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Research Article

A Hybrid Gradient-Projection Algorithm for Averaged Mappings in Hilbert Spaces

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It is well known that the gradient-projection algorithm (GPA) is very useful in solving constrained convex minimization problems. In this paper, we combine a general iterative method with the gradient-projection algorithm to propose a hybrid gradient-projection algorithm and prove that the sequence generated by the hybrid gradient-projection algorithm converges in norm to a minimizer of constrained convex minimization problems which solves a variational inequality.

1. Introduction

Let H be a real Hilbert space and C a nonempty closed and convex subset of H. Consider the following constrained convex minimization problem:

$$\text{minimize}_{x\in C} f(x), \tag{1.1}$$

where $f : C \to \mathbb{R}$ is a real-valued convex and continuously Fréchet differentiable function. The gradient ∇f satisfies the following Lipschitz condition:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \forall x, y \in C, \tag{1.2}$$

where $L > 0$. Assume that the minimization problem (1.1) is consistent, and let S denote its solution set.

It is well known that the gradient-projection algorithm is very useful in dealing with constrained convex minimization problems and has extensively been studied ([1–5] and the references therein). It has recently been applied to solve split feasibility problems [6–10]. Levitin and Polyak [1] consider the following gradient-projection algorithm:

$$x_{n+1} := \mathrm{Proj}_C\big(x_n - \lambda_n \nabla f(x_n)\big), \quad n \geq 0. \tag{1.3}$$

Let $\{\lambda_n\}_{n=0}^{\infty}$ satisfy

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}. \tag{1.4}$$

It is proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a minimizer of (1.1).

Xu proved that under certain appropriate conditions on $\{\alpha_n\}$ and $\{\lambda_n\}$ the sequence $\{x_n\}$ defined by the following relaxed gradient-projection algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathrm{Proj}_C\big(x_n - \lambda_n \nabla f(x_n)\big), \quad n \geq 0, \tag{1.5}$$

converges weakly to a minimizer of (1.1) [11].

Since the Lipschitz continuity of the gradient of f implies that it is indeed inverse strongly monotone (ism) [12, 13], its complement can be an averaged mapping. Recall that a mapping T is nonexpansive if and only if it is Lipschitz with Lipschitz constant not more than one, that a mapping is an averaged mapping if and only if it can be expressed as a proper convex combination of the identity mapping and a nonexpansive mapping, and that a mapping T is said to be ν-inverse strongly monotone if and only if $\langle x-y,\ Tx-Ty\rangle \geq \nu\|Tx-Ty\|^2$ for all $x, y \in H$, where the number $\nu > 0$. Recall also that the composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^{N}$ is averaged, then so is the composite $T_1 \cdots T_N$ [14]. In particular, an averaged mapping is a nonexpansive mapping [15]. As a result, the GPA can be rewritten as the composite of a projection and an averaged mapping which is again an averaged mapping.

Generally speaking, in infinite-dimensional Hilbert spaces, GPA has only weak convergence. Xu [11] provided a modification of GPA so that strong convergence is guaranteed. He considered the following hybrid gradient-projection algorithm:

$$x_{n+1} = \theta_n h(x_n) + (1-\theta_n)\mathrm{Proj}_C\big(x_n - \lambda_n \nabla f(x_n)\big). \tag{1.6}$$

It is proved that if the sequences $\{\theta_n\}$ and $\{\lambda_n\}$ satisfy appropriate conditions, the sequence $\{x_n\}$ generated by (1.6) converges in norm to a minimizer of (1.1) which solves the variational inequality

$$x^* \in S, \quad \langle (I-h)x^*,\ x - x^* \rangle \geq 0, \quad x \in S. \tag{1.7}$$

On the other hand, Ming Tian [16] introduced the following general iterative algorithm for solving the variational inequality

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad n \geq 0, \tag{1.8}$$

where F is a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$, $\eta > 0$ and f is a contraction with coefficient $0 < \alpha < 1$. Then, he proved that if $\{\alpha_n\}$ satisfying appropriate conditions, the $\{x_n\}$ generated by (1.8) converges strongly to the unique solution of variational inequality

$$\langle (\mu F - \gamma f)\tilde{x},\ \tilde{x} - z\rangle \leq 0, \quad z \in \operatorname{Fix}(T). \tag{1.9}$$

In this paper, motivated and inspired by the research work in this direction, we will combine the iterative method (1.8) with the gradient-projection algorithm (1.3) and consider the following hybrid gradient-projection algorithm:

$$x_{n+1} = \theta_n \gamma h(x_n) + (I - \mu\theta_n F)\operatorname{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \quad n \geq 0. \tag{1.10}$$

We will prove that if the sequence $\{\theta_n\}$ of parameters and the sequence $\{\lambda_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ generated by (1.10) converges in norm to a minimizer of (1.1) which solves the variational inequality (VI)

$$x^* \in S, \quad \langle (\mu F - \gamma h)x^*,\ x - x^*\rangle \geq 0, \quad \forall x \in S, \tag{1.11}$$

where S is the solution set of the minimization problem (1.1).

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$.

Lemma 2.1 (see [17]). *Assume that $\{a_n\}_{n=0}^{\infty}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n + \beta_n, \quad n \geq 0, \tag{2.1}$$

where $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ and $\{\delta_n\}_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \gamma_n|\delta_n| < \infty$;

(iii) $\sum_{n=0}^{\infty} \beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (see [18]). *Let C be a closed and convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with* $\operatorname{Fix} T \neq \emptyset$. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}_{n=1}^{\infty}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.3. *Let H be a Hilbert space, and let C be a nonempty closed and convex subset of H. $h : C \to C$ a contraction with coefficient $0 < \rho < 1$, and $F : C \to C$ a κ-Lipschitzian continuous operator and η-strongly monotone operator with $\kappa, \eta > 0$. Then, for $0 < \gamma < \mu\eta/\rho$,*

$$\langle x - y, (\mu F - \gamma h)x - (\mu F - \gamma h)y \rangle \geq (\mu\eta - \gamma\rho)\|x - y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

That is, $\mu F - \gamma h$ is strongly monotone with coefficient $\mu\eta - \gamma\rho$.

Lemma 2.4. *Let C be a closed subset of a real Hilbert space H, given $x \in H$ and $y \in C$. Then, $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y,\ y - z \rangle \geq 0, \quad \forall z \in C. \tag{2.3}$$

3. Main Results

Let H be a real Hilbert space, and let C be a nonempty closed and convex subset of H such that $C \pm C \subset C$. Assume that the minimization problem (1.1) is consistent, and let S denote its solution set. Assume that the gradient ∇f satisfies the Lipschitz condition (1.2). Since S is a closed convex subset, the nearest point projection from H onto S is well defined. Recall also that a contraction on C is a self-mapping of C such that $\|h(x) - h(y)\| \leq \rho\|x - y\|$, for all $x, y \in C$, where $\rho \in [0, 1)$ is a constant. Let F be a κ-Lipschitzian and η-strongly monotone operator on C with $\kappa, \eta > 0$. Denote by Π the collection of all contractions on C, namely,

$$\Pi = \{h : h \text{ is a contraction on } C\}. \tag{3.1}$$

Now given $h \in \Pi$ with $0 < \rho < 1$, $s \in (0, 1)$. Let $0 < \mu < 2\eta/\kappa^2$, $0 < \gamma < \mu(\eta - (\mu\kappa^2)/2)/\rho = \tau/\rho$. Assume that λ_s with respect to s is continuous and, in addition, $\lambda_s \in [a, b] \subset (0, 2/L)$. Consider a mapping X_s on C defined by

$$X_s(x) = s\gamma h(x) + (I - s\mu F)\mathrm{Proj}_C(I - \lambda_s \nabla f)(x), \quad x \in C. \tag{3.2}$$

It is easy to see that X_s is a contraction. Setting $V_s := \mathrm{Proj}_C(I - \lambda_s \nabla f)$. It is obvious that V_s is a nonexpansive mapping. We can rewrite $X_s(x)$ as

$$X_s(x) = s\gamma h(x) + (I - s\mu F)V_s(x). \tag{3.3}$$

First observe that for $s \in (0,1)$, we can get

$$
\begin{aligned}
&\left\|(I-s\mu F)V_s(x)-(I-s\mu F)V_s(y)\right\|^2 \\
&\quad = \left\|V_s(x)-V_s(y)-s\mu\left(FV_s(x)-FV_s(y)\right)\right\|^2 \\
&\quad = \left\|V_s(x)-V_s(y)\right\|^2 - 2s\mu\left\langle V_s(x)-V_s(y), FV_s(x)-FV_s(y)\right\rangle \\
&\qquad + s^2\mu^2\left\|FV_s(x)-FV_s(y)\right\|^2 \\
&\quad \leq \|x-y\|^2 - 2s\mu\eta\left\|V_s(x)-V_s(y)\right\|^2 + s^2\mu^2\kappa^2\left\|V_s(x)-V_s(y)\right\|^2 \\
&\quad \leq \left(1-s\mu\left(2\eta-s\mu\kappa^2\right)\right)\|x-y\|^2 \\
&\quad \leq \left(1-\frac{s\mu(2\eta-s\mu\kappa^2)}{2}\right)^2\|x-y\|^2 \\
&\quad \leq \left(1-s\mu\left(\eta-\frac{\mu\kappa^2}{2}\right)\right)^2\|x-y\|^2 \\
&\quad = (1-s\tau)^2\|x-y\|^2.
\end{aligned}
\tag{3.4}
$$

Indeed, we have

$$
\begin{aligned}
\left\|X_s(x)-X_s(y)\right\| &= \left\|s\gamma h(x)+(I-s\mu F)V_s(x)-s\gamma h(y)-(I-s\mu F)V_s(y)\right\| \\
&\leq s\gamma\left\|h(x)-h(y)\right\| + \left\|(I-s\mu F)V_s(x)-(I-s\mu F)V_s(y)\right\| \\
&\leq s\gamma\rho\|x-y\| + (1-s\tau)\|x-y\| \\
&= \left(1-s(\tau-\gamma\rho)\right)\|x-y\|.
\end{aligned}
\tag{3.5}
$$

Hence, X_s has a unique fixed point, denoted x_s, which uniquely solves the fixed-point equation

$$
x_s = s\gamma h(x_s) + \left(I-s\mu F\right)V_s(x_s).
\tag{3.6}
$$

The next proposition summarizes the properties of $\{x_s\}$.

Proposition 3.1. *Let x_s be defined by* (3.6).

(i) $\{x_s\}$ is bounded for $s \in (0,(1/\tau))$.

(ii) $\lim_{s\to 0}\|x_s - Proj_C(I-\lambda_s\nabla f)(x_s)\| = 0$.

(iii) x_s defines a continuous curve from $(0,1/\tau)$ into H.

Proof. (i) Take a $\overline{x} \in S$, then we have

$$\begin{aligned} \|x_s - \overline{x}\| &= \left\|s\gamma h(x_s) + (I - s\mu F)\operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s) - \overline{x}\right\| \\ &= \left\|(I - s\mu F)\operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s) - (I - s\mu F)\operatorname{Proj}_C(I - \lambda_s \nabla f)(\overline{x})\right. \\ &\quad \left. + s\left(\gamma h(x_s) - \mu F \operatorname{Proj}_C(I - \lambda_s \nabla f)(\overline{x})\right)\right\| \\ &\le (1 - s\tau)\|x_s - \overline{x}\| + s\left\|\gamma h(x_s) - \mu F(\overline{x})\right\| \\ &\le (1 - s\tau)\|x_s - \overline{x}\| + s\gamma\rho\|x_s - \overline{x}\| + s\left\|\gamma h(\overline{x}) - \mu F(\overline{x})\right\|. \end{aligned} \tag{3.7}$$

It follows that

$$\|x_s - \overline{x}\| \le \frac{\left\|\gamma h(\overline{x}) - \mu F(\overline{x})\right\|}{\tau - \gamma\rho}. \tag{3.8}$$

Hence, $\{x_s\}$ is bounded.

(ii) By the definition of $\{x_s\}$, we have

$$\begin{aligned} \left\|x_s - \operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s)\right\| &= \left\|s\gamma h(x_s) + (I - s\mu F)\operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s)\right. \\ &\quad \left. - \operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s)\right\| \\ &= s\left\|\gamma h(x_s) - \mu F \operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s)\right\| \longrightarrow 0, \end{aligned} \tag{3.9}$$

$\{x_s\}$ is bounded, so are $\{h(x_s)\}$ and $\{F \operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s)\}$.

(iii) Take $s, s_0 \in (0, 1/\tau)$, and we have

$$\begin{aligned} &\|x_s - x_{s_0}\| \\ &\quad = \left\|s\gamma h(x_s) + (I - s\mu F)\operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s) - s_0\gamma h(x_{s_0})\right. \\ &\qquad \left. - (I - s_0\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_{s_0})\right\| \\ &\quad \le \left\|(s - s_0)\gamma h(x_s) + s_0\gamma(h(x_s) - h(x_{s_0}))\right\| \\ &\qquad + \left\|(I - s_0\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - (I - s_0\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_{s_0})\right\| \\ &\qquad + \left\|(I - s\mu F)\operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s) - (I - s_0\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\right\| \\ &\quad \le \left\|(s - s_0)\gamma h(x_s) + s_0\gamma(h(x_s) - h(x_{s_0}))\right\| \\ &\qquad + \left\|(I - s_0\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - (I - s_0\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_{s_0})\right\| \\ &\qquad + \left\|(I - s\mu F)\operatorname{Proj}_C(I - \lambda_s \nabla f)(x_s) - (I - s\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\right\| \\ &\qquad + \left\|(I - s\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - (I - s_0\mu F)\operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\right\| \\ &\quad \le |s - s_0|\gamma\|h(x_s)\| + s_0\gamma\rho\|x_s - x_{s_0}\| + (1 - s_0\tau)\|x_s - x_{s_0}\| \\ &\qquad + |\lambda_s - \lambda_{s_0}|\left\|\nabla f(x_s)\right\| \\ &\qquad + \left\|s\mu F \operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - s_0\mu F \operatorname{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\right\| \end{aligned}$$

$$\begin{aligned}
&= |s-s_0|\gamma\|h(x_s)\| + s_0\gamma\rho\|x_s - x_{s_0}\| + (1-s_0\tau)\|x_s - x_{s_0}\| \\
&\quad + |\lambda_s - \lambda_{s_0}|\left\|\nabla f(x_s)\right\| + |s-s_0|\left\|\mu F\,\mathrm{Proj}_C(I-\lambda_{s_0}\nabla f)(x_s)\right\| \\
&= \left(\gamma\|h(x_s)\| + \mu\left\|F\,\mathrm{Proj}_C(I-\lambda_{s_0}\nabla f)(x_s)\right\|\right)|s-s_0| \\
&\quad + s_0\gamma\rho\|x_s - x_{s_0}\| + (1-s_0\tau)\|x_s - x_{s_0}\| + |\lambda_s - \lambda_{s_0}|\left\|\nabla f(x_s)\right\|.
\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}
\|x_s - x_{s_0}\| \le{}& \frac{\gamma\|h(x_s)\| + \mu\left\|F\,\mathrm{Proj}_C(I-\lambda_{s_0}\nabla f)(x_s)\right\|}{s_0(\tau-\gamma\rho)}|s-s_0| \\
&+ \frac{\left\|\nabla f(x_s)\right\|}{s_0(\tau-\gamma\rho)}|\lambda_s - \lambda_{s_0}|.
\end{aligned} \tag{3.11}$$

Therefore, $x_s \to x_{s_0}$ as $s \to s_0$. This means x_s is continuous. □

Our main result in the following shows that $\{x_s\}$ converges in norm to a minimizer of (1.1) which solves some variational inequality.

Theorem 3.2. *Assume that $\{x_s\}$ is defined by* (3.6)*, then x_s converges in norm as $s \to 0$ to a minimizer of* (1.1) *which solves the variational inequality*

$$\left\langle (\mu F - \gamma h)x^*,\ \tilde{x} - x^* \right\rangle \ge 0, \quad \forall \tilde{x} \in S. \tag{3.12}$$

Equivalently, we have $\mathrm{Proj}_s(I-(\mu F-\gamma h))x^* = x^*$.

Proof. It is easy to see that the uniqueness of a solution of the variational inequality (3.12). By Lemma 2.3, $\mu F - \gamma h$ is strongly monotone, so the variational inequality (3.12) has only one solution. Let $x^* \in S$ denote the unique solution of (3.12).

To prove that $x_s \to x^*$ $(s \to 0)$, we write, for a given $\tilde{x} \in S$,

$$\begin{aligned}
x_s - \tilde{x} &= s\gamma h(x_s) + (I - s\mu F)\mathrm{Proj}_C(I-\lambda_s\nabla f)(x_s) - \tilde{x} \\
&= s\left(\gamma h(x_s) - \mu F\tilde{x}\right) + (I - s\mu F)\mathrm{Proj}_C(I-\lambda_s\nabla f)(x_s) \\
&\quad - (I - s\mu F)\mathrm{Proj}_C(I-\lambda_s\nabla f)(\tilde{x}).
\end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}
\|x_s - \tilde{x}\|^2 &= s\left\langle \gamma h(x_s) - \mu F\tilde{x},\ x_s - \tilde{x} \right\rangle \\
&\quad + \left\langle (I - s\mu F)\mathrm{Proj}_C(I-\lambda_s\nabla f)(x_s) - (I - s\mu F)\mathrm{Proj}_C(I-\lambda_s\nabla f)(\tilde{x}), x_s - \tilde{x} \right\rangle \\
&\le (1 - s\tau)\|x_s - \tilde{x}\|^2 + s\left\langle \gamma h(x_s) - \mu F\tilde{x},\ x_s - \tilde{x} \right\rangle.
\end{aligned} \tag{3.14}$$

Hence,

$$\begin{aligned}\|x_s - \tilde{x}\|^2 &\le \frac{1}{\tau}\langle \gamma h(x_s) - \mu F\tilde{x},\, x_s - \tilde{x}\rangle \\ &\le \frac{1}{\tau}\left\{\gamma\rho\|x_s - \tilde{x}\|^2 + \langle \gamma h(\tilde{x}) - \mu F\tilde{x},\, x_s - \tilde{x}\rangle\right\}.\end{aligned} \tag{3.15}$$

To derive that

$$\|x_s - \tilde{x}\|^2 \le \frac{1}{\tau - \gamma\rho}\langle \gamma h(\tilde{x}) - \mu F\tilde{x},\, x_s - \tilde{x}\rangle. \tag{3.16}$$

Since $\{x_s\}$ is bounded as $s \to 0$, we see that if $\{s_n\}$ is a sequence in (0,1) such that $s_n \to 0$ and $x_{s_n} \rightharpoonup \overline{x}$, then by (3.16), $x_{s_n} \to \overline{x}$. We may further assume that $\lambda_{s_n} \to \lambda \in [0, 2/L]$ due to condition (1.4). Notice that $\mathrm{Proj}_C(I - \lambda\nabla f)$ is nonexpansive. It turns out that

$$\begin{aligned}&\left\|x_{s_n} - \mathrm{Proj}_C(I - \lambda\nabla f)x_{s_n}\right\| \\ &\quad\le \left\|x_{s_n} - \mathrm{Proj}_C(I - \lambda_{s_n}\nabla f)x_{s_n}\right\| + \left\|\mathrm{Proj}_C(I - \lambda_{s_n}\nabla f)x_{s_n} - \mathrm{Proj}_C(I - \lambda\nabla f)x_{s_n}\right\| \\ &\quad\le \left\|x_{s_n} - \mathrm{Proj}_C(I - \lambda_{s_n}\nabla f)x_{s_n}\right\| + \left\|(\lambda - \lambda_{s_n})\nabla f(x_{s_n})\right\| \\ &\quad= \left\|x_{s_n} - \mathrm{Proj}_C(I - \lambda_{s_n}\nabla f)x_{s_n}\right\| + |\lambda - \lambda_{s_n}|\left\|\nabla f(x_{s_n})\right\|.\end{aligned} \tag{3.17}$$

From the boundedness of $\{x_s\}$ and $\lim_{s\to 0}\|\mathrm{Proj}_C(I - \lambda_s\nabla f)x_s - x_s\| = 0$, we conclude that

$$\lim_{n\to\infty}\left\|x_{s_n} - \mathrm{Proj}_C(I - \lambda\nabla f)x_{s_n}\right\| = 0. \tag{3.18}$$

Since $x_{s_n} \rightharpoonup \overline{x}$, by Lemma 2.2, we obtain

$$\overline{x} = \mathrm{Proj}_C(I - \lambda\nabla f)\overline{x}. \tag{3.19}$$

This shows that $\overline{x} \in S$.

We next prove that $\overline{x}$ is a solution of the variational inequality (3.12). Since

$$x_s = s\gamma h(x_s) + (I - s\mu F)\mathrm{Proj}_C(I - \lambda_s\nabla f)(x_s), \tag{3.20}$$

we can derive that

$$\begin{aligned}&(\mu F - \gamma h)(x_s) \\ &= -\frac{1}{s}(I - \mathrm{Proj}_C(I - \lambda_s\nabla f)(x_s) + \mu(F(x_s) - F\,\mathrm{Proj}_C(I - \lambda_s\nabla f)(x_s)).\end{aligned} \tag{3.21}$$

Therefore, for $\tilde{x} \in S$,

$$\begin{aligned}
&\langle (\mu F - \gamma h)(x_s), x_s - \tilde{x} \rangle \\
&= -\frac{1}{s}\langle (I - \mathrm{Proj}_C(I - \lambda_s \nabla f))(x_s) - (I - \mathrm{Proj}_C(I - \lambda_s \nabla f))(\tilde{x}), x_s - \tilde{x} \rangle \\
&\quad + \mu \langle F(x_s) - F\,\mathrm{Proj}_C(I - \lambda_s \nabla f)(x_s), x_s - \tilde{x} \rangle \\
&\le \mu \langle F(x_s) - F\,\mathrm{Proj}_C(I - \lambda_s \nabla f)(x_s), x_s - \tilde{x} \rangle.
\end{aligned} \tag{3.22}$$

Since $\mathrm{Proj}_C(I - \lambda_s \nabla f)$ is nonexpansive, we obtain that $I - \mathrm{Proj}_C(I - \lambda_s \nabla f)$ is monotone, that is,

$$\langle (I - \mathrm{Proj}_C(I - \lambda_s \nabla f))(x_s) - (I - \mathrm{Proj}_C(I - \lambda_s \nabla f))(\tilde{x}), x_s - \tilde{x} \rangle \ge 0. \tag{3.23}$$

Taking the limit through $s = s_n \to 0$ ensures that $\overline{x}$ is a solution to (3.12). That is to say

$$\langle (\mu F - \gamma h)(\overline{x}), \overline{x} - \tilde{x} \rangle \le 0. \tag{3.24}$$

Hence $\overline{x} = x^*$ by uniqueness. Therefore, $x_s \to x^*$ as $s \to 0$. The variational inequality (3.12) can be written as

$$\langle (I - \mu F + \gamma h)x^* - x^*,\ \tilde{x} - x^* \rangle \le 0, \quad \forall \tilde{x} \in S. \tag{3.25}$$

So, by Lemma 2.4, it is equivalent to the fixed-point equation

$$P_S(I - \mu F + \gamma h)x^* = x^*. \tag{3.26}$$

□

Taking $F = A$, $\mu = 1$ in Theorem 3.2, we get the following

Corollary 3.3. *We have that $\{x_s\}$ converges in norm as $s \to 0$ to a minimizer of* (1.1) *which solves the variational inequality*

$$\langle (A - \gamma h)x^*,\ \tilde{x} - x^* \rangle \ge 0, \quad \forall \tilde{x} \in S. \tag{3.27}$$

Equivalently, we have $Proj_s(I - (A - \gamma h))x^* = x^*$.

Taking $F = I$, $\mu = 1$, $\gamma = 1$ in Theorem 3.2, we get the following.

Corollary 3.4. *Let $z_s \in H$ be the unique fixed point of the contraction $z \mapsto sh(z) + (1 - s)Proj_C(I - \lambda_s \nabla f)(z)$. Then, $\{z_s\}$ converges in norm as $s \to 0$ to the unique solution of the variational inequality*

$$\langle (I - h)x^*,\ \tilde{x} - x^* \rangle \ge 0, \quad \forall \tilde{x} \in S. \tag{3.28}$$

Finally, we consider the following hybrid gradient-projection algorithm,

$$\begin{cases} x_0 \in \text{Carbitrarily}, \\ x_{n+1} = \theta_n \gamma h(x_n) + (I - \mu\theta_n F)\text{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \forall n \geq 0. \end{cases} \tag{3.29}$$

Assume that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ satisfies the condition (1.4) and, in addition, that the following conditions are satisfied for $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty} \subset [0,1]$:

(i) $\theta_n \to 0$;

(ii) $\sum_{n=0}^{\infty} \theta_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\theta_{n+1} - \theta_n| < \infty$;

(iv) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Theorem 3.5. *Assume that the minimization problem* (1.1) *is consistent and the gradient* ∇f *satisfies the Lipschitz condition* (1.2). *Let* $\{x_n\}$ *be generated by algorithm* (3.29) *with the sequences* $\{\theta_n\}$ *and* $\{\lambda_n\}$ *satisfying the above conditions. Then, the sequence* $\{x_n\}$ *converges in norm to* x^* *that is obtained in Theorem 3.2.*

Proof. (1) The sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Setting

$$V_n := \text{Proj}_C(I - \lambda_n \nabla f). \tag{3.30}$$

Indeed, we have, for $\overline{x} \in S$,

$$\begin{aligned} \|x_{n+1} - \overline{x}\| &= \|\theta_n \gamma h(x_n) + (I - \mu\theta_n F)V_n x_n - \overline{x}\| \\ &= \|\theta_n(\gamma h(x_n) - \mu F(\overline{x})) + (I - \mu\theta_n F)V_n x_n - (I - \mu\theta_n F)V_n \overline{x}\| \\ &\leq (1 - \theta_n \tau)\|x_n - \overline{x}\| + \theta_n \rho\gamma \|x_n - \overline{x}\| + \theta_n \|\gamma h(\overline{x}) - \mu F(\overline{x})\| \\ &= (1 - \theta_n(\tau - \gamma\rho))\|x_n - \overline{x}\| + \theta_n \|\gamma h(\overline{x}) - \mu F(\overline{x})\| \\ &\leq \max\left\{ \|x_n - \overline{x}\|, \frac{1}{\tau - \gamma\rho}\|\gamma h(\overline{x}) - \mu F(\overline{x})\| \right\}, \quad \forall n \geq 0. \end{aligned} \tag{3.31}$$

By induction,

$$\|x_n - \overline{x}\| \leq \max\left\{ \|x_0 - \overline{x}\|, \frac{\|\gamma h(\overline{x}) - \mu F(\overline{x})\|}{\tau - \gamma\rho} \right\}. \tag{3.32}$$

In particular, $\{x_n\}_{n=0}^{\infty}$ is bounded.

(2) We prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Let M be a constant such that

$$M > \max\left\{ \sup_{n \geq 0} \gamma \|h(x_n)\|, \sup_{\kappa, n \geq 0} \mu \|F V_\kappa x_n\|, \sup_{n \geq 0} \|\nabla f(x_n)\| \right\}. \tag{3.33}$$

We compute

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \left\|\theta_n \gamma h(x_n) + (I - \mu\theta_n F)V_n x_n - \theta_{n-1}\gamma h(x_{n-1}) - (I - \mu\theta_{n-1}F)V_{n-1}x_{n-1}\right\| \\
&\quad = \big\|\theta_n \gamma (h(x_n) - h(x_{n-1})) + \gamma(\theta_n - \theta_{n-1})h(x_{n-1}) + (I - \mu\theta_n F)V_n x_n \\
&\qquad - (I - \mu\theta_n F)V_n x_{n-1} + (I - \mu\theta_n F)V_n x_{n-1} - (I - \mu\theta_{n-1}F)V_{n-1}x_{n-1}\big\| \\
&\quad = \big\|\theta_n \gamma (h(x_n) - h(x_{n-1})) + \gamma(\theta_n - \theta_{n-1})h(x_{n-1}) + (I - \mu\theta_n F)V_n x_n \\
&\qquad - (I - \mu\theta_n F)V_n x_{n-1} + (I - \mu\theta_n F)V_n x_{n-1} - (I - \mu\theta_n F)V_{n-1}x_{n-1} \\
&\qquad + (I - \mu\theta_n F)V_{n-1}x_{n-1} - (I - \mu\theta_{n-1}F)V_{n-1}x_{n-1}\big\| \\
&\quad \le \theta_n\gamma\rho\|x_n - x_{n-1}\| + \gamma|\theta_n - \theta_{n-1}|\|h(x_{n-1})\| + (1 - \theta_n\tau)\|x_n - x_{n-1}\| \\
&\qquad + \|V_n x_{n-1} - V_{n-1}x_{n-1}\| + \mu|\theta_n - \theta_{n-1}|\|FV_{n-1}x_{n-1}\| \\
&\quad \le \theta_n\gamma\rho\|x_n - x_{n-1}\| + M|\theta_n - \theta_{n-1}| + (1 - \theta_n\tau)\|x_n - x_{n-1}\| \\
&\qquad + \|V_n x_{n-1} - V_{n-1}x_{n-1}\| + M|\theta_n - \theta_{n-1}| \\
&\quad = (1 - \theta_n(\tau - \gamma\rho))\|x_n - x_{n-1}\| + 2M|\theta_n - \theta_{n-1}| + \|V_n x_{n-1} - V_{n-1}x_{n-1}\|,
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
\|V_n x_{n-1} - V_{n-1}x_{n-1}\| &= \left\|\mathrm{Proj}_C(I - \lambda_n\nabla f)x_{n-1} - \mathrm{Proj}_C(I - \lambda_{n-1}\nabla f)x_{n-1}\right\| \\
&\le \left\|(I - \lambda_n\nabla f)x_{n-1} - (I - \lambda_{n-1}\nabla f)x_{n-1}\right\| \\
&= |\lambda_n - \lambda_{n-1}|\left\|\nabla f(x_{n-1})\right\| \\
&\le M|\lambda_n - \lambda_{n-1}|.
\end{aligned}
\tag{3.35}
$$

Combining (3.34) and (3.35), we can obtain

$$
\|x_{n+1} - x_n\| \le (1 - (\tau - \gamma\rho)\theta_n)\|x_n - x_{n-1}\| + 2M(|\theta_n - \theta_{n-1}| + |\lambda_n - \lambda_{n-1}|). \tag{3.36}
$$

Apply Lemma 2.1 to (3.36) to conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

(3) We prove that $\omega_w(x_n) \subset S$. Let $\hat{x} \in \omega_w(x_n)$, and assume that $x_{n_j} \rightharpoonup \hat{x}$ for some subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$. We may further assume that $\lambda_{n_j} \to \lambda \in [0, 2/L]$ due to condition (1.4). Set $V := \mathrm{Proj}_C(I - \lambda\nabla f)$. Notice that V is nonexpansive and $\mathrm{Fix}\, V = S$. It turns out that

$$
\begin{aligned}
\left\|x_{n_j} - Vx_{n_j}\right\| &\le \left\|x_{n_j} - V_{n_j}x_{n_j}\right\| + \left\|V_{n_j}x_{n_j} - Vx_{n_j}\right\| \\
&\le \left\|x_{n_j} - x_{n_j+1}\right\| + \left\|x_{n_j+1} - V_{n_j}x_{n_j}\right\| + \left\|V_{n_j}x_{n_j} - Vx_{n_j}\right\| \\
&\le \left\|x_{n_j} - x_{n_j+1}\right\| + \theta_{n_j}\left\|\gamma h\left(x_{n_j}\right) - \mu F V_{n_j}x_{n_j}\right\| \\
&\quad + \left\|\mathrm{Proj}_C\left(I - \lambda_{n_j}\nabla f\right)x_{n_j} - \mathrm{Proj}_C(I - \lambda\nabla f)x_{n_j}\right\|
\end{aligned}
$$

$$\leq \left\|x_{n_j} - x_{n_j+1}\right\| + \theta_{n_j}\left\|\gamma h\left(x_{n_j}\right) - \mu F V_{n_j} x_{n_j}\right\| + \left|\lambda - \lambda_{n_j}\right|\left\|\nabla f\left(x_{n_j}\right)\right\|$$
$$\leq \left\|x_{n_j} - x_{n_j+1}\right\| + 2M\left(\theta_{n_j} + \left|\lambda - \lambda_{n_j}\right|\right) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.37}$$

So Lemma 2.2 guarantees that $\omega_w(x_n) \subset \operatorname{Fix} V = S$.

(4) We prove that $x_n \to x^*$ as $n \to \infty$, where x^* is the unique solution of the VI (3.12). First observe that there is some $\hat{x} \in \omega_w(x_n) \subset S$ Such that

$$\limsup_{n\to\infty}\langle(\mu F - \gamma h)x^*,\ x_n - x^*\rangle = \langle(\mu F - \gamma h)x^*,\ \hat{x} - x^*\rangle \geq 0. \tag{3.38}$$

We now compute

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\theta_n\gamma h(x_n) + (I - \mu\theta_n F)\operatorname{Proj}_C(I - \lambda_n\nabla f)(x_n) - x^*\right\|^2 \\
&= \left\|\theta_n(\gamma h(x_n) - \mu F x^*) + (I - \mu\theta_n F)V_n(x_n) - (I - \mu\theta_n F)V_n x^*\right\|^2 \\
&= \left\|\theta_n\gamma(h(x_n) - h(x^*)) + (I - \mu\theta_n F)V_n(x_n) - (I - \mu\theta_n F)V_n x^* + \theta_n(\gamma h(x^*) - \mu F x^*)\right\|^2 \\
&\leq \left\|\theta_n\gamma(h(x_n) - h(x^*)) + (I - \mu\theta_n F)V_n(x_n) - (I - \mu\theta_n F)V_n x^*\right\|^2 \\
&\quad + 2\theta_n\langle(\gamma h - \mu F)x^*, x_{n+1} - x^*\rangle \\
&= \left\|\theta_n\gamma(h(x_n) - h(x^*))\right\|^2 + \left\|(I - \mu\theta_n F)V_n(x_n) - (I - \mu\theta_n F)V_n x^*\right\|^2 \\
&\quad + 2\theta_n\gamma\langle h(x_n) - h(x^*), (I - \mu\theta_n F)V_n(x_n) - (I - \mu\theta_n F)V_n x^*\rangle \\
&\quad + 2\theta_n\langle(\gamma h - \mu F)x^*,\ x_{n+1} - x^*\rangle \\
&\leq \theta_n^2\gamma^2\rho^2\|x_n - x^*\|^2 + (1 - \theta_n\tau)^2\|x_n - x^*\|^2 + 2\theta_n\gamma\rho(1 - \theta_n\tau)\|x_n - x^*\|^2 \\
&\quad + 2\theta_n\langle(\gamma h - \mu F)x^*,\ x_{n+1} - x^*\rangle \\
&= \left(\theta_n^2\gamma^2\rho^2 + (1 - \theta_n\tau)^2 + 2\theta_n\gamma\rho(1 - \theta_n\tau)\right)\|x_n - x^*\|^2 \\
&\quad + 2\theta_n\langle(\gamma h - \mu F)x^*,\ x_{n+1} - x^*\rangle \\
&\leq \left(\theta_n\gamma^2\rho^2 + 1 - 2\theta_n\tau + \theta_n\tau^2 + 2\theta_n\gamma\rho\right)\|x_n - x^*\|^2 \\
&\quad + 2\theta_n\langle(\gamma h - \mu F)x^*,\quad x_{n+1} - x^*\rangle \\
&= \left(1 - \theta_n\left(2\tau - \gamma^2\rho^2 - \tau^2 - 2\gamma\rho\right)\right)\|x_n - x^*\|^2 + 2\theta_n\langle(\gamma h - \mu F)x^*,\ x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.39}$$

Applying Lemma 2.1 to the inequality (3.39), together with (3.38), we get $\|x_n - x^*\| \to 0$ as $n \to \infty$. □

Corollary 3.6 (see [11]). *Let* $\{x_n\}$ *be generated by the following algorithm:*

$$x_{n+1} = \theta_n h(x_n) + (1 - \theta_n) Proj_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0. \tag{3.40}$$

Assume that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ *satisfies the conditions* (1.4) *and (iv) and that* $\{\theta_n\} \subset [0,1]$ *satisfies the conditions (i)–(iii). Then* $\{x_n\}$ *converges in norm to* x^* *obtained in* Corollary 3.4.

Corollary 3.7. *Let* $\{x_n\}$ *be generated by the following algorithm:*

$$x_{n+1} = \theta_n \gamma h(x_n) + (I - \theta_n A) Proj_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0. \tag{3.41}$$

Assume that the sequences $\{\theta_n\}$ *and* $\{\lambda_n\}$ *satisfy the conditions contained in* Theorem 3.5, *then* $\{x_n\}$ *converges in norm to* x^* *obtained in* Corollary 3.3.

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