show that $J \in \mathcal{C}$. Pick x in J. Then $J = [x]$ and $K = [f(x)]$. So $f(x) \in I - L^{**}$. Hence, $\phi(x) + \alpha = \phi(f(x)) \in \phi(L^*)$. Or $\phi(x) \in \phi(L^*) - \alpha$. Since $-\alpha \in \mathrm{Las}_{\mathbb{R}}(\phi(L^*))$, we have $\phi(L^*) - \alpha = \phi(L^*)$ and $\phi(x) \in \phi(L^*)$. Therefore $x \in I - L^{**}$. This shows that $J \in \mathcal{C}$.

(d)⇒(c): to show that $f(L^{**}) \subseteq L^{**}$, let $x \in L^{**}$. Assume to get a contradiction that $f(x) \in I - L^{**}$. In this case $[f(x)] \in \mathcal{C}$. By (d) one has $f(J) \subseteq [f(x)]$ for some $J \in \mathcal{C}$. On the other hand, $f([x]) \subseteq [f(x)]$. Part (a) of Proposition 3.2 implies $J = [x] = \{x\}$. But this is impossible because $J \subseteq I - L^{**}$ and $x \in L^{**}$.

Now we show that $L^{**} \subseteq f(L^{**})$. Let $y \in L^{**}$. By Proposition 3.2 there exists $J \in \widehat{I}$ such that $f(J) \subseteq [y]$. So that $f(J) \subseteq L^{**}$. Hence by (d), J cannot be a component of $I - L^{**}$. That is, $J = [x] = \{x\}$ for some $x \in L^{**}$. Clearly $y = f(x)$. Thus $y \in f(L^{**})$.

(c)⇒(b): To show that $\phi(L^{**}) + \alpha \subseteq \phi(L^{**})$ let $x \in L^{**}$. Then $f(x) \in L^{**}$. So

$$\phi(x) + \alpha = \phi\big(f(x)\big) \in \phi(L^{**}). \tag{3.15}$$

We now show that $\phi(L^{**}) \subseteq \phi(L^{**}) + \alpha$. Let $y \in \phi(L^{**})$. Then $y = \phi(x)$ for some $x \in L^{**}$. Since $f(L^{**}) = L^{**},\ x = f(a)$ for some $a \in L^{**}$. Thus

$$y = \phi(x) = \phi\big(f(a)\big) = \phi(a) + \alpha \in \phi(L^{**}) + \alpha. \tag{3.16}$$

This proves (b).

(b)⇒(a): noticing that $\phi(L^*) = \mathbb{R} - \phi(L^{**})$, this follows from part (c) of Proposition 3.5.

□

Lemma 3.6. *Let L and M be two Cantor-like sets fitted in I. If there exists a Cantor function $\psi : I \to \mathbb{R}$ which lives on both L and M, then $L = M$.*

Proof. Let $x \in I - L$. Then x lies in the interior of a component J of $I - L^{**}$. Since ψ lives on L, it is constant on J. On the other hand, ψ lives also on M. So J is included in a component of $I - M^{**}$. Hence $x \in I - M$. This shows that $M \subseteq L$. By the symmetry we conclude that $L = M$. □

Proposition 3.7. *(a) If f is a continuous function in $\rho(\phi)$, then the image under f of every component of $I - L^{**}$ is another one. Moreover, f is surjective.*

(b) If G is a noncyclic disjoint subgroup of $B_i(I)$ such that $G \subseteq Realm(\phi)$, then $G \subseteq \rho(\phi)$ and $L(G) = L$. Moreover, ind_ϕ establishes an isomorphism of $(G, \circ, <)$ onto $(\mathrm{ind}_\phi(G), +, <)$.

Proof. (a) First we show that f is surjective. By Proposition 3.3 we have $L^{**} = f(L^{**}) \subseteq f(I)$. Since $\sup L^{**} = \sup I$ and $\inf L^{**} = \inf I$, we have $\sup f(I) = \sup I$ and $\inf f(I) = \inf I$. On the other hand, since f is continuous, $f(I)$ is connected. Therefore, $f(I) = I$ and f is surjective.

Now let J a component of $I - L^{**}$. According to Proposition 3.3, $f(J) \subseteq K$ for some component K of $I - L^{**}$. Since f is surjective and since by Proposition 3.2(a) the function f does not map any point of $I - J$ into K, one concludes $f(J) = K$.

(b) By Theorems 1.4 and 1.5 the function ϕ lives on $L(G)$. On the other hand by the hypothesis ϕ lives on L. So by Lemma 3.6, $L(G) = L$. Proposition 2.4 now gives $G \subseteq \rho(\phi)$.

Finally, we turn to the map ind_ϕ. By Proposition 3.1(a), ind_ϕ is a homomorphism of $(G, \circ)$ onto $(\mathrm{ind}_\phi(G), +)$. Let f and g be in G and $f < g$. Pick a point a in L^{**}. Noticing that $f(a)$ and $g(a)$ are in L^{**} we have

$$\phi(a) + \mathrm{ind}_\phi(f) = \phi(f(a)) < \phi(g(a)) = \phi(a) + \mathrm{ind}_\phi(g). \tag{3.17}$$

Therefore $\mathrm{ind}_\phi(f) < \mathrm{ind}_\phi(g)$. This completes the proof. □

4. Spoiled Disjoint Groups

In this section we use Cantor functions to determine the structure of spoiled disjoint (iteration) subgroups of $B_i(I)$.

Proposition 4.1. *Let G and H be two spoiled disjoint subgroups of $B_i(I)$. If there exists a Cantor function $\phi : I \to \mathbb{R}$ such that G and H are both contained in the realm of ϕ and $\mathrm{ind}_\phi(G) = \mathrm{ind}_\phi(H)$, then $L(G) = L(H)$ and $H = \gamma \circ G \circ \gamma^{-1}$ for some $\gamma \in B_i(I)$ which is identity on $L(G)$.*

Proof. Since $G \subseteq \mathrm{Realm}(\phi)$ and $H \subseteq \mathrm{Realm}(\phi)$, we conclude by Proposition 3.7(b) that ϕ lives on both $L(G)$ and $L(H)$. By Lemma 3.6 we have $L(G) = L(H)$. Put $L := L(G)$ and suppose that $\widehat{I}$ and $\mathcal{C}$ are as in Section 3. On $\mathcal{C}$ define $J \sim K$ provided $f(J) = K$ for some $f \in G$ (note that in view of Proposition 3.7 relation $\sim$ is well defined). Clearly $\sim$ is an equivalence relation. Form a set $\mathcal{A}$ by choosing one element in each equivalence class modulo $\sim$.

Let λ_1 and λ_2 be the restrictions of the map ind_ϕ to G and H, respectively. Proposition 3.7(b) now implies that the maps λ_1 and λ_2 are monotone isomorphisms. For each $f \in G$ there exists a unique $f_0 \in H$ such that $\mathrm{ind}_\phi(f) = \mathrm{ind}_\phi(f_0)$; in fact $f_0 = \lambda_2^{-1} \circ \lambda_1(f)$. Note that the map $f \mapsto f_0$ is therefore an isomorphism of G onto H. It is easy to see that for each $x \in I$, $f(x)$ and $f_0(x)$ belong to the same element of $\widehat{I}$. In particular the functions f and f_0 agree on L. Define the map γ as follows:

the definition of γ on L^{**}: For each $x \in L^{**}$ define $\gamma(x) = x$;

the definition of γ on $I - L^{**}$: Let $K \in \mathcal{C}$. Then there exists a unique element $J \in \mathcal{A}$ such that $J \sim K$. We denote by $f_{J,K}$ the unique function in G that maps J onto K. Now for every $x \in J$ define

$$\gamma\left(f_{J,K}(x)\right) = \left(f_{J,K}\right)_0(x). \tag{4.1}$$

This implies that $\gamma(y) = (f_{J,K})_0 \circ (f_{J,K})^{-1}(y)$ for all $y \in K$. It follows that γ maps K in a strictly increasing manner onto itself.

We claim that γ satisfies the conditions of the theorem. First we show that γ is strictly increasing. Let x and y be in I and $x < y$. If $[x] = [y]$, then since γ is strictly increasing on $[x]$, one has $\gamma(x) < \gamma(y)$. If $[x] < [y]$, then noticing that $\gamma(x) \in [x]$ and $\gamma(y) \in [y]$, we get $\gamma(x) < \gamma(y)$.

Since γ maps each element of $\widehat{I}$ onto itself, it follows that γ is surjective.

We now show that $\gamma \circ G \circ \gamma^{-1} = H$. It suffices to show that $\gamma \circ f \circ \gamma^{-1}(y) = f_0(y)$ for all $f \in G$, $y \in I$. Or equivalently $\gamma \circ f(y) = f_0 \circ \gamma(y)$ for all $f \in G$, $y \in I$. To do this let $f \in G$ and $y \in I$. First suppose that $y \in L^{**}$. Then $f(y) \in L^{**}$. So

$$\gamma \circ f(y) = \gamma\left(f(y)\right) = f(y) = f_0(y) = f_0\left(\gamma(y)\right) = f_0 \circ \gamma(y). \tag{4.2}$$

Next suppose that $y \in I - L^{**}$. Put $K = [y]$ and $M = f(K)$. There exists a unique $J \in \mathcal{A}$ such that $J \sim K$. Moreover, $y = f_{J,K}(x)$ for some $x \in J$. We have

$$\begin{aligned} \gamma \circ f(y) = \gamma \circ f \circ f_{J,K}(x) = \gamma \circ f_{J,M}(x) = \left(f_{J,M}\right)_0(x) = \left(f \circ f_{J,K}\right)_0(x) \\ = f_0 \circ \left(f_{J,K}\right)_0(x) = f_0 \circ \gamma\left(f_{J,K}(x)\right) = f_0 \circ \gamma(y). \end{aligned} \tag{4.3}$$

Therefore $\gamma \circ f(y) = f_0 \circ \gamma(y)$ for all $f \in G$, $y \in I$ as asserted. □

We continue with introducing the piecewise linear group generated by a Cantor function. Let $\phi : I \to \mathbb{R}$ be an increasing Cantor function which lives on a Cantor-like set L fitted in I. Let $\alpha \in \operatorname{Ind}(\phi)$. To this α we correspond the piecewise linear function $\delta_{\phi,\alpha}$ as follows. By the definition of $\operatorname{Ind}(\phi)$ there exists an $f \in \rho(\phi)$ such that $\operatorname{ind}_\phi(f) = \alpha$. If $x \in L^{**}$, put $\delta_{\phi,\alpha}(x) = f(x)$. If $x \in I - L^{**}$ let $\delta_{\phi,\alpha}$ be the line which maps the closed interval $[x]$ increasingly onto the closed interval $[f(x)]$. We claim that $\delta_{\phi,\alpha} \in \rho(\phi) \bigcap B_i(I)$ and $\operatorname{ind}_\phi(\delta_{\phi,\alpha}) = \alpha$.

First we must show that the definition of $\delta_{\phi,\alpha}$ does not depend on the choice of f with $\operatorname{ind}_\phi(f) = \alpha$. To see this let g be another element of $\rho(\phi)$ such that $\operatorname{ind}_\phi(g) = \alpha$. In this case for every $x \in I$ one has

$$\phi\left(f(x)\right) = \phi(x) + \alpha = \phi\left(g(x)\right), \tag{4.4}$$

hence $[f(x)] = [g(x)]$. In particular f and g agree on L^{**} since being elements of $\rho(\phi)$, for every $x \in L^{**}$ we have $\{f(x)\} = [f(x)] = [g(x)] = \{g(x)\}$. Thus $\delta_{\phi,\alpha}$ is well defined.

Clearly $[\delta_{\phi,\alpha}(x)] = [f(x)]$ for every $x \in I$. Hence

$$\phi(\delta_{\phi,\alpha}(x)) = \phi(f(x)) = \phi(x) + \alpha. \tag{4.5}$$

So that $\delta_{\phi,\alpha} \in \mathrm{Realm}(\phi)$ and $\mathrm{ind}_\phi(\delta_{\phi,\alpha}) = \alpha$.

That $\delta_{\phi,\alpha}$ is surjective is clear by Proposition 3.2(c).

To show that $\delta_{\phi,\alpha}$ is strictly increasing, let x and y be in I and $x < y$. If $[x] = [y]$, then $\delta_{\phi,\alpha}(x) < \delta_{\phi,\alpha}(y)$ by the definition of $\delta_{\phi,\alpha}$. If $[x] < [y]$, then $[\delta_{\phi,\alpha}(x)] < [\delta_{\phi,\alpha}(y)]$ by Proposition 3.2(a). So that $\delta_{\phi,\alpha}(x) < \delta_{\phi,\alpha}(y)$.

Finally, $\delta_{\phi,\alpha}(L^{**}) = f(L^{**}) = L^{**}$. Therefore $\delta_{\phi,\alpha} \in \rho(\phi) \bigcap B_i(I)$ as asserted.

Now put $\Delta(\phi) := \{\delta_{\phi,\alpha} : \alpha \in \mathrm{Ind}(\phi)\}$. We claim that $\Delta(\phi)$ is a disjoint group and call it the *piecewise linear group generated by* ϕ.

Theorem 4.2. *With ϕ as above, $\Delta(\phi)$ is a disjoint subgroup of $B_i(I)$ which is isomorphic to* $\mathrm{Ind}(\phi)$ *as an ordered group. Moreover, if $\Delta(\phi)$ is noncyclic, it is a spoiled disjoint group such that $L(\Delta(\phi)) = L$ and in particular $\Delta(\phi)$ is a maximal element in the set of all disjoint subgroups of $B_i(I)$.*

Proof. First we show that $\Delta(\phi)$ is a group. Clearly $\delta_{\phi,0} = i \in \Delta(\phi)$. Let α and β be in $\mathrm{Ind}(\phi)$. Proposition 3.1 gives

$$\mathrm{ind}_\phi\left(\delta_{\phi,\alpha} \circ \delta_{\phi,\beta}^{-1}\right) = \alpha - \beta = \mathrm{ind}_\phi(\delta_{\phi,\alpha-\beta}). \tag{4.6}$$

In other words $\delta_{\phi,\alpha} \circ \delta_{\phi,\beta}^{-1}$ and $\delta_{\phi,\alpha-\beta}$ are two piecewise linear functions in $\mathrm{Realm}(\phi)$ whose indices with respect to ϕ are the same. Since such a function is unique,

$$\delta_{\phi,\alpha} \circ \delta_{\phi,\beta}^{-1} = \delta_{\phi,\alpha-\beta} \in \Delta(\phi). \tag{4.7}$$

Therefore $\Delta(\phi)$ is a subgroup of $B_i(I)$.

To show that $\Delta(\phi)$ is disjoint, let α and β be in $\mathrm{Ind}(\phi)$ and $\delta_{\phi,\alpha}(a) = \delta_{\phi,\beta}(a)$ for some $a \in I$. Then

$$\phi(a) + \alpha = \phi(\delta_{\phi,\alpha}(a)) = \phi(\delta_{\phi,\beta}(a)) = \phi(a) + \beta, \tag{4.8}$$

hence $\alpha = \beta$. So that $\delta_{\phi,\alpha} = \delta_{\phi,\beta}$. It follows that $\Delta(\phi)$ is disjoint.

Suppose that $\Delta(\phi)$ is not cyclic. By Proposition 3.7(b) we get $L(\Delta(\phi)) = L$. The same Proposition implies that ind_ϕ is an isomorphism of $(\Delta(\phi), \circ, <)$ onto $(\mathrm{Ind}(\phi), +, <)$.

To show the maximality of $\Delta(\phi)$ when it is noncyclic, suppose to get a contradiction that there exists a disjoint subgroup G of $B_i(I)$ which strictly contains $\Delta(\phi)$. So G is spoiled by Proposition 2.2. Thus $G \subseteq \mathrm{Realm}(\psi)$ for some increasing Cantor function $\psi : I \to \mathbb{R}$. Accordingly $\Delta(\phi)$ is contained in both $\mathrm{Realm}(\phi)$ and $\mathrm{Realm}(\psi)$. Since $\Delta(\phi)$ is noncyclic, $\mathrm{Realm}(\phi) = \mathrm{Realm}(\psi)$ by Theorem 1.5. Thus $\mathrm{Realm}(\phi) = \mathrm{Realm}(\psi)$ and $\mathrm{ind}_\phi = \mathrm{ind}_\psi$. Hence $\mathrm{Ind}(\phi)$ is a strict subset of $\mathrm{ind}_\phi(G)$ because $\mathrm{Ind}(\phi) = \mathrm{ind}_\phi(\Delta(\phi))$. But on the other hand by Proposition 3.7, we have $\mathrm{ind}_\phi(G) \subseteq \mathrm{Ind}(\phi)$. This gives a contradiction and confirms the maximality of $\Delta(\phi)$. □

4.1. The Structure of Spoiled Disjoint Subgroups of $B_i(I)$

Let L be a Cantor-like set fitted in I and $\mathcal{C}$ be the set of all components of $I - L^{**}$. Let E be a countable dense subset of $\mathbb{R}$ and θ an order preserving bijection of $\mathcal{C}$ onto E (such a θ exists for both structures $(\mathcal{C},<)$ and $(E,<)$ are of order type η). Since $\mathcal{C}$ is dense in $\widehat{I} = \{[x] : x \in I\}$, θ extends to an order preserving bijection $\overline{\theta} : \widehat{I} \to \mathbb{R}$. Put $\phi = \overline{\theta} \circ \pi$. Then ϕ is an increasing Cantor function which lives on L and $\phi(L^*) = E$. Define

$$\Delta(L,E,\theta) := \Delta(\phi). \tag{4.9}$$

Note that by Corollary 3.4 we have

$$\mathrm{ind}_{\phi}(\Delta(\phi)) = \mathrm{Ind}(\phi) = \mathrm{Las}_{\mathbb{R}}(\phi(L^*)) = \mathrm{Las}_{\mathbb{R}}(E), \tag{4.10}$$

so $(\Delta(\phi),\circ,<)$ is isomorphic to $(\mathrm{Las}_{\mathbb{R}}(E),+<)$ via ind_{ϕ}.

If E is an additive subgroup of $\mathbb{R}$, then $\mathrm{Las}_{\mathbb{R}}(E) = E$; therefore $(\Delta(\phi),\circ,<)$ is isomorphic to $(E,+,<)$. Combining this fact with Theorem 4.2 the following existence theorem is concluded.

Theorem 4.3. *For every Cantor-like set L fitted in I and every countable dense subgroup A of* $(\mathbb{R},+)$ *there exists a maximal spoiled disjoint subgroup G of* $B_i(I)$ *such that G is (as an ordered group) isomorphic to A and* $L(G) = L$.

Now one is ready to determine the structure of spoiled disjoint subgroups of $B_i(I)$.

Theorem 4.4. *Let L be a Cantor-like set fitted in I and* $\mathcal{C}$ *be the set of all components of* $I - L^{**}$. *The general form of all spoiled disjoint subgroups of* $B_i(I)$ *such that* $L(G) = L$ *is given by* $G = \gamma \circ \Delta \circ \gamma^{-1}$ *where* γ *is a homeomorphism of I onto itself such that* $\gamma(x) = x$ *for every* $x \in L$ *and* Δ *is a noncyclic subgroup of* $\Delta(L,E,\theta)$ *for some countable dense subset E of* $\mathbb{R}$ *for which* $\mathrm{Las}_{\mathbb{R}}(\mathrm{E})$ *is noncyclic and some order preserving bijection* θ *of* $\mathcal{C}$ *onto E.*

Proof. Part 1

Let G be a spoiled disjoint subgroup of $B_i(I)$ such that $L(G) = L$. Then by Theorem 1.4, $G \subsetneq \mathrm{Realm}(\phi)$ for some Cantor function $\phi : I \to \mathbb{R}$ which lives on L. Put $E := \phi(L^*)$ and define $\theta : \mathcal{C} \to E$ by $\theta([x]) = \phi(x)$. This θ is an order preserving bijection (see Section 3). Put $\Delta := \{\delta_{\phi,\alpha} : \alpha \in \mathrm{ind}_{\phi}(G)\}$. Then Δ is a subgroup of $\Delta(L,E,\theta)$ such that $\mathrm{ind}_{\phi}(G) = \mathrm{ind}_{\phi}(\Delta)$. In particular Δ is noncyclic. By Proposition 4.1 there exists a $\gamma \in B_i(I)$ which is identity on L and such that $G = \gamma \circ \Delta \circ \gamma^{-1}$.

Part 2

Suppose that γ, Δ, L, E, and θ are as in the statement of the theorem and assume $G = \gamma\circ\Delta\circ\gamma^{-1}$. Since $\mathcal{C}$ is dense in $\widehat{I}$ and E is dense in $\mathbb{R}$, the map θ extends uniquely to an order preserving bijection $\overline{\theta} : \widehat{I} \to \mathbb{R}$. Put $\phi = \overline{\theta} \circ \pi$. Then ϕ is an increasing Cantor function which lives on L and $\Delta(L,E,\theta) = \Delta(\phi) \subsetneq \mathrm{Realm}(\phi)$. Since both γ and γ^{-1} are in $\mathrm{Realm}(\phi)$ and $\mathrm{Realm}(\phi)$ is a monoid, one has $G \subsetneq \mathrm{Realm}(\phi)$.

To show that G is disjoint suppose that f and g are in G and $f(a) = g(a)$ for some $a \in I$. Then

$$\phi(a) + \operatorname{ind}_\phi(f) = \phi(f(a)) = \phi(g(a)) = \phi(a) + \operatorname{ind}_\phi(g), \tag{4.11}$$

thus $\operatorname{ind}_\phi(f) = \operatorname{ind}_\phi(g)$. The functions $\gamma^{-1} \circ f \circ \gamma$ and $\gamma^{-1} \circ g \circ \gamma$ are in Δ and we have

$$\begin{aligned}\operatorname{ind}_\phi\left(\gamma^{-1} \circ f \circ \gamma\right) &= -\operatorname{ind}_\phi(\gamma) + \operatorname{ind}_\phi(f) + \operatorname{ind}_\phi(\gamma) = \operatorname{ind}_\phi(f) = \operatorname{ind}_\phi(g)\\ &= -\operatorname{ind}_\phi(\gamma) + \operatorname{ind}_\phi(g) + \operatorname{ind}_\phi(\gamma) = \operatorname{ind}_\phi\left(\gamma^{-1} \circ g \circ \gamma\right).\end{aligned} \tag{4.12}$$

By the definition of $\Delta(\phi)$ for every $\alpha \in \operatorname{Ind}(\phi)$ only one element of $\Delta(\phi)$ has index α with respect to ϕ. Hence $\gamma^{-1} \circ f \circ \gamma = \gamma^{-1} \circ g \circ \gamma$. Canceling γ and γ^{-1} we get $f = g$. Therefore, G is disjoint.

Since G is noncyclic and $G \subseteq \operatorname{Realm}(\phi)$, Proposition 3.7(b) implies $L(G) = L$.

This completes the proof. □

4.2. *The Structure of Spoiled Disjoint Iteration Subgroups of* $B_i(I)$

Let L be a Cantor-like set fitted in I, E be a countable dense subset of $\mathbb{R}$, and $c : \mathbb{R} \to \mathbb{R}$ be an additive function such that $E + \operatorname{Im}(c) = E$. Let $\theta : \mathcal{C} \to E$ be an order preserving bijection and $\overline{\theta} : \widehat{I} \to \mathbb{R}$ be its extension. Put $\phi = \overline{\theta} \circ \pi$. Then ϕ is an increasing Cantor function which lives on $L(G)$ and $\phi(L^*) = E$. So that $\operatorname{Im}(c) \subseteq \operatorname{Las}_{\mathbb{R}}(E) = \operatorname{Ind}(\phi)$. This implies that for all $t \in \mathbb{R}$, $c(t) \in \operatorname{Ind}(\phi)$. This allows us to set $\delta^t := \delta_{\phi,c(t)}$ for every $t \in \mathbb{R}$. Put

$$\Delta(L, E, \theta, c) := \left\{\delta^t : t \in \mathbb{R}\right\}. \tag{4.13}$$

The group $\Delta := \Delta(L, E, \theta, c)$ is a spoiled disjoint iteration subgroup of $B_i(I)$ such that $L(\Delta) = L$: because for all reals s and t

$$\delta^s \circ \delta^t = \delta_{\phi,c(s)} \circ \delta_{\phi,c(t)} = \delta_{\phi,c(s)+c(t)} = \delta^{s+t}. \tag{4.14}$$

If E is in addition a divisible additive subgroup of $\mathbb{R}$, then there exists an additive function $c : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{Im}(c) = E$. We have $E = \operatorname{Im}(c) = \operatorname{Las}_{\mathbb{R}}(E) = \operatorname{Ind}(\phi)$. So $\Delta(L, E, \theta, c)$ is isomorphic, as an ordered group, to E. This yields the following existence theorem.

Theorem 4.5. *For every Cantor-like set L fitted in I and every countable divisible subgroup A of $(\mathbb{R}, +)$ there exists a spoiled disjoint iteration subgroup $G = \{f^t : t \in \mathbb{R}\}$ of $B_i(I)$ such that G is (as an ordered group) isomorphic to A and $L(G) = L$.*

The next theorem determines the structure of spoiled disjoint iteration subgroups of $B_i(I)$. It was already stated and proved by Zdun in [6, Theorem 4]. Here is given a new proof as well as a little modification in the statement.

Theorem 4.6. *Let L be a Cantor-like set fitted in I and $\mathcal{C}$ be the set of all components of $I - L^{**}$. The general form of all spoiled disjoint iteration subgroups $G = \{f^t : t \in \mathbb{R}\}$ of $B_i(I)$ such that $L(G) = L$ is given by the relation $f^t = \gamma \circ \delta^t \circ \gamma^{-1} (t \in \mathbb{R})$ where γ is a homeomorphism of I onto itself such that $\gamma(x) = x$ for all $x \in L$ and $\{\delta^t : t \in \mathbb{R}\} = \Delta(L, E, \theta, c)$ for some countable dense subset E of $\mathbb{R}$, an order preserving bijection $\theta : \mathcal{C} \to E$ and an additive function $c : \mathbb{R} \to \mathbb{R}$ such that $E + \operatorname{Im}(c) = E$.*

Proof. *Part* 1

Let $G = \{f^t : t \in \mathbb{R}\}$ be a spoiled disjoint iteration subgroup of $B_i(I)$ such that $L(G) = L$. By Theorem 4.4, $G = \gamma \circ \Delta \circ \gamma^{-1}$ where γ is a homeomorphism in $\mathfrak{F}(I)$ such that $\gamma|_L = i|_L$, and Δ is a noncyclic subgroup of $\Delta(L, E, \theta)$ for some countable dense subset E of $\mathbb{R}$ such that $\operatorname{Las}_{\mathbb{R}}(E)$ is noncyclic and some order preserving bijection $\theta : \mathcal{C} \to E$. Then

$$\Delta = \gamma^{-1} \circ G \circ \gamma = \left\{\gamma^{-1} \circ f^t \circ \gamma : t \in \mathbb{R}\right\}. \tag{4.15}$$

For every $t \in \mathbb{R}$ define $\delta^t := \gamma^{-1} \circ f^t \circ \gamma$. It is straightforward to see that $\Delta = \{\delta^t : t \in \mathbb{R}\}$ is an iteration group. Put $\phi := \overline{\theta} \circ \pi$. Then the map ind_ϕ is in isomorphism of $(\Delta, \circ, <)$ onto an additive subgroup of $\operatorname{Las}_{\mathbb{R}}(E)$. Define $c : \mathbb{R} \to \mathbb{R}$ by $c(t) := \operatorname{ind}_\phi(\delta^t)$. Then $\delta^t = \delta_{\phi, c(t)}$. Moreover, c is an additive function such that $\operatorname{Im}(c)$ is an additive subgroup of $\operatorname{Las}_{\mathbb{R}}(E)$. So $E + \operatorname{Im}(c) = E$.

Part 2

Suppose that $G = \gamma \circ \Delta \circ \gamma^{-1}$ where $\Delta = \Delta(L, E, \theta, c)$ and γ, E, θ, and c are as in the statement of the theorem. Then by Theorem 4.4, G is a spoiled disjoint subgroup of G and $L(G) = L$. For every $t \in \mathbb{R}$ put $f^t := \gamma \circ \delta^t \circ \gamma^{-1}$. Then for all reals s and t, $f^s \circ f^t = f^{s+t}$. Furthermore, $G = \{f^t : t \in \mathbb{R}\}$.

This completes the proof. □

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