Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 763728, 14 pages
doi:10.1155/2012/763728

Research Article

On the Stability Problem in Fuzzy Banach Space

G. Zamani Eskandani,[1] P. Găvruţa,[2] and Gwang Hui Kim[3]

[1] *Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran*
[2] *Department of Mathematics, University Politehnica of Timisoara, Piata Victoriei 2, 300006 Timisoara, Romania*
[3] *Department of Mathematics, Kangnam University, Suwon, Kyunggi 449-702, Republic of Korea*

Correspondence should be addressed to Gwang Hui Kim, ghkim@kangnam.ac.kr

Received 6 February 2012; Accepted 17 May 2012

Academic Editor: Nicole Brillouet-Belluot

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We investigate the generalized Ulam-Hyers stability of the Cauchy functional equation and pose two open problems in fuzzy Banach space.

1. Introduction and Preliminaries

In 1940, Ulam [1] asked the first question on the stability problem. In 1941, Hyers [2] solved the problem of Ulam. This result was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an *unbounded Cauchy difference*.

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality:*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \epsilon\left(\|x\|^p + \|y\|^p \right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then, the limit $L(x) = \lim_{n\to\infty}(1/2^n) f(2^n x)$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\left\| f(x) - L(x) \right\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

In 1990, Th. M. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Th. M. Rassias and Šemrl [7], that one cannot prove a Th. M. Rassias type theorem when $p = 1$. Găvruţa [8] proved that the function $f(x) = x \ln|x|$, if $x \neq 0$ and $f(0) = 0$ satisfies (1.1) with $\epsilon = p = 1$ but

$$\sup_{x\neq 0}\frac{|f(x)-A(x)|}{|x|} \geq \sup_{n\in\mathbb{N}}\frac{|n\ln n - A(n)|}{n} = \sup_{n\in\mathbb{N}}|\ln n - A(1)| = \infty \tag{1.3}$$

for any additive function $A : \mathbb{R} \to \mathbb{R}$. J. M. Rassias [9] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^{p_1}\|y\|^{p_2}$ for $p_1, p_2 \in \mathbb{R}$ with $p_1 + p_2 \neq 1$ (see also [10, 11]) and has obtained the following theorem.

Theorem 1.2. *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p = p_1 + p_2 \neq 1$ such that f satisfies the inequality:*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\|x\|^{p_1}\|y\|^{p_2} \tag{1.4}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}\|x\|^p \tag{1.5}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

In the case $p = 1$, we do not have stability [12]. In 1994, a further generalization of Th. M. Rassias' Theorem was obtained by Găvruţa [13], in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Isac and Th. M. Rassias [14] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^{p_1} + \|y\|^{p_2}$ in Theorem 1.1 and solved stability problem when $p_2 \leq p_1 < 1$ or $1 < p_2 \leq p_1$, also they asked the question whether such a theorem can be proved for $p_2 < 1 < p_1$. Găvruţa [8] gave a negative answer to this question. Isac and Th. M. Rassias [15] applied the Ulam-Hyers-Rassias stability theory to prove fixed point theorems and study some new applications in nonlinear analysis. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Ulam-Hyers stability to a number of functional equations and mappings (see [16–40]). We also refer the readers to the books of Czerwik [41] and Hyers et al. [42].

Th. M. Rassias [43] has obtained the following theorem and posed a problem.

Theorem 1.3. *Let E_1 and E_2 be two Banach spaces, and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.6}$$

for all $x, y \in X$. Let k be a positive integer $k > 2$. Then, there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{k\theta}{k - k^p} \|x\|^p s(k,p) \tag{1.7}$$

for all $x \in X$, where

$$s(k,p) = 1 + \frac{1}{k} \sum_{m=2}^{k-1} m^p. \tag{1.8}$$

Th. M. Rassias Problem

What is the best possible value of k in Theorem 1.3?

Găvruţa et al. have given a generalization of [13] and have answered to Th. M. Rassias problem [44].

In [45], J. M. Rassias et al. have investigated the generalized Ulam-Hyers "product-sum" stability of functional equations and have obtained the following theorem.

Theorem 1.4 (see [45]). *Let $f : E \to F$ be a mapping which satisfies the inequality*

$$\begin{aligned} &\left\| f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2\left(m^2 - 2\right) f(x) + 2f(y) \right\|_F \\ &\quad \leq \epsilon\left(\|x\|_E^p \|y\|_E^p + \|x\|_E^{2p} + \|y\|_E^{2p} \right) \end{aligned} \tag{1.9}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either $m > 1, p < 1$ or $m < 1$, $p > 1$ with $m \neq 0$, $m \neq \pm 1, m \neq \sqrt{\pm 2}$, and $-1 \neq |m|^{p-1} < 1$. Then, the limit $\lim_{n\to\infty} m^{-2n} f(m^n x)$ exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x) - Q(x)\|_F \leq \frac{\epsilon}{2\left|m^2 - m^{2p}\right|} \|x\|_E^{2p} \tag{1.10}$$

for all $x \in E$.

Note that the mixed "product-sum" function was introduced by J. M. Rassias in 2008-2009 [46–48].

We recall some basic facts concerning fuzzy normed space.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $c, t \in \mathbb{R}$,

($N1$) $N(x,c) = 0$ for $c \leq 0$;

($N2$) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

($N3$) $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$;

($N4$) $N(x + y, \mathrm{t}) \geq \min\{N(x,t), N(y,t)\}$;

(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and

$$\lim_{t\to\infty} N(x,t) = 1. \tag{1.11}$$

The pair (X,N) is called a fuzzy normed linear space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [49–51].

Let (X,N) be a fuzzy normed space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in a fuzzy normed space (X,N) is called Cauchy if, for each $\epsilon > 0$ and $\delta > 0$, one can find some n_0 such that

$$N(x_m - x_n, \delta) > 1 - \epsilon \tag{1.12}$$

for all $n, m \geq n_0$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If, in a fuzzy-normed space, each Cauchy sequence is convergent, then the fuzzy-norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Stability of Cauchy, Jensen, quadratic, and cubic function equation in fuzzy normed spaces have first been investigated in [50–53].

In this paper, we give a generalization of the results from [13] and pose two open problems in fuzzy Banach space. For convenience, we use the following abbreviation for a given mapping f:

$$Df(x,y) =: f(x+y) - f(x) - f(y). \tag{1.13}$$

2. Stability of the Cauchy Functional Equation

Hereafter, unless otherwise stated, we will assume that X is real vector space, (Y,N) is a complete fuzzy norm space and k is a fixed integer greater than 1.

Theorem 2.1. *Let (Z,N') be a fuzzy normed space and $\varphi : X\times X \to Z$ be a mapping such that, $\varphi(kx,ky) = \alpha\varphi(x,y)$ for some α with $0 < \alpha < k$. Suppose that $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\varphi(x,y),t) \tag{2.1}$$

for all $x,y \in X$ and all positive real number t. Then, there is a unique additive mapping $T_k : X \to Y$ such that $T_k(x) = \lim_{n\to\infty} f(k^n x)/k^n$ and

$$N(T_k(x) - f(x),t) \geq M_k(x,(k-\alpha)t), \tag{2.2}$$

where $M_k(x,t) := \min\{N'(\varphi(x,ix),t) : 1 \leq i < k\}$.

Proof. By induction on k, we show that

$$N(f(kx) - kf(x),t) \geq M_k(x,t) := \min\{N'(\varphi(x,ix),t) : 1 \leq i < k\} \tag{2.3}$$

for all $x \in X$ and all positive real number t. Letting $y = x$ in (2.1), we get

$$N(f(2x) - 2f(x), t) \geq N'(\varphi(x, x), t). \tag{2.4}$$

So we get (2.3) for $k = 2$.

Assume that (2.3) holds for k with $k > 2$. Letting $y = kx$ in (2.1), we get

$$N(f((k+1)x) - f(x) - f(kx), t) \geq N'(\varphi(x, kx), t). \tag{2.5}$$

for all $x \in X$. By using (2.3) and (2.5), we get (2.3) for $k + 1$ and this completes the induction argument. Replacing x by $k^n x$ in (2.3), we get

$$N\left(f\left(k^{n+1}x\right) - kf(k^n x), t\right) \geq M_k(k^n x, t). \tag{2.6}$$

Thus

$$N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(k^n x)}{k^n}, \frac{t}{k^{n+1}}\right) \geq M_k\left(x, \frac{t}{\alpha^n}\right) \tag{2.7}$$

for all $x \in X$ and all positive real number t. Hence,

$$\begin{aligned} N&\left(\frac{1}{k^{n+1}}f\left(k^{n+1}x\right) - \frac{1}{k^m}f(k^m x), \sum_{i=m}^{n}\frac{\alpha^i}{k^{i+1}}t\right) \\ &\geq N\left(\sum_{i=m}^{n}\frac{1}{k^{i+1}}f\left(k^{i+1}x\right) - \frac{1}{k^i}f\left(k^i x\right), \sum_{i=m}^{n}\frac{\alpha^i}{k^{i+1}}t\right) \\ &\geq \min\bigcup_{i=m}^{n}\left\{N\left(\frac{1}{k^{i+1}}f\left(k^{i+1}x\right) - \frac{1}{k^i}f\left(k^i x\right), \frac{\alpha^i}{k^{i+1}}t\right)\right\} \\ &\geq M_k(x, t). \end{aligned} \tag{2.8}$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M_k(x, t) = 1$, there is some $t_0 > 0$ such that $M_k(x, t_0) > 1 - \epsilon$. Since $\sum_{n=0}^{\infty}(\alpha^n/k^n)t_0 < \infty$, there is some $n_0 \in N$ such that $\sum_{i=m}^{n}(\alpha^i/k^i)t_0 < k\delta$ for all $n > m \geq n_0$. It follows that

$$\begin{aligned} N&\left(\frac{1}{k^{n+1}}f\left(k^{n+1}x\right) - \frac{1}{k^m}f(k^m x), \delta\right) \\ &\geq N\left(\frac{1}{k^{n+1}}f\left(k^{n+1}x\right) - \frac{1}{k^m}f(k^m x), \sum_{i=m}^{n}\frac{\alpha^i}{k^{i+1}}t_0\right) \\ &\geq M_k(x, t_0) > 1 - \epsilon \end{aligned} \tag{2.9}$$

for all $x \in X$ and all nonnegative integers n and m with $n > m \geq n_0$. Therefore, the sequence $\{(1/k^n)f(k^n x)\}$ is a Cauchy sequence in (Y, N) for all $x \in X$. Since (Y, N) is complete, the sequence $\{(1/k^n)f(k^n x)\}$ converges in Y for all $x \in X$. So one can define the mapping $T_k : X \to Y$ by

$$T_k(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x) \tag{2.10}$$

for all $x \in X$. Now, we show that T_k is an additive mapping. It follows from (2.1) and (2.10) that

$$\begin{aligned}
N(DT_k(x,y),t) &= \lim_{n\to\infty} N\left(\frac{Df(k^n x, k^n y)}{k^n}, t\right) \\
&\geq \lim_{n\to\infty} N'\left(\frac{\varphi(k^n x, k^n y)}{k^n}, t\right) \\
&= \lim_{n\to\infty} N'\left(\varphi(x,y), \frac{k^n}{\alpha^n} t\right) \\
&= 1
\end{aligned} \tag{2.11}$$

for all $x, y \in X$ and all positive real number t. Therefore, the mapping T_k is additive. Moreover, if we put $m = 0$ in (2.8), we observe that

$$N\left(\frac{1}{k^{n+1}} f\left(k^{n+1} x\right) - f(x), \sum_{i=0}^{n} \frac{\alpha^i}{k^{i+1}} t\right) \geq M_k(x,t). \tag{2.12}$$

Therefore,

$$N\left(\frac{1}{k^{n+1}} f\left(k^{n+1} x\right) - f(x), t\right) \geq M_k\left(x, \frac{t}{\sum_{i=0}^{n} (\alpha^i / k^{i+1})}\right). \tag{2.13}$$

It follows from (2.13), for large enough n, that

$$\begin{aligned}
N(T_k(x) - f(x), t) &\geq \min\left\{N\left(\frac{f(k^{n+1} x)}{k^{n+1}} - f(x), t\right), N\left(T_k(x) - \frac{f(k^{n+1} x)}{k^{n+1}}, t\right)\right\} \\
&\geq M_k\left(x, \frac{t}{\sum_{i=0}^{n} (\alpha^i / k^{i+1})}\right) \\
&\geq M_k(x, (k-\alpha)t).
\end{aligned} \tag{2.14}$$

Now, we show that T_k is unique. Let T' be another additive mapping from X into Y, which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$, we have

$$\begin{aligned}
N(T_k(x) - T'(x), t) &\geq \min\{N(T_k(x) - f(x), t), N(f(x) - T'(x), t)\} \\
&\geq M_k(x, (k-\alpha)t).
\end{aligned} \tag{2.15}$$

So,

$$N(T_k(x) - T'(x), t) = N\left(\frac{T_k(k^n x)}{k^n} - \frac{T'(k^n x)}{k^n}, t\right)$$
$$= N(T_k(k^n x) - T'(k^n x), k^n t)$$
$$\geq M_k(k^n x, (k - \alpha)k^n t)$$
$$\geq M_k\left(x, (k - \alpha)\frac{k^n}{\alpha^n} t\right). \tag{2.16}$$

Hence, the right-hand side of the above inequality tends to 1 as $n \to \infty$. It follows that $T_k(x) = T'(x)$ for all $x \in X$. □

Theorem 2.2. *Let (Z, N') be a fuzzy normed space and, $\Phi : X \times X \to Z$ be a mapping such that $\Phi(k^{-1}x, k^{-1}y) = \alpha^{-1}\Phi(x, y)$ for some α with $\alpha > k$. Suppose that $f : X \to Y$ be mapping such that*

$$N(Df(x, y), t) \geq N'(\Phi(x, y), t) \tag{2.17}$$

for all $x, y \in X$ and all positive real number t. Then, there is a unique additive mapping $T_k : X \to Y$ such that $T_k(x) = \lim_{n\to\infty} k^n f(x/k^n)$ and

$$N(T_k(x) - f(x), t) \geq M_k(x, (\alpha - k)t), \tag{2.18}$$

where $M_k(x, t) := \min\{N'(\Phi(x, ix), t) : 1 \leq i < k\}$.

Proof. Similarly to the proof of Theorem 2.1, we have

$$N(f(kx) - kf(x), t) \geq M_k(x, t) \tag{2.19}$$

for all $x \in X$ and all positive real number t. Replacing x by x/k^{n+1} in (2.19), we get

$$N\left(f\left(\frac{x}{k^n}\right) - kf\left(\frac{x}{k^{n+1}}\right), t\right) \geq M_k\left(\frac{x}{k^{n+1}}, t\right). \tag{2.20}$$

Thus,

$$N\left(k^n f\left(\frac{x}{k^n}\right) - k^{n+1} f\left(\frac{x}{k^{n+1}}\right), k^n t\right) \geq M_k\left(x, \alpha^{n+1} t\right) \tag{2.21}$$

for all $x \in X$ and all positive real number t. Hence,

$$N\left(k^{n+1}f\left(\frac{x}{k^{n+1}}\right)-k^m f\left(\frac{x}{k^m}\right),\sum_{i=m}^{n}\frac{k^i}{\alpha^{i+1}}t\right)\geq N\left(\sum_{i=m}^{n}k^{i+1}f\left(\frac{x}{k^{i+1}}\right)-k^i f\left(\frac{x}{k^i}\right),\sum_{i=m}^{n}\frac{k^i}{\alpha^{i+1}}t\right)$$
$$\geq \min\bigcup_{i=m}^{n}\left\{N\left(k^{i+1}f\left(\frac{x}{k^{i+1}}\right)-k^i f\left(\frac{x}{k^i}\right),\frac{k^i}{\alpha^{i+1}}t\right)\right\}$$
$$\geq M_k(x,t). \tag{2.22}$$

Let $\epsilon>0$ and $\delta>0$ be given. Since $\lim_{t\to\infty}M_k(x,t)=1$, there is some $t_0>0$ such that $M_k(x,t_0)>1-\epsilon$. Since $\sum_{n=0}^{\infty}(k^n/\alpha^n)t_0<\infty$, there is some $n_0\in N$ such that $\sum_{i=m}^{n}(k^i/\alpha^i)t_0<\alpha\delta$ for all $n>m\geq n_0$. It follows from (2.22) that

$$N\left(k^{n+1}f\left(\frac{x}{k^{n+1}}\right)-k^m f\left(\frac{x}{k^m}\right),\delta\right)\geq N\left(k^{n+1}f\left(\frac{x}{k^{n+1}}\right)-k^m f\left(\frac{x}{k^m}\right),\sum_{i=m}^{n}\frac{k^i}{\alpha^{i+1}}t_0\right)$$
$$\geq M_k(x,t_0)>1-\epsilon \tag{2.23}$$

for all $x\in X$ and all nonnegative integers n and m with $n>m\geq n_0$. Therefore, the sequence $\{k^n f(x/k^n)\}$ is a Cauchy sequence in (Y,N) for all $x\in X$. Since (Y,N) is complete, the sequence $\{k^n f(x/k^n)\}$ converges in Y for all $x\in X$. So one can define the mapping $T_k: X\to Y$ by

$$T_k(x):=\lim_{n\to\infty}k^n f\left(\frac{x}{k^n}\right) \tag{2.24}$$

for all $x\in X$. The rest of the proof is similar to the proof of Theorem 2.1 □

Theorem 2.3. *Let X be a normed space, let (Z,N') be a fuzzy normed space, and let $\psi:[0,\infty)\to[0,\infty)$ be a function such that*

(1) $\psi(ts)=\psi(t)\psi(s)$,

(2) $\psi(t)<t$ *for all* $t>1$.

Suppose that a mapping $f: X\to Y$ satisfies the inequality:

$$N(Df(x,y),t)\geq N'((\psi(\|x\|)+\psi(\|y\|))z_0,t) \tag{2.25}$$

for all $x,y\in X$ and all positive real number t, where z_0 is a fixed vector of Z. Then, there exists a unique additive mapping $T_k: X\to Y$ satisfying $T_k(x):=\lim_{n\to\infty}(f(k^n x)/k^n)$ and

$$N(T_k(x)-f(x),t)\geq N'\left(\psi(\|x\|)z_0,\frac{k-\psi(k)}{\sigma_k(\psi)}t\right) \tag{2.26}$$

for all $x\in X$, where $\sigma_k(\psi)=\max\{1+\psi(i):1\leq i<k\}$. Moreover, $T_k=T_2$ for all $k\geq 2$.

Proof. Let

$$\varphi(x,y) = (\psi(\|x\|) + \psi(\|y\|))z_0 \tag{2.27}$$

for all $x, y \in X$. So,

$$\varphi(kx,ky) = \psi(k)\varphi(x,y). \tag{2.28}$$

where $\psi(k) < k$. By using Theorem 2.1, we can get (2.26). Now, we show that $T_k = T_2$. It follows from (1) that $\psi(k^n) = (\psi(k))^n$. Replacing x by $2^n x$ in (2.26), we get

$$N(T_k(2^n x) - f(2^n x), t) \geq N'\left(\psi(\|2^n x\|)z_0, \frac{k-\psi(k)}{\sigma_k(\psi)}t\right) \tag{2.29}$$

for all $x \in X$. So we have

$$N\left(T_k(x) - \frac{f(2^n x)}{2^n}, t\right) \geq N'\left(\psi(\|x\|)z_0, \frac{k-\psi(k)}{\sigma_k(\psi)\psi(2^n)}2^n t\right) \tag{2.30}$$

Using (2) and passing the limit $n \to \infty$ in (2.30), we get $T_k = T_2$. □

Theorem 2.4. *Let X be a normed space, let (Z, N') be a fuzzy normed space, and let $\psi : [0,\infty) \to [0,\infty)$ be a function such that*

(1) $\psi(ts) = \psi(t)\psi(s)$,

(2) $\psi(t) > t$ *for all* $t > 1$.

Suppose that a mapping $f : X \to Y$ satisfies the inequality:

$$N(Df(x,y),t) \geq N'((\psi(\|x\|) + \psi(\|y\|))z_0, t) \tag{2.31}$$

for all $x, y \in X$ and all positive real number t, where z_0 is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ satisfying $T_k(x) := \lim_{n\to\infty} k^n f(x/k^n)$ and

$$N(T_k(x) - f(x), t) \geq N'\left(\psi(\|x\|)z_0, \frac{\psi(k)-k}{\sigma_k(\psi)}t\right) \tag{2.32}$$

for all $x \in X$, where

$$\sigma_k(\psi) = \max\{1 + \psi(i) : 1 \leq i < k\}. \tag{2.33}$$

Moreover, $T_k = T_2$ for all $k \geq 2$.

Proof. Let

$$\Phi(x,y) = (\psi(\|x\|) + \psi(\|y\|))z_0 \tag{2.34}$$

for all $x, y \in X$. So, we have

$$\Phi\left(k^{-1}x, k^{-1}y\right) = \psi\left(k^{-1}\right)\Phi(x, y), \tag{2.35}$$

where $\psi(k^{-1}) = \psi(k)^{-1} < k^{-1}$. It follows from (1) that $\psi(k^{-n}) = (\psi(k))^{-n}$. By using Theorem 2.2, we can get (2.32). Now, we show that $T_k = T_2$. Replacing x by $x/2^n$ in (2.32), we get

$$N\left(T_k\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right), t\right) \geq N'\left(\psi\left(\left\|\left(\frac{x}{2^n}\right)\right\|\right)z_0, \frac{\psi(k) - k}{\sigma_k(\psi)}t\right). \tag{2.36}$$

for all $x \in X$. So we have

$$N\left(T_k(x) - 2^n f\left(\frac{x}{2^n}\right), t\right) \geq N'\left(\psi(\|x\|)z_0, \frac{\psi(k) - k}{2^n \sigma_k(\psi)\psi(2^{-n})}t\right). \tag{2.37}$$

Using (2) and passing the limit $n \to \infty$ in (2.37), we get $T_k = T_2$. □

Theorem 2.5. *Let X be a normed space, let p be a nonnegative real number such that $p \neq 1$, and let $H : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a homogeneous function of degree p. Suppose that (Z, N') be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x, y), t) \geq N'(H(\|x\|, \|y\|)z_0, t) \tag{2.38}$$

for all $x, y \in X$ and all positive real number t, where z_0 is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq M_k(x, |k^p - k|t), \tag{2.39}$$

where $M_k(x, t) := \min\{N'(\|x\|^p H(1, i)z_0, t) : 1 \leq i < k\}$.

Proof. The proof follows from Theorems 2.1 and 2.2. □

For the particular cases $H(x, y) = \theta(x^p + y^p)$, $H(x, y) = x^r y^s$, $H(x, y) = x^r y^s + x^{r+s} + y^{r+s} (r + s = p)$, and $H(x, y) = \min\{x^p, y^p\}$, we have the following corollaries.

Corollary 2.6. *Let X be a normed space, let p be a nonnegative real number such that $p \neq 1$. Suppose that (Z, N') be a fuzzy normed space and $f : X \to Y$ be mapping such that*

$$N(Df(x, y), t) \geq N'((\|x\|^p + \|y\|^p)\theta, t) \tag{2.40}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p\theta, \frac{|k^p - k|}{1 + (k - 1)^p}t\right). \tag{2.41}$$

Corollary 2.7. *Let X be a normed space, r, s be non-negative real numbers such that $p := r + s \neq 1$. Suppose that (Z, N') be a fuzzy normed space and $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\|x\|^r \|y\|^s \theta, t) \tag{2.42}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^s} t\right). \tag{2.43}$$

Corollary 2.8. *Let X be a normed space, and let r, s be nonnegative real numbers such that $p := r + s \neq 1$. Suppose that (Z, N') be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'\left(\theta\|x\|^r \|y\|^s + \theta\|x\|^{r+s} + \theta\|y\|^{r+s}, t\right) \tag{2.44}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^s + (k-1)^p + 1} t\right). \tag{2.45}$$

Corollary 2.9. *Let X be a normed space, let p be a nonnegative real number such that $p \neq 1$. Suppose that (Z, N') be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\min\{\|x\|^p, \|y\|^p\}\theta, t) \tag{2.46}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'(\|x\|^p \theta, |k^p - k| t). \tag{2.47}$$

Problem 1. Whether Theorem 2.5 and/or such Corollaries can be proved for $p = 1$?

Problem 2. What is the best possible value of k in Corollaries 2.6 and 2.7?

Acknowledgment

G. H. Kim was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant no. 2011-0005197).

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York, NY, USA, 1960.
[2] D. H. Hyers, “On the stability of the linear functional equation,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
[3] T. Aoki, “On the stability of the linear transformation in Banach spaces,” *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[4] Th. M. Rassias, “On the stability of the linear mapping in Banach spaces,” *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[5] Th. M. Rassias, “Problem 16; 2, report of the 27th International Symposium on functional equations,” *Aequationes Mathematicae*, vol. 39, pp. 292–293, 1990.
[6] Z. Gajda, “On stability of additive mappings,” *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.
[7] Th. M. Rassias and P. Šemrl, “On the behavior of mappings which do not satisfy Hyers-Ulam stability,” *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.
[8] P. Gǎvruţa, “On a problem of G. Isac and Th. M. Rassias concerning the stability of mappings,” *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 543–553, 2001.
[9] J. M. Rassias, “On approximation of approximately linear mappings by linear mappings,” *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
[10] J. M. Rassias, “On approximation of approximately linear mappings by linear mappings,” *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[11] J. M. Rassias, “Solution of a problem of Ulam,” *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[12] P. Gǎvruţǎ, “An answer to question of John M. Rassias concerning the stability of Cauchy equation,” *Advanced in Equation and Inequality*, pp. 67–71, 1999.
[13] P. Gǎvruţa, “A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[14] G. Isac and Th. M. Rassias, “Functional inequalities for approximately additive mappings,” in *Stability of Mappings of Hyers-Ulam Type*, Hadronic Press Collection of Original Articles, pp. 117–125, Hadronic Press, Palm Harbor, Fla, USA, 1994.
[15] G. Isac and Th. M. Rassias, “Stability of ψ-additive mappings: applications to nonlinear analysis,” *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.
[16] C. Baak and M. S. Moslehian, “On the stability of J^*-homomorphisms,” *Nonlinear Analysis*, vol. 63, no. 1, pp. 42–48, 2005.
[17] B. Bouikhalene, E. Elqorachi, and J. M. Rassias, “The superstability of d’Alembert’s functional equation on the Heisenberg group,” *Applied Mathematics Letters*, vol. 23, no. 1, pp. 105–109, 2010.
[18] L. Cǎdariu and V. Radu, “The fixed points method for the stability of some functional equations,” *Carpathian Journal of Mathematics*, vol. 23, no. 1-2, pp. 63–72, 2007.
[19] G. Z. Eskandani, “On the Hyers-Ulam-Rassias stability of an additive functional equation in quasi-Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 405–409, 2008.
[20] G. Z. Eskandani, H. Vaezi, and Y. N. Dehghan, “Stability of a mixed additive and quadratic functional equation in non-Archimedean Banach modules,” *Taiwanese Journal of Mathematics*, vol. 14, no. 4, pp. 1309–1324, 2010.
[21] G. Z. Eskandani, H. Vaezi, and F. Moradlou, “On the Hyers-Ulam-Rassias stability of functional equations in quasi-Banach spaces,” *International Journal of Applied Mathematics & Statistics*, vol. 15, pp. 1–15, 2009.
[22] P. Gǎvruţǎ, “On the Hyers-Ulam-Rassias stability of mappings,” in *Recent Progress in Inequalities*, G. V. Milovanovic, Ed., vol. 430, pp. 465–469, Kluwer Academic, Dordrecht, The Netherlands, 1998.
[23] P. Gǎvruţǎ and L. Cǎdariu, “General stability of the cubic functional equation,” *Buletinul Stiintific al Universitǎtii “Politehnica” din Timişoara. Seria Matematica-Fizica*, vol. 47(61), no. 1, pp. 59–70, 2002.
[24] K. W. Jun and Y. H. Lee, “On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality,” *Mathematical Inequalities & Applications*, vol. 4, no. 1, pp. 93–118, 2001.
[25] K. Jun, H. Kim, and J. Rassias, “Extended Hyers-Ulam stability for Cauchy-Jensen mappings,” *Journal of Difference Equations and Applications*, pp. 1–15, 2007.
[26] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[27] S. M. Jung, "Asymptotic properties of isometries," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 2, pp. 642–653, 2002.

[28] Pl. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.

[29] H. M. Kim, J. M. Rassias, and Y. S. Cho, "Stability problem of Ulam for Euler-Lagrange quadratic mappings," *Journal of Inequalities and Applications*, vol. 2007, Article ID 10725, 15 pages, 2007.

[30] Y. S. Lee and S. Y. Chung, "Stability of an Euler-Lagrange-Rassias equation in the spaces of generalized functions," *Applied Mathematics Letters of Rapid Publication*, vol. 21, no. 7, pp. 694–700, 2008.

[31] D. Miheţ, "The fixed point method for fuzzy stability of the Jensen functional equation," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1663–1667, 2009.

[32] F. Moradlou, H. Vaezi, and G. Z. Eskandani, "Hyers-Ulam-Rassias stability of a quadratic and additive functional equation in quasi-Banach spaces," *Mediterranean Journal of Mathematics*, vol. 6, no. 2, pp. 233–248, 2009.

[33] M. S. Moslehian, "On the orthogonal stability of the Pexiderized quadratic equation," *Journal of Difference Equations and Applications*, vol. 11, no. 11, pp. 999–1004, 2005.

[34] P. Nakmahachalasint, "On the generalized Ulam-Gavruta-Rassias stability of mixed-type linear and Euler-Lagrange-Rassias functional equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 63239, 10 pages, 2007.

[35] C. Park and J. M. Rassias, "Stability of the Jensen-type functional equation in C^*-algebras: a fixed point approach," *Abstract and Applied Analysis*, vol. 2009, Article ID 360432, 17 pages, 2009.

[36] A. Pietrzyk, "Stability of the Euler-Lagrange-Rassias functional equation," *Demonstratio Mathematica*, vol. 39, no. 3, pp. 523–530, 2006.

[37] J. M. Rassias, J. Lee, and H. M. Kim, "Refinned Hyers-Ulam stability for Jensen type mappings," *Journal of the Chungcheong Mathematical Society*, vol. 22, pp. 101–116, 2009.

[38] J. M. Rassias, "Complete solution of the multi-dimensional problem of Ulam," *Discussiones Mathematicae*, vol. 14, pp. 101–107, 1994.

[39] K. Ravi and M. Arunkumar, "On the Ulam-Gavruta-Rassias stability of the orthogonally Euler-Lagrange type functional equation," *International Journal of Applied Mathematics & Statistics*, vol. 7, pp. 143–156, 2007.

[40] K. Ravi, J. M. Rassias, M. Arunkumar, and R. Kodandan, "Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 4, article 114, pp. 1–29, 2009.

[41] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[42] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications, 34, Birkhäuser, Basle, Switzerland , 1998.

[43] Th. M. Rassias, "On a modified Hyers-Ulam sequence," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 1, pp. 106–113, 1991.

[44] P. Găvruţa, M. Hossu, D. Popescu, and C. Căprău, "On the stability of mappings and an answer to a problem of Th. M. Rassias," *Annales Mathématiques Blaise Pascal*, vol. 2, no. 2, pp. 55–60, 1995.

[45] K. Ravi, M. Arunkumar, and J. M. Rassias, "Ulam stability for the orthogonally general Euler-Lagrange type functional equation," *International Journal of Mathematics and Statistics*, vol. 3, no. A08, pp. 36–46, 2008.

[46] H. X. Cao, J. R. Lv, and J. M. Rassias, "Superstability for generalized module left derivations and generalized module derivations on a Banach module. I," *Journal of Inequalities and Applications*, vol. 2009, Article ID 718020, 10 pages, 2009.

[47] H. X. Cao, J. R. Lv, and J. M. Rassias, "Superstability for generalized module left derivations and generalized module derivations on a Banach module. II," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 3, article 85, pp. 1–8, 2009.

[48] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, Article ID 042303, pp. 1–9, 2009.

[49] T. Bag and S. K. Samanta, "Finite dimensional fuzzy normed linear spaces," *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 687–705, 2003.

[50] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy versions of Hyers-Ulam-Rassias theorem," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 720–729, 2008.

[51] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, "Fuzzy stability of the Jensen functional equation," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 730–738, 2008.

[52] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy approximately cubic mappings," *Information Sciences*, vol. 178, no. 19, pp. 3791–3798, 2008.
[53] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy almost quadratic functions," *Results in Mathematics*, vol. 52, no. 1-2, pp. 161–177, 2008.