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Research Article

Fixed Point and Weak Convergence Theorems for (α, β)-Hybrid Mappings in Banach Spaces

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We introduce the class of (α, β)-hybrid mappings relative to a Bregman distance D_f in a Banach space, and then we study the fixed point and weak convergence problem for such mappings.

1. Introduction

Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to H$ is said to be

(1.1) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;

(1.2) nonspreading if $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$, for all $x, y \in C$, cf. [1, 2];

(1.3) hybrid if $3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$, for all $x, y \in C$, cf. [3].

Recently, Kocourek et al. [4] introduced a new class of nonlinear mappings in a Hilbert space containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings and established some fixed point and ergodic theorems for mappings in this new class. For $\alpha, \beta \in \mathbb{R}$, they call a mapping $T : C \to H$

(1.4) (α,β)-hybrid if

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2\le\beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2 \tag{1.1}$$

for all $x,y\in C$.

Obviously, T is nonexpansive if and only if it is $(1,0)$-hybrid; T is nonspreading if and only if it is $(2,1)$-hybrid; T is hybrid if and only if it is $(3/2,1/2)$-hybrid.

Motivated by the above works, we extend the concept of (α,β)-hybrid from Hilbert spaces to Banach spaces. For a nonempty subset C of a Banach space X, a Gâteaux differentiable convex function $f:X\to(-\infty,\infty]$ and $\alpha,\beta\in\mathbb{R}$, a mapping $T:C\to X$ is said to be

(1.5) (α,β)-hybrid relative to D_f if there are $\alpha,\beta\in\mathbb{R}$ such that

$$\alpha D_f(Tx,Ty)+(1-\alpha)D_f(x,Ty)\le\beta D_f(Tx,y)+(1-\beta)D_f(x,y) \tag{1.2}$$

for all $x,y\in C$, where D_f is the Bregman distance associated with f. Sections 3 and 4 are devoted to investigation of the fixed point and weak convergence problem for such type of mappings, respectively. Our fixed point theorem extends that of [4].

2. Preliminaries

In what follows, X will be a real Banach space with topological dual X^*, and $f:X\to(-\infty,\infty]$ will be a convex function. $\mathfrak{D}$ denotes the domain of f, that is,

$$\mathfrak{D}=\{x\in X:f(x)<\infty\}, \tag{2.1}$$

and $\mathfrak{D}^\circ$ denotes the algebraic interior of $\mathfrak{D}$, that is, the subset of $\mathfrak{D}$ consisting of all those points $x\in\mathfrak{D}$ such that, for any $y\in X\setminus\{x\}$, there is z in the open segment (x,y) with $[x,z]\subseteq\mathfrak{D}$. The topological interior of $\mathfrak{D}$, denoted by $\mathrm{Int}(\mathfrak{D})$, is contained in $\mathfrak{D}^\circ$. f is said to be proper provided that $\mathfrak{D}\neq\emptyset$. f is called lower semicontinuous (l.s.c.) at $x\in X$ if $f(x)\le\liminf_{y\to x}f(y)$. f is strictly convex if

$$f(\alpha x+(1-\alpha)y)<\alpha f(x)+(1-\alpha)f(y) \tag{2.2}$$

for all $x,y\in X$ and $\alpha\in(0,1)$.

The function $f:X\to(-\infty,\infty]$ is said to be Gâteaux differentiable at $x\in X$ if there is $f'(x)\in X^*$ such that

$$\lim_{t\to0}\frac{f(x+ty)-f(x)}{t}=\langle y,f'(x)\rangle \tag{2.3}$$

for all $y\in X$.

The Bregman distance D_f associated with a proper convex function f is the function $D_f:\mathfrak{D}\times\mathfrak{D}\to[0,\infty]$ defined by

$$D_f(y,x)=f(y)-f(x)+f^\circ(x,x-y), \tag{2.4}$$

where $f^\circ(x, x-y) = \lim_{t\to 0^+}(f(x+t(x-y)) - f(x))/t$. $D_f(y,x)$ is finite valued if and only if $x \in \mathfrak{D}^\circ$, compare Proposition 1.1.2 (iv) of [5]. When f is Gâteaux differentiable on $\mathfrak{D}$, (2.4) becomes

$$D_f(y,x) = f(y) - f(x) - \langle y-x, f'(x)\rangle, \tag{2.5}$$

and then the modulus of total convexity is the function $\nu_f : \mathfrak{D}^\circ \times [0,\infty) \to [0,\infty]$ defined by

$$\nu_f(x,t) = \inf\{D_f(y,x) : y \in \mathfrak{D}, \|y-x\| = t\}. \tag{2.6}$$

It is known that

$$\nu_f(x,ct) \geq c\nu_f(x,t) \tag{2.7}$$

for all $t \geq 0$ and $c \geq 1$, compare Proposition 1.2.2 (ii) of [5]. By definition it follows that

$$D_f(y,x) \geq \nu_f(x, \|y-x\|). \tag{2.8}$$

The modulus of uniform convexity of f is the function $\delta_f : [0,\infty) \to [0,\infty]$ defined by

$$\delta_f(t) = \inf\left\{f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) : x,y \in \mathfrak{D}, \|x-y\| \geq t\right\}. \tag{2.9}$$

The function f is called uniformly convex if $\delta_f(t) > 0$ for all $t > 0$. If f is uniformly convex then for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)}{2} + \frac{f(y)}{2} - \delta \tag{2.10}$$

for all $x, y \in \mathfrak{D}$ with $\|x-y\| \geq \varepsilon$.

Note that for $y \in \mathfrak{D}$ and $x \in \mathfrak{D}^\circ$, we have

$$\begin{aligned} f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) &= f(y) - f(x) - \frac{f(x + ((y-x)/2)) - f(x)}{1/2} \\ &\leq f(y) - f(x) - f^\circ(x, y-x) \leq D_f(y,x), \end{aligned} \tag{2.11}$$

where the first inequality follows from the fact that the function $t \to (f(x+tz) - f(x))/t$ is nondecreasing on $(0,\infty)$. Therefore

$$\nu_f(x,t) \geq \delta_f(t), \tag{2.12}$$

whenever $x \in \mathfrak{D}^\circ$ and $t \geq 0$. For other properties of the Bregman distance D_f, we refer readers to [5].

The normalized duality mapping J from X to 2^{X^*} is defined by

$$Jx = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\} \tag{2.13}$$

for all $x \in X$.

When $f(x) = \|x\|^2$ in a smooth Banach space, it is known that $f'(x) = 2J(x)$ for $x \in X$, compare Corollaries 1.2.7 and 1.4.5 of [6]. Hence we have

$$\begin{aligned} D_f(y, x) &= \|y\|^2 - \|x\|^2 - \langle y - x, f'(x) \rangle \\ &= \|y\|^2 - \|x\|^2 - 2\langle y - x, Jx \rangle \\ &= \|y\|^2 + \|x\|^2 - 2\langle y, Jx \rangle. \end{aligned} \tag{2.14}$$

Moreover, as the normalized duality mapping J in a Hilbert space H is the identity operator, we have

$$D_f(y, x) = \|y\|^2 + \|x\|^2 - 2\langle y, x \rangle = \|y - x\|^2. \tag{2.15}$$

Thus, in case $f(x) = \|x\|^2$ in a Hilbert space, (1.5) coincides with (1.4). However, in general they are different as the following example shows.

Example 2.1. Let $f(x) = |x|$ for $x \in \mathbb{R}$. f is a continuous convex function with

$$D_f(y, x) = \begin{cases} 0, & \text{if } x > 0,\ y \geq 0 \text{ or } x < 0,\ y \leq 0, \\ -2y, & \text{if } x \geq 0,\ y \leq 0, \\ 2y, & \text{if } x \leq 0,\ y \geq 0. \end{cases} \tag{2.16}$$

Let $C = [0, 1]$ and define $T : C \to C$ by

$$T(x) = \begin{cases} x, & \text{if } x \in (0, 1], \\ \dfrac{1}{2}, & \text{if } x = 0. \end{cases} \tag{2.17}$$

If T were (α, β)-hybrid for some $\alpha, \beta \in \mathbb{R}$, then we would have

$$\alpha\|Tx - T0\|^2 + (1 - \alpha)\|x - T0\|^2 \leq \beta\|Tx - 0\|^2 + (1 - \beta)\|x - 0\|^2, \quad \forall x \in (0, 1]. \tag{2.18}$$

Since

$$
\begin{aligned}
(2.18) &\Longleftrightarrow \alpha\left|x-\frac{1}{2}\right|^2+(1-\alpha)\left|x-\frac{1}{2}\right|^2 \leq \beta|x|^2+(1-\beta)|x|^2, \quad \forall x \in (0,1] \\
&\Longleftrightarrow \left|x-\frac{1}{2}\right|^2 \leq |x|^2, \quad \forall x \in (0,1] \\
&\Longleftrightarrow \frac{1}{4} \leq x, \quad \forall x \in (0,1],
\end{aligned} \tag{2.19}
$$

we see that T is not (α,β)-hybrid for any $\alpha,\beta \in \mathbb{R}$. But some simple computations show that T is $(1,0)$-hybrid relative to D_f.

A function $g : X \to (-\infty,\infty]$ is said to be subdifferentiable at a point $x \in X$ if there exists a linear functional $x^* \in X^*$ such that

$$
g(y)-g(x) \geq \langle y-x, x^*\rangle, \quad \forall y \in X. \tag{2.20}
$$

We call such x^* the subgradient of g at x. The set of all subgradients of g at x is denoted by $\partial g(x)$, and the mapping $\partial g : X \to 2^{X^*}$ is called the subdifferential of g. For a l.s.c. convex function $f, \partial f$ is bounded on bounded subsets of $\operatorname{Int}(\mathfrak{D})$ if and only if f is bounded on bounded subsets there, compare Proposition 1.1.11 of [5]. A proper convex l.s.c. function f is Gâteaux differentiable at $x \in \operatorname{Int}(\mathfrak{D})$ if and only if it has a unique subgradient at x; in such case $\partial f(x) = f'(x)$, compare Corollary 1.2.7 of [6].

The following lemma will be quoted in the sequel.

Lemma 2.2 (see Proposition 1.1.9 of [5]). *If a proper convex function $f : X \to (-\infty,\infty]$ is Gâteaux differentiable on* $\operatorname{Int}(\mathfrak{D})$ *in a Banach space X, then the following statements are equivalent.*

(a) *The function f is strictly convex on* $\operatorname{Int}(\mathfrak{D})$.

(b) *For any two distinct points $x, y \in \operatorname{Int}(\mathfrak{D})$, one has $D_f(y,x) > 0$.*

(c) *For any two distinct points $x, y \in \operatorname{Int}(\mathfrak{D})$, one has*

$$
\langle x-y, f'(x)-f'(y)\rangle > 0. \tag{2.21}
$$

Throughout this paper, $F(T)$ will denote the set of all fixed points of a mapping T.

3. Fixed Point Theorems

In this section, we apply Lemma 2.2 to study the fixed point problem for mappings satisfying (1.5).

Theorem 3.1. *Let X be a reflexive Banach space and let $f : X \to (-\infty,\infty]$ be a l.s.c. strictly convex function so that it is Gâteaux differentiable on* $\operatorname{Int}(\mathfrak{D})$ *and is bounded on bounded subsets of* $\operatorname{Int}(\mathfrak{D})$. *Suppose $C \subseteq \operatorname{Int}(\mathfrak{D})$ is a nonempty closed convex subset of X and $T : C \to C$ is (α,β)-hybrid relative to D_f. Then the following two statements are equivalent.*

(a) *There is a point* $x \in C$ *such that* $\{T^n x\}_{n\in\mathbb{N}}$ *is bounded.*

(b) $F(T) \neq \emptyset$.

Proof. If $F(T) \neq \emptyset$, then $\{T^n v\}_{n\in\mathbb{N}} = \{v\}$ is bounded for any $v \in F(T)$. Now assume that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded for some $x \in C$, and for any $n \in \mathbb{N}$, let $S_n(x) = (1/n)\sum_{k=0}^{n-1} T^k x$. Then $\{S_n x\}_{n\in\mathbb{N}}$ is bounded, and so, in view of X being reflexive, it has a subsequence $\{S_{n_i} x\}_{i\in\mathbb{N}}$ so that $S_{n_i} x$ converges weakly to some $v \in C$ as $n_i \to \infty$. Since T is (α,β)-hybrid relative to D_f, we have, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$,

$$0 \leq \beta D_f\left(T^{k+1}x, y\right) + (1-\beta) D_f\left(T^k x, y\right) - \left[\alpha D_f\left(T^{k+1}x, Ty\right) + (1-\alpha) D_f\left(T^k x, Ty\right)\right]. \tag{3.1}$$

Rewrite $\alpha D_f(T^{k+1}x, Ty) + (1-\alpha) D_f(T^k x, Ty)$ as

$$\begin{aligned}
\alpha D_f\left(T^{k+1}x, Ty\right) + (1-\alpha) D_f\left(T^k x, Ty\right) &= \alpha\left[f\left(T^{k+1}x\right) - f(Ty) - \left\langle T^{k+1}x - Ty, f'(Ty)\right\rangle\right] \\
&\quad + (1-\alpha)\left[f\left(T^k x\right) - f(Ty) - \left\langle T^k x - Ty, f'(Ty)\right\rangle\right] \\
&= \alpha f\left(T^{k+1}x\right) + (1-\alpha) f\left(T^k x\right) - f(Ty) \\
&\quad - \left\langle \alpha\left(T^{k+1}x - Ty\right) + (1-\alpha)\left(T^k x - Ty\right), f'(Ty)\right\rangle \\
&= \alpha f\left(T^{k+1}x\right) + (1-\alpha) f\left(T^k x\right) - f(Ty) \\
&\quad - \left\langle \alpha T^{k+1}x + (1-\alpha) T^k x - Ty, f'(Ty)\right\rangle.
\end{aligned} \tag{3.2}$$

Similarly, we also have

$$\begin{aligned}
&\beta D_f\left(T^{k+1}x, y\right) + (1-\beta) D_f\left(T^k x, y\right) \\
&\quad = \beta f\left(T^{k+1}x\right) + (1-\beta) f\left(T^k x\right) - f(y) - \left\langle \beta T^{k+1}x + (1-\beta) T^k x - y, f'(y)\right\rangle.
\end{aligned} \tag{3.3}$$

Consequently, we obtain from (3.1) that

$$\begin{aligned}
0 &\leq (\beta - \alpha)\left[f\left(T^{k+1}x\right) - f\left(T^k x\right)\right] + [f(Ty) - f(y)] \\
&\quad + \left\langle T^k x + \alpha\left(T^{k+1}x - T^k x\right) - Ty, f'(Ty)\right\rangle - \left\langle T^k x + \beta\left(T^{k+1}x - T^k x\right) - y, f'(y)\right\rangle.
\end{aligned} \tag{3.4}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n-1$, we get

$$\begin{aligned} 0 &\le (\beta - \alpha)\left[f(T^n x) - f(x)\right] + n\left[f(Ty) - f(y)\right] \\ &\quad + \langle nS_n x + \alpha(T^n x - x) - nTy, f'(Ty)\rangle - \langle nS_n x + \beta(T^n x - x) - ny, f'(y)\rangle. \end{aligned} \tag{3.5}$$

Dividing the above inequality by n, we have

$$\begin{aligned} 0 &\le \frac{\beta - \alpha}{n}\left[f(T^n x) - f(x)\right] + f(Ty) - f(y) + \left\langle S_n x + \frac{\alpha}{n}(T^n x - x) - Ty, f'(Ty)\right\rangle \\ &\quad - \left\langle S_n x + \frac{\beta}{n}(T^n x - x) - y, f'(y)\right\rangle. \end{aligned} \tag{3.6}$$

Replacing n by n_i and letting $n_i \to \infty$, we obtain from the fact that $\{f(T^n x)\}_{n\in\mathbb{N}}$ is bounded that

$$0 \le f(Ty) - f(y) + \langle v - Ty, f'(Ty)\rangle - \langle v - y, f'(y)\rangle. \tag{3.7}$$

Putting $y = v$ in (3.7), we get

$$0 \le f(Tv) - f(v) + \langle v - Tv, f'(Tv)\rangle, \tag{3.8}$$

that is,

$$0 \le -D_f(v, Tv), \tag{3.9}$$

from which follows that $D_f(v, Tv) = 0$. Therefore $Tv = v$ by Lemma 2.2. □

Since the function $f(x) = \|x\|^2$ in a Hilbert space H satisfies all the requirements of Theorem 3.1, the corollary below follows immediately.

Corollary 3.2 (see [4]). *Let C be a nonempty closed convex subset of Hilbert space H and Suppose $T : C \to C$ is (α, β)-hybrid. Then the following two statements are equivalent.*

(a) *There exists $x \in C$ such that $\{T^n(x)\}_{n\in\mathbb{N}}$ is bounded.*

(b) *T has a fixed point.*

We now show that the fixed point set $F(T)$ is closed and convex under the assumptions of Theorem 3.1.

A mapping $T : C \to X$ is said to be quasi-nonexpansive with respect to D_f if $F(T) \ne \emptyset$ and $D_f(v, Tx) \le D_f(v, x)$ for all $x \in C$ and all $v \in F(T)$.

The following lemma is shown in Huang et al. [7].

Lemma 3.3. *Let $f : X \to (-\infty, \infty]$ be a proper strictly convex function on a Banach space X so that it is Gâteaux differentiable on $\mathrm{Int}(\mathfrak{D})$, and let $C \subseteq \mathrm{Int}(\mathfrak{D})$ be a nonempty closed convex subset of X. If $T : C \to C$ is quasi-nonexpansive with respect to D_f, then $F(T)$ is a closed convex subset.*

Proposition 3.4. *Let $f : X \to (-\infty, \infty]$ be a proper strictly convex function on a reflexive Banach space X so that it is Gâteaux differentiable on $\operatorname{Int}(\mathfrak{D})$ and is bounded on bounded subsets of $\operatorname{Int}(\mathfrak{D})$, and let $C \subseteq \operatorname{Int}(\mathfrak{D})$ be a nonempty closed convex subset of X. Suppose $T : C \to C$ is (α,β)-hybrid relative to D_f and has a point $x_0 \in C$ such that $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is bounded. Then T is quasi-nonexpansive with respect to D_f, and therefore $F(T)$ is a nonempty closed convex subset of C.*

Proof. In view of Theorem 3.1, $F(T) \neq \emptyset$. Now, for any $v \in F(T)$ and any $y \in C$, as T is (α,β)-hybrid relative to D_f, we have

$$\begin{aligned} D_f(v,Ty) &= \alpha D_f(v,Ty) + (1-\alpha) D_f(v,Ty) \\ &= \alpha D_f(Tv,Ty) + (1-\alpha) D_f(v,Ty) \\ &\leq \beta D_f(Tv,y) + (1-\beta) D_f(v,y) \\ &= \beta D_f(v,y) + (1-\beta) D_f(v,y) = D_f(v,y), \end{aligned} \tag{3.10}$$

so T is quasi-nonexpansive with respect to D_f, and hence $F(T)$ is a nonempty closed convex subset of C by Lemma 3.3. □

For the remainder of this section, we establish a common fixed point theorem for a commutative family of (α,β)-hybrid mappings relative to D_f.

Lemma 3.5. *Let X be a reflexive Banach space and let $f : X \to (-\infty,\infty]$ be a l.s.c. strictly convex function so that it is Gâteaux differentiable on $\operatorname{Int}(\mathfrak{D})$ and is bounded on bounded subsets of $\operatorname{Int}(\mathfrak{D})$. Suppose $C \subseteq \operatorname{Int}(\mathfrak{D})$ is a nonempty bounded closed convex subset of X and $\{T_1, T_2, \ldots, T_N\}$ is a commutative finite family of (α,β)-hybrid mappings relative to D_f from C into itself. Then $\{T_1, T_2, \ldots, T_N\}$ has a common fixed point.*

Proof. We prove this lemma by induction with respect to N. To begin with, we deal with the case that $N = 2$. By Proposition 3.4, we see that $F(T_1)$ and $F(T_2)$ are nonempty bounded closed convex subsets of X. Moreover, $F(T_1)$ is T_2-invariant. Indeed, for any $v \in F(T_1)$, it follows from $T_1T_2 = T_2T_1$ that $T_1T_2v = T_2T_1v = T_2v$, which shows that $T_2v \in F(T_1)$. Consequently, the restriction of T_2 to $F(T_1)$ is (α,β)-hybrid relative to D_f, and so by Theorem 3.1, T_2 has a fixed point $u \in F(T_1)$, that is, $u \in F(T_1) \cap F(T_2)$.

By induction hypothesis, assume that for some $n \geq 2$, $E = \cap_{k=1}^{n} F(T_k)$ is nonempty. Then E is a nonempty closed convex subset of X, and the restriction of T_{n+1} to E is a (α,β)-hybrid mapping relative to D_f from E into itself. By Theorem 3.1, T_{n+1} has a fixed point in X. This shows that $E \cap F(T_{n+1}) \neq \emptyset$, that is, $\cap_{k=1}^{n+1} F(T_k) \neq \emptyset$, completing the proof. □

Here we would like to remark that in the above lemma, the assumption f is bounded on bounded subsets of $\operatorname{Int}(\mathfrak{D})$ can be dropped by checking the proof of Theorem 3.1 and noting that we have assumed C is bounded.

Theorem 3.6. *Let X be a reflexive Banach space and let $f : X \to (-\infty,\infty]$ be a l.s.c. strictly convex function so that it is Gâteaux differentiable on $\operatorname{Int}(\mathfrak{D})$. Suppose $C \subseteq \operatorname{Int}(\mathfrak{D})$ is a nonempty bounded closed convex subset of X and $\{T_i\}_{i\in I}$ is a commutative family of (α,β)-hybrid mappings relative to D_f from C into itself. Then $\{T_i\}_{i\in I}$ has a common fixed point.*

Proof. Since C is a nonempty bounded closed convex subset of the reflexive Banach space X, it is weakly compact. By Proposition 3.4, each $F(T_i)$ is a nonempty weakly compact subset of C. Therefore, the conclusion follows once we note that $\{F(T_i)\}_{i\in I}$ has the finite intersection property by Lemma 3.5. □

4. Weak Convergence Theorems

In this section, we discuss the demiclosedness and the weak convergence problem of (α,β)-hybrid mappings relative to D_f. We denote the weak convergence and strong convergence of a sequence $\{x_n\}$ to v in a Banach space X by $x_n \rightharpoonup v$ and $x_n \to v$, respectively. For a nonempty closed convex subset C of a Banach space X, a mapping $T : C \to X$ is demiclosed if for any sequence $\{x_n\}$ in C with $x_n \rightharpoonup v$ and $x_n - Tx_n \to 0$, one has $Tv = v$.

The following Opial-like inequality for the Bregman distance is proved in [7]. For the Opial's inequality we refer readers to Lemma 1 of [8].

Lemma 4.1. *Suppose $f : X \to (-\infty,\infty]$ is a proper strictly convex function so that it is Gâteaux differentiable on $\mathrm{Int}(\mathfrak{D})$ in a Banach space X and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{D}$ such that $x_n \rightharpoonup v$ for some $v \in \mathrm{Int}(\mathfrak{D})$. Then*

$$\liminf_{n\to\infty} D_f(x_n,v) < \liminf_{n\to\infty} D_f(x_n,y), \quad \forall y \in \mathrm{Int}(\mathfrak{D}) \text{ with } y \neq v. \tag{4.1}$$

Proposition 4.2. *Suppose $f : X \to (-\infty,\infty]$ is a strictly convex function so that it is Gâteaux differentiable on $\mathrm{Int}(\mathfrak{D})$ and is bounded on bounded subsets of $\mathrm{Int}(\mathfrak{D})$ in a Banach space X, and suppose C is a closed convex subset of $\mathrm{Int}(\mathfrak{D})$. If $T : C \to X$ is (α,β)-hybrid relative to D_f, then T is demiclosed.*

Proof. Let $\{x_n\}$ be any sequence in C with $x_n \rightharpoonup v$ and $x_n - Tx_n \to 0$. We have to show that $Tv = v$. Since f is bounded, by Proposition 1.1.11 of [5] there exists a constant $M > 0$ such that

$$\max\{\sup\{\|f'(x_n)\| : n \in \mathbb{N}\}, \|f'(Tv)\|, \|f'(v)\|\} \le M. \tag{4.2}$$

Note that, for $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\alpha D_f(Tx_n,Tv) + (1-\alpha)D_f(x_n,Tv) &= \alpha\left[f(Tx_n) - f(Tv) - \langle Tx_n - Tv, f'(Tv)\rangle\right] \\
&\quad + (1-\alpha)\left[f(x_n) - f(Tv) - \langle x_n - Tv, f'(Tv)\rangle\right] \\
&= \alpha\left[f(Tx_n) - f(Tv) + f(Tv) - f(x_n)\right] \\
&\quad - \langle Tx_n - Tv - x_n + Tv, f'(Tv)\rangle \\
&\quad + f(x_n) - f(Tv) - \langle x_n - Tv, f'(Tv)\rangle \\
&= \alpha\left[f(Tx_n) - f(x_n) - \langle Tx_n - x_n, f'(Tv)\rangle\right] + D_f(x_n,Tv).
\end{aligned} \tag{4.3}$$

Similarly, for $\beta \in \mathbb{R}$, we have

$$\beta D_f(Tx_n, v) + (1-\beta) D_f(x_n, v) = \beta\left[f(Tx_n) - f(x_n) - \langle Tx_n - x_n, f'(v)\rangle\right] + D_f(x_n, v). \tag{4.4}$$

Thus we obtain from the (α, β)-hybrid of T that

$$\begin{aligned} &\alpha\left[f(Tx_n) - f(x_n) - \langle Tx_n - x_n, f'(Tv)\rangle\right] + D_f(x_n, Tv) \\ &\quad \le \beta\left[f(Tx_n) - f(x_n) - \langle Tx_n - x_n, f'(v)\rangle\right] + D_f(x_n, v), \end{aligned} \tag{4.5}$$

which implies that

$$\begin{aligned} D_f(x_n, Tv) &\le D_f(x_n, v) + (\alpha - \beta)\left(f(x_n) - f(Tx_n)\right) + \langle Tx_n - x_n, \alpha f'(Tv) - \beta f'(v)\rangle \\ &\le D_f(x_n, v) + (\alpha - \beta)\langle x_n - Tx_n, f'(x_n)\rangle + \langle Tx_n - x_n, \alpha f'(Tv) - \beta f'(v)\rangle \\ &\le D_f(x_n, v) + 3M(|\alpha| + |\beta|)\|x_n - Tx_n\|. \end{aligned} \tag{4.6}$$

Consequently, if $Tv \neq v$, then Lemma 4.1 implies that

$$\begin{aligned} \liminf_{n\to\infty} D_f(x_n, v) &< \liminf_{n\to\infty} D_f(x_n, Tv) \\ &= \liminf_{n\to\infty} \left[D_f(x_n, v) + 3M(|\alpha| + |\beta|)\|x_n - Tx_n\|\right] \\ &= \liminf_{n\to\infty} D_f(x_n, v), \end{aligned} \tag{4.7}$$

a contradiction. This completes the proof. □

A mapping $T : C \to C$ is said to be asymptotically regular if, for any $x \in C$, the sequence $\{T^{n+1}x - T^n x\}$ tends to zero as $n \to \infty$.

Theorem 4.3. *Suppose the following conditions hold.*

(4.3.1) $f : X \to (-\infty, \infty]$ *is l.s.c. uniformly convex function so that it is Gâteaux differentiable on* $\mathrm{Int}(\mathfrak{D})$ *and is bounded on bounded subsets of* $\mathrm{Int}(\mathfrak{D})$ *in a reflexive Banach space* X.

(4.3.2) $C \subsetneq \mathrm{Int}(\mathfrak{D})$ *is a closed convex subset of* X.

(4.3.3) $T : C \to C$ *is* (α, β)*-hybrid relative to* D_f *and is asymptotically regular with a bounded sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *for some* $x_0 \in C$.

(4.3.4) *The mapping* $x \to f'(x)$ *for* $x \in X$ *is weak-to-weak* continuous.*

Then for any $x \in C$, $\{T^n x\}_{n\in\mathbb{N}}$ *is weakly convergent to an element* $v \in F(T)$.

Proof. Let $v \in F(T)$ and $x \in C$. If $\{T^n x\}_{n\in\mathbb{N}}$ is not bounded, then there is a subsequence $\{T^{n_i} x\}_{i\in\mathbb{N}}$ such that $\|v - T^{n_i}x\| \geq 1$ for all $i \in \mathbb{N}$ and $\|v - T^{n_i}x\| \to \infty$ as $i \to \infty$. From (4.3.3), for any $n \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned} D_f\left(v, T^{n+1}x\right) &= \alpha D_f\left(Tv, T^{n+1}x\right) + (1-\alpha) D_f\left(v, T^{n+1}x\right) \\ &\leq \beta D_f(Tv, T^n x) + (1-\beta) D_f(v, T^n x) = D_f(v, T^n x) \\ &\leq D_f(v, x), \end{aligned} \tag{4.8}$$

which in conjunction with (2.7), (2.8), and (2.12) implies that

$$\begin{aligned} D_f(v, x) \geq D_f(v, T^{n_i}x) &\geq \nu_f(T^{n_i}x, \|v - T^{n_i}x\|) \\ &\geq \|v - T^{n_i}x\| \nu_f(T^{n_i}x, 1) \\ &\geq \|v - T^{n_i}x\| \delta_f(1) \longrightarrow \infty, \quad \text{as } i \longrightarrow \infty, \end{aligned} \tag{4.9}$$

a contradiction. Therefore, for any $x \in X$, $\{T^n x\}_{n\in\mathbb{N}}$ is bounded, and so it has a subsequence $\{T^{n_j} x\}_{j\in\mathbb{N}}$ which is weakly convergent to w for some $w \in C$. As $T^{n_j}x - T^{n_j+1}x \to 0$, it follows from the demiclosedness of T that $w \in F(T)$. It remains to show that $T^n x \rightharpoonup w$ as $n \to \infty$. Let $\{T^{n_k} x\}_{k\in\mathbb{N}}$ be any subsequence of $\{T^n x\}_{n\in\mathbb{N}}$ so that $T^{n_k}x \rightharpoonup u$ for some $u \in C$. Then $u \in F(T)$. Since both of $\{D_f(w, T^n x)\}_{n\in\mathbb{N}}$ and $\{D_f(u, T^n x)\}_{n\in\mathbb{N}}$ are decreasing, we have

$$\lim_{n\to\infty}\left[D_f(w, T^n x) - D_f(u, T^n x)\right] = \lim_{n\to\infty}\left[f(w) - f(u) - \langle w - u, f'(T^n x)\rangle\right] = a \tag{4.10}$$

for some $a \in \mathbb{R}$. Particularly, from (4.3.4) we obtain

$$\begin{aligned} a &= \lim_{n_j\to\infty}\left[f(w) - f(u) - \langle w - u, f'(T^{n_j}x)\rangle\right] = f(w) - f(u) - \langle w - u, f'(w)\rangle, \\ a &= \lim_{n_k\to\infty}\left[f(w) - f(u) - \langle w - u, f'(T^{n_k}x)\rangle\right] = f(w) - f(u) - \langle w - u, f'(u)\rangle. \end{aligned} \tag{4.11}$$

Consequently, $\langle w - u, f'(w) - f'(u)\rangle = 0$, and hence $w = u$ by the strict convexity of f. This shows that $T^n x \rightharpoonup w$ for some $w \in F(T)$. □

5. Conclusion

In this paper, we have introduced the Bregman distance D_f and a new class of mappings, (α, β)-hybrid mappings relative to D_f in Banach spaces. We also have given and proved a necessary and sufficient condition for the existence of fixed points of the introduced mappings and some properties of the mappings. In fact, our result properly extends the Kocourek-Takahashi-Yao fixed point theorems for (α, β)-hybrid mappings in Hilbert spaces in 2010 [4]. Since the Kocourek-Takahashi-Yao fixed point theorems can be applied to study the nonexpansive mappings, the nonspreading mappings, and the hybrid mappings, our theorems in this paper are also good for these famous mappings in the field of fixed point theory.

According to [7], the fixed point theorems in this paper can be expected to discuss in a wider class of mappings, called point-dependent (α, β)-hybrid mappings relative to D_f in Banach spaces. In a point-dependent (α, β)-hybrid mapping, the α and the β are not constant again but two functions from a nonempty subset of a Banach space to real numbers. Therefore, inequality (1.4) for point-dependent (α, β)-hybrid mappings becomes

$$\alpha(y)\|Tx - Ty\|^2 + (1 - \alpha(y))\|x - Ty\|^2 \leq \beta(y)\|Tx - y\|^2 + (1 - \beta(y))\|x - y\|^2 \tag{5.1}$$

for all $x, y \in C$.

In addition, Noor [9–11] provides algorithms to search the fixed points of nonexpansive mappings and then combines the result with general variational inequalities to study applied mathematical problems. We are motivated by that and expect to develop algorithms from the theorems of this paper to approach the fixed points of the introduced mappings. Through the combination of the fixed point theorems and the corresponding algorithms, the introduced mappings of this paper would be able to be applied to more fields of applied mathematics.

References

[1] S. Iemoto and W. Takahashi, "Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 12, pp. e2082–e2089, 2009.

[2] F. Kohsaka and W. Takahashi, "Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces," *Archiv der Mathematik*, vol. 91, no. 2, pp. 166–177, 2008.

[3] W. Takahashi, "Fixed point theorems for new nonlinear mappings in a Hilbert space," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 1, pp. 79–88, 2010.

[4] P. Kocourek, W. Takahashi, and J.-C. Yao, "Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces," *Taiwanese Journal of Mathematics*, vol. 14, no. 6, pp. 2497–2511, 2010.

[5] D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, vol. 40, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[6] I. Ciorănescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[7] Y. Y. Huang, J. C. Jeng, T. Y. Kuo, and C. C. Hong, "Fixed point and weak convergence theorems for point-dependent λ-hybrid mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2011, article 105, 2011.

[8] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[9] M. A. Noor, "New approximation schemes for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.

[10] M. Aslam Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.

[11] M. A. Noor, "General variational inequalities and nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 810–822, 2007.