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Research Article

Generalizations of Wendroff Integral Inequalities and Their Discrete Analogues

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Generalizations of Wendroff type integral inequalities with four dependent limits and their discrete analogues are obtained. In applications, these results are used to establish the stability estimates for the solution of the Goursat problem.

1. Introduction

Integral inequalities play a significant role in the theory of ordinary and partial differential equations. They are useful to investigate some properties of the solutions of equations, such as existence, uniqueness, and stability, see for instance [1–6].

Most scientific and technical problems can be solved by using mathematical modelling and new numerical methods. This is based on the mathematical description of real processes and the subsequent solving of the appropriate mathematical problems on the computer. The mathematical models of many scientific and technical problems lead to already known or new problems of partial differential equations. In most of the cases it is difficult to find the exact solutions of the problems for partial differential equations. For this reason discrete methods play a significant role, especially due to the increasing role of mathematical methods of solving problems in various areas of science and engineering. A well-known and widely applied method of approximate solutions for problems of differential equations is the method of difference schemes. Modern computers allow us to implement highly accurate difference schemes. Hence, the task is to construct and investigate highly accurate difference schemes for various types of partial differential equations. The investigation of stability and convergence of these difference schemes is based on the discrete analogues of integral inequalities, see for instance [1, 7–9].

Due to various motivations, several generalizations and applications of Wendroff-type integral inequality have been obtained and used extensively, see for instance [10–12]. In [12], the following generalizations of Wendroff-type integral inequality in two independent variables are obtained.

Theorem 1.1. *Assume that* $u(x, y) \geq 0$ *and* $a(x, y) \geq 0$ *are continuous functions on* $0 \leq x \leq L$, $0 \leq y \leq M$ *and the inequalities*

$$u(x, y) \leq C + \int_0^x \int_0^y a(s, t) u(s, t) dt\, ds, \quad 0 \leq x \leq L,\ 0 \leq y \leq M \tag{1.1}$$

hold, where $C = \text{const} \geq 0$. *Then, for* $u(x, y)$ *the inequalities:*

$$u(x, y) \leq C \exp\left[\int_0^x \int_0^y a(s, t) dt\, ds\right], \quad 0 \leq x \leq L,\ 0 \leq y \leq M \tag{1.2}$$

are satisfied.

Theorem 1.2. *Assume that* $u(x, y) \geq 0$ *is a continuous function on* $0 \leq x \leq L$, $0 \leq y \leq M$ *and the inequalities:*

$$u(x, y) \leq f(x, y) + \int_0^x a(s) u(s, y) ds + \int_0^y b(t) u(x, t) dt, \quad 0 \leq x \leq L,\ 0 \leq y \leq M \tag{1.3}$$

hold, where $f(x, y) > 0$ *is a continuous function on* $0 \leq x \leq L$, $0 \leq y \leq M$ *and increasing with respect to each variable,* $a(x) \geq 0$ *and* $b(y) \geq 0$ *are integrable functions on* $0 \leq x \leq L$ *and* $0 \leq y \leq M$, *respectively. Then, for* $u(x, y)$ *the inequalities:*

$$u(x, y) \leq f(x, y) \exp\left[\int_0^x a(s) ds + \int_0^y b(t) dt + \int_0^x \int_0^y a(s) b(t) dt\, ds\right], \quad 0 \leq x \leq L,\ 0 \leq y \leq M \tag{1.4}$$

are satisfied.

In this paper, generalizations of Wendroff-type integral inequalities in two independent variables with four dependent limits and their discrete analogues are obtained. In applications, these results are used to obtain the stability estimates of solutions for the Goursat problem.

2. Wendroff-type Integral Inequalities with Four Dependent Limits and Their Discrete Analogues

First of all, let us give the discrete analogue of the Gronwall-type integral inequality with two dependent limits. We will need this result in the remaining part of the paper.

Theorem 2.1. *Assume that $v_i \geq 0$, $a_i \geq 0$, $\delta_i \geq 0$, $i = -N, \ldots, N$ are the sequences of real numbers and the inequalities:*

$$v_i \leq \delta_i + h \sum_{\substack{j=-|i|+1 \\ j \neq 0}}^{|i|-1} a_j v_j, \quad i = -N, \ldots, N \tag{2.1}$$

hold. Then, for v_i the inequalities:

$$v_i \leq \delta_i + \exp\left[h \sum_{\substack{j=-|i|+1 \\ j \neq 0}}^{|i|-1} a_j\right] h \sum_{\substack{j=-|i|+1 \\ j \neq 0}}^{|i|-1} a_j \delta_j \exp\left[-h \sum_{\substack{n=-|j| \\ n \neq 0}}^{|j|} a_n\right], \quad i = -N, \ldots, N \tag{2.2}$$

are satisfied.

Proof. The proof of (2.2) for $i = -1, 0, 1$ follows directly from (2.1). Let us prove (2.2) for $i = -N, \ldots, -2, 2, \ldots, N$. We denote

$$y_i = h \sum_{\substack{j=-|i|+1 \\ j \neq 0}}^{|i|-1} a_j v_j, \quad i = -N, \ldots, N. \tag{2.3}$$

Then, (2.1) gets the form

$$v_i \leq \delta_i + y_i, \quad i = -N, \ldots, N. \tag{2.4}$$

Moreover, we have

$$y_{-i} = y_i, \quad i = -N, \ldots, N. \tag{2.5}$$

Then, using (2.3)–(2.5) for $i = 1, \ldots, N-1$, we obtain

$$\begin{aligned} y_{i+1} - y_i &= h(a_i v_i + a_{-i} v_{-i}) \leq h a_i (y_i + \delta_i) + h a_{-i} (y_{-i} + \delta_{-i}) \\ &= h(a_i + a_{-i}) y_i + h(a_i \delta_i + a_{-i} \delta_{-i}). \end{aligned} \tag{2.6}$$

So,

$$y_{i+1} \leq [1 + h(a_i + a_{-i})] y_i + h(a_i \delta_i + a_{-i} \delta_{-i}), \quad i = 1, \ldots, N-1. \tag{2.7}$$

Then by induction, we can prove that

$$y_i \le \prod_{j=1}^{i-1}\left[1 + h\left(a_j + a_{-j}\right)\right] y_1 + \sum_{j=1}^{i-2} h\left(a_j\delta_j + a_{-j}\delta_{-j}\right) \prod_{n=j+1}^{i-1}\left[1 + h(a_n + a_{-n})\right] + h(a_{i-1}\delta_{i-1} + a_{-i+1}\delta_{-i+1}), \tag{2.8}$$

for $i = 2,\dots,N$. Since $y_1 = 0$, using (2.4) and the inequality $1 + x < \exp[x]$, $x > 0$, we obtain

$$\begin{aligned}
v_i &\le \delta_i + h\sum_{j=1}^{i-2}\left(a_j\delta_j + a_{-j}\delta_{-j}\right)\exp\left[h\sum_{n=j+1}^{i-1}(a_n + a_{-n})\right] + h(a_{i-1}\delta_{i-1} + a_{-i+1}\delta_{-i+1}) \\
&= \delta_i + \exp\left[h\sum_{n=1}^{i-1}(a_n + a_{-n})\right] h\sum_{j=1}^{i-2}\left(a_j\delta_j + a_{-j}\delta_{-j}\right)\exp\left[-h\sum_{n=1}^{j}(a_n + a_{-n})\right] \\
&\quad + h(a_{i-1}\delta_{i-1} + a_{-i+1}\delta_{-i+1}) \\
&= \delta_i + \exp\left[h\sum_{j=1}^{i-1}\left(a_j + a_{-j}\right)\right] h\sum_{j=1}^{i-1}\left(a_j\delta_j + a_{-j}\delta_{-j}\right)\exp\left[-h\sum_{n=1}^{j}(a_n + a_{-n})\right] \\
&= \delta_i + \exp\left[h\sum_{\substack{j=-i+1 \\ j\neq 0}}^{i-1} a_j\right] h\sum_{j=1}^{i-1}\left(a_j\delta_j + a_{-j}\delta_{-j}\right)\exp\left[-h\sum_{\substack{n=-j \\ n\neq 0}}^{j} a_n\right] \\
&= \delta_i + \exp\left[h\sum_{\substack{j=-i+1 \\ j\neq 0}}^{i-1} a_j\right] h\sum_{\substack{j=-i+1 \\ j\neq 0}}^{i-1} a_j\delta_j\exp\left[-h\sum_{\substack{n=-|j| \\ n\neq 0}}^{|j|} a_n\right].
\end{aligned} \tag{2.9}$$

So, we proved (2.2) for $i = 2,\dots,N$.

Let us prove (2.2) for $i = -N,\dots,-2$. Using (2.3)–(2.5) for $i = -N+1,\dots,-1$, we have

$$\begin{aligned}
y_{i-1} - y_i &= h(a_i v_i + a_{-i}v_{-i}) \le h a_i\left(y_i + \delta_i\right) + h a_{-i}\left(y_{-i} + \delta_{-i}\right) \\
&= h(a_i + a_{-i})y_i + h(a_i\delta_i + a_{-i}\delta_{-i}).
\end{aligned} \tag{2.10}$$

So,

$$y_{i-1} \le \left[1 + h(a_i + a_{-i})\right] y_i + h(a_i\delta_i + a_{-i}\delta_{-i}), \quad i = -N+1,\dots,-1. \tag{2.11}$$

Then by induction, we can prove that

$$
\begin{aligned}
y_i \leq{}& \prod_{n=1}^{-i-1}[1+h(a_n+a_{-n})]y_{-1}+\sum_{j=1}^{-i-2}h\left(a_j\delta_j+a_{-j}\delta_{-j}\right)\prod_{n=j+1}^{-i-1}[1+h(a_n+a_{-n})]\\
&+h(a_{-i-1}\delta_{-i-1}+a_{i+1}\delta_{i+1}),
\end{aligned}
\tag{2.12}
$$

for $i=-N,\ldots,-2$. Since $y_{-1}=0$, using (2.4) and the inequality $1+x<\exp[x]$, $x>0$, we obtain

$$
\begin{aligned}
v_i &\leq \delta_i+h\sum_{j=1}^{-i-2}\left(a_j\delta_j+a_{-j}\delta_{-j}\right)\exp\left[h\sum_{n=j+1}^{-i-1}(a_n+a_{-n})\right]+h(a_{-i-1}\delta_{-i-1}+a_{i+1}\delta_{i+1})\\
&=\delta_i+\exp\left[h\sum_{n=1}^{-i-1}(a_n+a_{-n})\right]h\sum_{j=1}^{-i-2}\left(a_j\delta_j+a_{-j}\delta_{-j}\right)\exp\left[-h\sum_{n=1}^{j}(a_n+a_{-n})\right]\\
&\quad+h(a_{-i-1}\delta_{-i-1}+a_{i+1}\delta_{i+1})\\
&=\delta_i+\exp\left[h\sum_{j=1}^{-i-1}(a_j+a_{-j})\right]h\sum_{j=1}^{-i-1}\left(a_j\delta_j+a_{-j}\delta_{-j}\right)\exp\left[-h\sum_{n=1}^{j}(a_n+a_{-n})\right]\\
&=\delta_i+\exp\left[h\sum_{\substack{j=i+1\\ j\neq 0}}^{-i-1}a_j\right]h\sum_{j=1}^{-i-1}\left(a_j\delta_j+a_{-j}\delta_{-j}\right)\exp\left[-h\sum_{\substack{n=-j\\ n\neq 0}}^{j}a_n\right]\\
&=\delta_i+\exp\left[h\sum_{\substack{j=i+1\\ j\neq 0}}^{-i-1}a_j\right]h\sum_{\substack{j=i+1\\ j\neq 0}}^{-i-1}a_j\delta_j\exp\left[-h\sum_{\substack{n=-|j|\\ n\neq 0}}^{|j|}a_n\right].
\end{aligned}
\tag{2.13}
$$

So, we proved (2.2) for $i=-N,\ldots,-2$. □

By putting $Nh=1$ and passing to limit $h\to 0$ in the Theorem 2.1, we obtain the following generalization of Gronwall's integral inequality with two dependent limits.

Theorem 2.2. *Assume that $v(t)\geq 0$, $\delta(t)\geq 0$ are continuous functions on $[-1,1]$, $a(t)\geq 0$ is an integrable function on $[-1,1]$, and the inequalities*

$$
v(t)\leq\delta(t)+\operatorname{sgn}(t)\int_{-t}^{t}a(s)v(s)ds,\quad -1\leq t\leq 1 \tag{2.14}
$$

hold. Then for $v(t)$, the inequalities:

$$v(t) \leq \delta(t) + \exp\left[\operatorname{sgn}(t) \int_{-t}^{t} a(s)ds\right] \operatorname{sgn}(t) \int_{-t}^{t} a(s)\delta(s) \exp\left[-\operatorname{sgn}(s) \int_{-s}^{s} a(\tau)d\tau\right] ds, \quad -1 \leq t \leq 1 \tag{2.15}$$

are satisfied.

Now, we consider the generalizations of Wendroff inequality for integrals in two independent variables with four dependent limits and their discrete analogues.

Theorem 2.3. *Assume that $v_n^k \geq 0$, $a_n^k \geq 0$, $n = -N, \ldots, N$, $k = -K, \ldots, K$ are sequences of real numbers, and the inequalities:*

$$v_n^k \leq C + h_1 h_2 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} a_s^\tau v_s^\tau, \quad n = -N, \ldots, N, \ k = -K, \ldots, K \tag{2.16}$$

hold, where $C = \text{const} > 0$, $h_1 = \text{const} > 0$, $h_2 = \text{const} > 0$. *Then for v_n^k, the following inequalities:*

$$v_n^k \leq C \exp\left[h_1 h_2 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} a_s^\tau\right], \quad n = -N, \ldots, N, \ k = -K, \ldots, K. \tag{2.17}$$

are satisfied.

Proof. The proof of (2.17) for $n = -1, 0, 1$, $k = -K, \ldots, K$ and $k = -1, 0, 1$, $n = -N, \ldots, N$ follows directly from (2.16). Let us prove (2.17) for $n = -N, \ldots, -2, 2, \ldots, N$, $k = -K, \ldots, -2, 2, \ldots, K$. We denote

$$\omega_n^k = C + h_1 h_2 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} a_s^\tau v_s^\tau, \quad n = -N, \ldots, N, \ k = -K, \ldots, K. \tag{2.18}$$

Then, (2.16) gets the form:

$$v_n^k \leq \omega_n^k, \quad n = -N, \ldots, N, \ k = -K, \ldots, K. \tag{2.19}$$

Furthermore, we have

$$\omega_n^k = \omega_{-n}^k = \omega_n^{-k} = \omega_{-n}^{-k}, \quad n = -N, \ldots, N, \ k = -K, \ldots, K. \tag{2.20}$$

From (2.18) for $n = -N, \ldots, N$, $k = 1, \ldots, K-1$, we have

$$\omega_n^{k+1} - \omega_n^k = h_1 h_2 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} \left(a_s^k v_s^k + a_s^{-k} v_s^{-k} \right) \geq 0. \tag{2.21}$$

So,

$$\omega_n^k \leq \omega_n^{k+1}, \quad n = -N, \ldots, N, \; k = 1, \ldots, K-1. \tag{2.22}$$

Then using (2.18)–(2.22) for $n = 1, \ldots, N-1$, $k = -K, \ldots, -2, 2, \ldots, K$, we obtain

$$\begin{aligned}
\omega_{n+1}^k - \omega_n^k &= h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} \left(a_n^\tau v_n^\tau + a_{-n}^\tau v_{-n}^\tau \right) \leq h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} \left(a_n^\tau \omega_n^\tau + a_{-n}^\tau \omega_{-n}^\tau \right) \\
&= h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} \omega_n^\tau \left(a_n^\tau + a_{-n}^\tau \right) = h_1 h_2 \sum_{\tau=1}^{|k|-1} \omega_n^\tau \left(a_n^\tau + a_{-n}^\tau \right) + h_1 h_2 \sum_{\tau=1}^{|k|-1} \omega_n^{-\tau} \left(a_n^{-\tau} + a_{-n}^{-\tau} \right) \\
&= h_1 h_2 \sum_{\tau=1}^{|k|-1} \omega_n^\tau \left(a_n^\tau + a_{-n}^\tau + a_n^{-\tau} + a_{-n}^{-\tau} \right) \leq h_1 h_2 \omega_n^{|k|} \sum_{\tau=1}^{|k|-1} \left(a_n^\tau + a_{-n}^\tau + a_n^{-\tau} + a_{-n}^{-\tau} \right) \\
&= h_1 h_2 \omega_n^k \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} \left(a_n^\tau + a_{-n}^\tau \right).
\end{aligned} \tag{2.23}$$

So,

$$\omega_{n+1}^k \leq \omega_n^k \left[1 + h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} \left(a_n^\tau + a_{-n}^\tau \right) \right], \quad n = 1, \ldots, N-1, \; k = -K, \ldots, -2, 2, \ldots, K. \tag{2.24}$$

Then by induction, we can prove that

$$\omega_n^k \leq \omega_1^k \prod_{s=1}^{n-1} \left[1 + h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} \left(a_s^\tau + a_{-s}^\tau \right) \right], \quad n = 2, \ldots, N, \; k = -K, \ldots, -2, 2, \ldots, K. \tag{2.25}$$

Since $\omega_1^k = C$, $k = -K, \dots, K$, using (2.19) and the inequality $1 + x < \exp[x]$, $x > 0$, we obtain

$$v_n^k \le \omega_n^k \le C \exp\left[h_1 h_2 \sum_{s=1}^{n-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} \left(a_s^\tau + a_{-s}^\tau\right) \right] = C \exp\left[h_1 h_2 \sum_{\substack{s=-n+1 \\ s \ne 0}}^{n-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} a_s^\tau \right]. \tag{2.26}$$

So, we proved (2.17) for $n = 2, \dots, N$, $k = -K, \dots, -2, 2, \dots, K$.

Using (2.18)–(2.22) for $n = -N + 1, \dots, -1$, $k = -K, \dots, -2, 2, \dots, K$, we obtain

$$\begin{aligned}
\omega_{n-1}^k - \omega_n^k &= h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} \left(a_n^\tau v_n^\tau + a_{-n}^\tau v_{-n}^\tau\right) \le h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} \left(a_n^\tau \omega_n^\tau + a_{-n}^\tau \omega_{-n}^\tau\right) \\
&= h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} \omega_n^\tau \left(a_n^\tau + a_{-n}^\tau\right) = h_1 h_2 \sum_{\tau=1}^{|k|-1} \omega_n^\tau \left(a_n^\tau + a_{-n}^\tau\right) + h_1 h_2 \sum_{\tau=1}^{|k|-1} \omega_n^{-\tau} \left(a_n^{-\tau} + a_{-n}^{-\tau}\right) \\
&= h_1 h_2 \sum_{\tau=1}^{|k|-1} \omega_n^\tau \left(a_n^\tau + a_{-n}^\tau + a_n^{-\tau} + a_{-n}^{-\tau}\right) \le h_1 h_2 \omega_n^{|k|} \sum_{\tau=1}^{|k|-1} \left(a_n^\tau + a_{-n}^\tau + a_n^{-\tau} + a_{-n}^{-\tau}\right) \\
&= h_1 h_2 \omega_n^k \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} \left(a_n^\tau + a_{-n}^\tau\right).
\end{aligned} \tag{2.27}$$

So,

$$\omega_{n-1}^k \le \omega_n^k \left[1 + h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} \left(a_n^\tau + a_{-n}^\tau\right) \right], \quad n = -N + 1, \dots, -1, \; k = -K, \dots, -2, 2, \dots, K. \tag{2.28}$$

Then by induction, we can prove that

$$\omega_n^k \le \omega_{-1}^k \prod_{s=1}^{-n-1} \left[1 + h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \ne 0}}^{|k|-1} \left(a_s^\tau + a_{-s}^\tau\right) \right], \quad n = -N, \dots, -2, \; k = -K, \dots, -2, 2, \dots, K. \tag{2.29}$$

Since $\omega_{-1}^k = C$, $k = -K, \dots, K$ using (2.19) and the inequality $1 + x < \exp[x]$, $x > 0$, we obtain

$$v_n^k \le \omega_n^k \le C \exp\left[h_1 h_2 \sum_{s=1}^{-n-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} \left(a_s^\tau + a_{-s}^\tau \right) \right] = C \exp\left[h_1 h_2 \sum_{\substack{s=n+1 \\ s \neq 0}}^{-n-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} a_s^\tau \right]. \tag{2.30}$$

So, we proved (2.17) for $n = -N, \dots, -2$, $k = -K, \dots, -2, 2, \dots, K$. □

Theorem 2.4. *Assume that $v_n^k \ge 0$, $\delta_n^k > 0$, $a_n \ge 0$, $b_k \ge 0$, $n = -N, \dots, N$, $k = -K, \dots, K$ are sequences of real numbers, and the inequalities:*

$$v_n^k \le \delta_n^k + h_1 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} a_s v_s^k + h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} b_\tau v_n^\tau, \quad n = -N, \dots, N, \ k = -K, \dots, K, \tag{2.31}$$

$$\begin{aligned} &\delta_n^k \le \delta_{n+1}^k, \quad n = -N, \dots, N-1, \ k = -K, \dots, K, \\ &\delta_n^k \le \delta_n^{k+1}, \quad n = -N, \dots, N, \ k = -K, \dots, K-1 \end{aligned} \tag{2.32}$$

hold, where $h_1 = \text{const} > 0$, $h_2 = \text{const} > 0$. Then for v_n^k, $n = -N, \dots, N$, $k = -K, \dots, K$, the inequalities:

$$v_n^k \le \delta_n^k \exp\left[h_1 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} a_s + h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} b_\tau + h_1 h_2 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} a_s b_\tau \right] \tag{2.33}$$

are satisfied.

Proof. We denote $u_n^k = v_n^k / \delta_n^k$, $n = -N, \dots, N$, $k = -K, \dots, K$. Then, (2.31) takes the form

$$u_n^k \le 1 + h_1 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} a_s u_s^k + h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} b_\tau u_n^\tau, \quad n = -N, \dots, N, \ k = -K, \dots, K. \tag{2.34}$$

Next, by denoting

$$T_n^k = 1 + h_1 \sum_{\substack{s=-|n|+1 \\ s \neq 0}}^{|n|-1} a_s u_s^k, \quad n = -N, \dots, N, \ k = -K, \dots, K, \tag{2.35}$$

we have

$$u_n^k \le T_n^k + h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau \neq 0}}^{|k|-1} b_\tau u_n^\tau, \quad n = -N, \dots, N, \ k = -K, \dots, K. \tag{2.36}$$

By using Theorem 2.1, we obtain

$$u_n^k \le T_n^k + \exp\left[h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau\right] h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau T_n^\tau \exp\left[-h_2 \sum_{\substack{j=-|\tau| \\ j\neq 0}}^{|\tau|} b_j\right]. \tag{2.37}$$

Inserting (2.35) yields

$$\begin{aligned}
u_n^k &\le 1 + h_1 \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s u_s^k + \exp\left[h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau\right] h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau \exp\left[-h_2 \sum_{\substack{j=-|\tau| \\ j\neq 0}}^{|\tau|} b_j\right] \\
&\quad + \exp\left[h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau\right] h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s b_\tau u_s^\tau \exp\left[-h_2 \sum_{\substack{j=-|\tau| \\ j\neq 0}}^{|\tau|} b_j\right] \\
&\le \exp\left[h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau\right] + h_1 \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s u_s^k \\
&\quad + \exp\left[h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau\right] h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s b_\tau u_s^\tau \exp\left[-h_2 \sum_{\substack{j=-|\tau| \\ j\neq 0}}^{|\tau|} b_j\right].
\end{aligned} \tag{2.38}$$

Now, by denoting

$$w_n^k = u_n^k \exp\left[-h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} b_\tau\right], \quad n = -N, \ldots, N, \ k = -K, \ldots, K, \tag{2.39}$$

we have

$$w_n^k \le 1 + h_1 \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s w_s^k + h_1 h_2 \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s b_\tau w_s^\tau. \tag{2.40}$$

Let us denote the right-hand side of (2.40) by R_n^k. Then, (2.40) gets the form

$$w_n^k \le R_n^k, \quad n = -N, \ldots, N, \ k = -K, \ldots, K. \tag{2.41}$$

Then by using induction, we can prove that

$$R_n^k \exp\left[-h_1 \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s\right] \leq 1 + h_1 h_2 \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} a_s b_\tau R_s^\tau \exp\left[-h_1 \sum_{\substack{j=-|s|+1 \\ j\neq 0}}^{|s|-1} a_j\right]. \tag{2.42}$$

By using Theorem 2.3, we obtain

$$R_n^k \exp\left[-h_1 \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} a_s\right] \leq \exp\left[h_1 h_2 \sum_{\substack{s=-|n|+1 \\ s\neq 0}}^{|n|-1} \sum_{\substack{\tau=-|k|+1 \\ \tau\neq 0}}^{|k|-1} a_s b_\tau\right], \quad n=-N,\ldots,N,\ k=-K,\ldots,K. \tag{2.43}$$

Finally, by combining (2.39), (2.41), and (2.43), we finish the proof of (2.33). □

By putting $Nh_1 = L$, $Kh_2 = M$ and passing to limit as $h_1 \to 0$, $h_2 \to 0$ in Theorems 2.3 and 2.4, we obtain the following two theorems about the generalizations of Wendroff integral inequality with four dependent limits.

Theorem 2.5. *Assume that $v(x,y) \geq 0$ and $b(x,y) \geq 0$ are continuous functions on $-L \leq x \leq L$, $-M \leq y \leq M$ and the inequalities*

$$v(x,y) \leq C + \operatorname{sgn}(xy) \int_{-x}^{x} \int_{-y}^{y} v(s,t) b(s,t)\, dt\, ds, \quad -L \leq x \leq L,\ -M \leq y \leq M \tag{2.44}$$

hold, where C = const ≥ 0. *Then for $v(x,y)$ the inequalities*

$$v(x,y) \leq C \exp\left[\operatorname{sgn}(xy) \int_{-x}^{x} \int_{-y}^{y} b(s,t)\, dt\, ds\right], \quad -L \leq x \leq L,\ -M \leq y \leq M \tag{2.45}$$

are satisfied.

Theorem 2.6. *Assume that $v(x,y) \geq 0$ is a continuous function on $-L \leq x \leq L$, $-M \leq y \leq M$, and the inequalities:*

$$\begin{aligned} v(x,y) \leq{} & f(x,y) + \operatorname{sgn}(x) \int_{-x}^{x} v(s,y) a(s)\, ds \\ & + \operatorname{sgn}(y) \int_{-y}^{y} v(x,t) b(t)\, dt, \quad -L \leq x \leq L,\ -M \leq y \leq M \end{aligned} \tag{2.46}$$

hold, where $f(x,y) > 0$ is a continuous function on $-L \le x \le L$, $-M \le y \le M$ and increasing with respect to each variable, $a(x) \ge 0$ and $b(y) \ge 0$ are integrable functions on $-L \le x \le L$ and $-M \le y \le M$, respectively. Then for $v(x,y)$, the inequalities

$$v(x,y) \le f(x,y) \exp\left[\operatorname{sgn}(x) \int_{-x}^{x} a(s)ds + \operatorname{sgn}(y) \int_{-y}^{y} b(t)dt + \operatorname{sgn}(xy) \int_{-x}^{x} \int_{-y}^{y} a(s)b(t)ds\,dt\right] \tag{2.47}$$

are satisfied.

Finally, we formulate (without proofs) the generalizations of Wendroff-type inequalities for the integrals in three independent variables with six dependent limits.

Theorem 2.7. *Assume that $v(x,y,z) \ge 0$, $b(x,y,z) \ge 0$ $(-l_1 \le x \le l_1,\ -l_2 \le y \le l_2,\ -l_3 \le z \le l_3)$ are continuous functions, and the inequalities:*

$$v(x,y,z) \le C + \operatorname{sgn}(xyz) \int_{-x}^{x} \int_{-y}^{y} \int_{-z}^{z} v(s,t,\tau) b(s,t,\tau) d\tau\, dt\, ds \tag{2.48}$$

hold, where $C = \text{const} \ge 0$. Then for $v(x,y,z)$, the inequalities

$$\begin{gathered} v(x,y,z) \le C \exp\left[\operatorname{sgn}(xyz) \int_{-x}^{x} \int_{-y}^{y} \int_{-z}^{z} b(s,t,\tau) d\tau\, dt\, ds\right], \\ -l_1 \le x \le l_1, \quad -l_2 \le y \le l_2, \quad -l_3 \le z \le l_3 \end{gathered} \tag{2.49}$$

are satisfied.

Theorem 2.8. *Assume that $v(x,y,z) \ge 0$ $(-l_1 \le x \le l_1,\ -l_2 \le y \le l_2,\ -l_3 \le z \le l_3)$ is a continuous function, and the inequalities:*

$$\begin{aligned} v(x,y,z) \le \delta(x,y,z) &+ \operatorname{sgn}(x) \int_{-x}^{x} v(s,y,z) a(s) ds + \operatorname{sgn}(y) \int_{-y}^{y} v(x,t,z) b(t) dt \\ &+ \operatorname{sgn}(z) \int_{-z}^{z} v(s,t,\tau) c(\tau) d\tau \end{aligned} \tag{2.50}$$

hold, where $a(x) \ge 0$ $(-l_1 \le x \le l_1)$, $b(y) \ge 0$ $(-l_2 \le y \le l_2)$, and $c(z) \ge 0$ $(-l_3 \le z \le l_3)$ are integrable functions, $\delta(x,y,z) > 0$ is continuous and increasing with respect to each variable. Then for $v(x,y,z)$, the inequalities:

$$\begin{gathered} v(x,y,z) \le \delta(x,y,z) \exp\left[T(x,y,z) + \sqrt[3]{\operatorname{sgn}(xyz) \int_{-x}^{x} \int_{-y}^{y} \int_{-z}^{z} a(s)b(t)c(\tau) d\tau\, dt\, ds\, e^{T(x,y,z)}}\right] \\ -l_1 \le x \le l_1, \quad -l_2 \le y \le l_2, \quad -l_3 \le z \le l_3 \end{gathered} \tag{2.51}$$

are satisfied, where

$$T(x,y,z) = \operatorname{sgn}(x)\int_{-x}^{x} a(s)ds + \operatorname{sgn}(y)\int_{-y}^{y} b(t)dt + \operatorname{sgn}(z)\int_{-z}^{z} c(\tau)d\tau$$
$$+ \operatorname{sgn}(xy)\int_{-x}^{x}\int_{-y}^{y} a(s)b(t)dt\,ds + \operatorname{sgn}(xz)\int_{-x}^{x}\int_{-z}^{z} a(s)c(\tau)d\tau\,ds \tag{2.52}$$
$$+ \operatorname{sgn}(yz)\int_{-y}^{y}\int_{-z}^{z} b(t)c(\tau)d\tau\,dt.$$

3. Applications

In applications, we consider the Goursat problem for hyperbolic equations:

$$u_{xy} = f(x,y) + \frac{\partial}{\partial y}\left(a(x,y)u(x,y) + a(-x,y)u(-x,y)\right)$$
$$+ \frac{\partial}{\partial x}\left(b(x,y)u(x,y) + b(x,-y)u(x,-y)\right), \quad -l_1 < x < l_1,\ -l_2 < y < l_2, \tag{3.1}$$
$$u(x,0) = \phi(x), \quad -l_1 \le x \le l_1,$$
$$u(0,y) = \varphi(y), \quad -l_2 \le y \le l_2.$$

A function $u(x,y)$ is called a solution of the Goursat problem (3.1) if the following conditions are satisfied:$u(x,y)$ is twice continuously differentiable on the region $[-l_1,l_1] \times [-l_2,l_2]$, and the derivatives at the endpoints are understood as the appropriate unilateral derivatives;

Theorem 3.1. *Assume that the functions $\phi(x)$ and $\varphi(y)$ are continuously differentiable and $\phi(0) = \varphi(0)$. Let $a(x,y)$, $b(x,y)$, and $f(x,y)$ be continuously differentiable functions. Then, there is a unique solution of the problem* (3.1) *and the stability inequalities:*

$$|u(x,y)| \le \left(l_1 l_2 f + \frac{3}{2}(\phi+\varphi) + 2al_1\phi + 2bl_2\varphi\right)e^{2a|x|+2b|y|+4ab|xy|},$$
$$|u_x(x,y)|, |u_y(x,y)| \le M_1, \qquad |u_{xy}(x,y)| \le M_2 \tag{3.2}$$

hold, where M_1 and M_2 do not depend on x and y and

$$f = \max_{\substack{|x|\le l_1\\ |y|\le l_2}}|f(x,y)|, \qquad a = \max_{\substack{|x|\le l_1\\ |y|\le l_2}}|a(x,y)|, \qquad b = \max_{\substack{|x|\le l_1\\ |y|\le l_2}}|b(x,y)|,$$
$$\phi = \max_{|x|\le l_1}|\phi(x)|, \qquad \varphi = \max_{|y|\le l_2}|\varphi(y)|. \tag{3.3}$$

The proof of this theorem is based on the formula:

$$u(x,y) = \phi(x) + \psi(y) - \phi(0) - \int_{-x}^{x} a(s,0)\phi(s)ds - \int_{-y}^{y} b(0,t)\psi(t)dt + \int_{0}^{x}\int_{0}^{y} f(s,t)dt\,ds$$
$$+ \int_{-x}^{x} a(s,y)u(s,y)ds + \int_{-y}^{y} b(x,t)u(x,t)dt, \quad -l_1 \le x \le l_1,\ -l_2 \le y \le l_2 \tag{3.4}$$

and on the Theorem 2.6.

Now, we consider the difference schemes for approximate solutions of problem (3.1):

$$\begin{aligned}
\frac{u^{k}_{\overline{y},n} - u^{k-1}_{\overline{y},n}}{h_1} &= \frac{a^{k}_{n}u^{k}_{n} - a^{k-1}_{n}u^{k-1}_{n}}{h_2} + \frac{a^{k}_{-n}u^{k}_{-n} - a^{k-1}_{-n}u^{k-1}_{-n}}{h_2} + \frac{b^{k}_{n}u^{k}_{n} - b^{k}_{n-1}u^{k}_{n-1}}{h_1}\\
&\quad + \frac{b^{-k}_{n}u^{-k}_{n} - b^{-k}_{n-1}u^{-k}_{n-1}}{h_1} + f^{k}_{n}, \quad -N+1 \le n \le N,\ -K+1 \le k \le K,\\
a^{k}_{n} &= a(x_n, y_k), \quad b^{k}_{n} = b(x_n, y_k), \quad f^{k}_{n} = f(x_n, y_k), \quad -N \le n \le N,\ -K \le k \le K,\\
x_n &= nh_1, \quad -N \le n \le N, \quad Nh_1 = l_1, \quad y_k = kh_2, \quad -K \le k \le K, \quad Kh_2 = l_2,\\
u^{0}_{n} &= \phi(x_n), \quad -N \le n \le N, \qquad u^{k}_{0} = \psi(y_k), \quad -K \le k \le K,
\end{aligned} \tag{3.5}$$

where $u^{k}_{\overline{y},n} = (u^{k}_{n} - u^{k}_{n-1})/h_2$.

Theorem 3.2. *For the solution of difference schemes* (3.5)*, the following estimates are satisfied:*

$$\max_{\substack{-N\le n\le N\\ -K\le k\le K}} \left|u^{k}_{n}\right| \le M\left[\max_{\substack{-N+1\le n\le N\\ -K+1\le k\le K}}\left|f^{k}_{n}\right| + \max_{-N\le n\le N}\left|\phi_n\right| + \max_{-K\le k\le K}\left|\psi_k\right|\right], \tag{3.6}$$

where M does not depend on h_1, h_2, f^{k}_{n}, ϕ_n, $\psi_k(-N \le n \le N,\ -K \le k \le K)$.

The proof of this theorem is based on the following formula:

$$u^{k}_{n} = \phi_n + \psi_k - \phi_0 - h_1\sum_{\substack{s=-n\\ s\ne 0}}^{n} a^{0}_{s}\phi_s - h_2\sum_{\substack{\tau=-k\\ \tau\ne 0}}^{k} b^{\tau}_{0}\psi_\tau + h_1h_2\sum_{s=1}^{n}\sum_{\tau=1}^{k} f^{\tau}_{s}$$
$$+ h_1\sum_{\substack{s=-n\\ s\ne 0}}^{n} a^{k}_{s}u^{k}_{s} + h_2\sum_{\substack{\tau=-k\\ \tau\ne 0}}^{k} b^{\tau}_{n}u^{\tau}_{n}, \quad -N \le n \le N,\ -K \le k \le K \tag{3.7}$$

and on Theorem 2.4.

4. Conclusion

In this paper, generalizations of Wendroff-type inequalities for the integrals in two independent variables with four dependent limits and their discrete analogues are studied. The generalizations of Wendroff-type integral inequalities are used to establish stability estimates for the solution of the Goursat problem. A difference scheme approximately solving the Goursat problem is presented. Bu using the discrete analogues of the generalizations of Wendroff-type integral inequalities, stability estimates for the solution of this difference scheme are established.

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