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Research Article

Spectral Approach to Derive the Representation Formulae for Solutions of the Wave Equation

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Using spectral properties of the Laplace operator and some structural formula for rapidly decreasing functions of the Laplace operator, we offer a novel method to derive explicit formulae for solutions to the Cauchy problem for classical wave equation in arbitrary dimensions. Among them are the well-known d'Alembert, Poisson, and Kirchhoff representation formulae in low space dimensions.

1. Introduction

The wave equation for a function $u(x_1, \ldots, x_n, t) = u(x, t)$ of n space variables $x_1, \ldots, x_n$ and the time t is given by

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{1.2}$$

is the Laplacian. The wave equation is encountered often in applications. For $n = 1$ the equation can represent sound waves in pipes or vibrations of strings, for $n = 2$ waves on the surface of water, for $n = 3$ waves in acoustics or optics. Therefore, formulae that give the solution of the Cauchy problem in explicit form are of great significance. In the Cauchy problem (initial value problem) one asks for a solution $u(x,t)$ of (1.1) defined for $x \in \mathbb{R}^n$, $t \geq 0$ that satisfies (1.1) for $x \in \mathbb{R}^n$, $t > 0$ and the initial conditions

$$u(x,0) = \varphi(x), \qquad \frac{\partial u(x,0)}{\partial t} = \psi(x) \quad (x \in \mathbb{R}^n). \tag{1.3}$$

If $n = 1$ and $\varphi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$, then the classical solution of problem (1.1), (1.3) is given by *d'Alembert's formula*

$$u(x,t) = \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\,dy. \tag{1.4}$$

If $n = 2$ and $\varphi \in C^3(\mathbb{R}^2)$, $\psi \in C^2(\mathbb{R}^2)$, then the solution of problem (1.1), (1.3) is given by *Poisson's formula*

$$u(x,t) = \frac{1}{2\pi}\int_{|y-x|<t} \frac{\psi(y)\,dy}{\sqrt{t^2 - |y-x|^2}} + \frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{|y-x|<t} \frac{\varphi(y)\,dy}{\sqrt{t^2 - |y-x|^2}}\right], \tag{1.5}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $|y-x|^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2$.

If $n = 3$ and $\varphi \in C^3(\mathbb{R}^2)$, $\psi \in C^2(\mathbb{R}^2)$, then the solution of problem (1.1), (1.3) is given by *Kirchhoff's formula*

$$u(x,t) = \frac{1}{4\pi t}\int_{|y-x|=t} \psi(y)\,dS_y + \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{|y-x|=t} \varphi(y)\,dS_y\right], \tag{1.6}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $|y-x|^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2$, and dS_y is the surface element of the sphere $\{y \in \mathbb{R}^3 : |y-x| = t\}$.

Passing to an arbitrary n let us denote by $u(x,t) = N_\varphi(x,t)$ the solution of the problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.7}$$

$$u(x,0) = \varphi(x), \qquad \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in \mathbb{R}^n. \tag{1.8}$$

It is easy to see that then the function

$$v(x,t) = \int_0^t u(x,\tau)\,d\tau \tag{1.9}$$

is the solution of the problem

$$\frac{\partial^2 v}{\partial t^2} = \Delta v, \quad x \in \mathbb{R}^n, \ t > 0, \tag{1.10}$$

$$v(x,0) = 0, \qquad \frac{\partial v(x,0)}{\partial t} = \varphi(x), \quad x \in \mathbb{R}^n. \tag{1.11}$$

Indeed, integrating (1.7) we get

$$\int_0^t \frac{\partial^2 u(x,\tau)}{\partial \tau^2} d\tau = \int_0^t \Delta u(x,\tau) d\tau = \Delta \int_0^t u(x,\tau) d\tau = \Delta v(x,t). \tag{1.12}$$

Hence,

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial u(x,0)}{\partial t} = \Delta v(x,t) \quad \text{or} \quad \frac{\partial u(x,t)}{\partial t} = \Delta v(x,t), \tag{1.13}$$

by the second condition in (1.8). On the other hand, from (1.9),

$$\frac{\partial v(x,t)}{\partial t} = u(x,t), \qquad \frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial u(x,t)}{\partial t}. \tag{1.14}$$

Comparing (1.13) and (1.14), we get (1.10). Besides,

$$v(x,0) = 0, \qquad \frac{\partial v(x,0)}{\partial t} = u(x,0) = \varphi(x) \tag{1.15}$$

so that initial conditions in (1.11) are also satisfied.

Consequently, the solution $u(x,t)$ of problem (1.1), (1.3) is represented in the form

$$u(x,t) = N_\varphi(x,t) + \int_0^t N_\psi(x,\tau) d\tau. \tag{1.16}$$

It follows that it is sufficient to know an explicit form of the solution $N_\varphi(x,t)$ of problem (1.7), (1.8). It is known [1, 2] that

$$N_\varphi(x,t) = \frac{1}{2^{m+1}\pi^m} \left(\frac{\partial}{\partial t} \frac{1}{t} \right)^m \int_{|y-x|=t} \varphi(y) dS_y \quad \text{if } n = 2m+1, \tag{1.17}$$

$$N_\varphi(x,t) = \frac{1}{2^m \pi^m} \left(\frac{\partial}{\partial t} \frac{1}{t} \right)^{m-1} \frac{\partial}{dt} \int_{|y-x|<t} \frac{\varphi(y) dy}{\sqrt{t^2 - |y-x|^2}} \quad \text{if } n = 2m, \tag{1.18}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $|y - x|^2 = (y_1 - x_1)^2 + \cdots + (y_n - x_n)^2$, and dS_y is the surface element of the sphere $\{y \in \mathbb{R}^n : |y - x| = t\}$.

In the present paper, we give a new proof of formulae (1.17), (1.18) for the solution of problem (1.7), (1.8). Our method of the proof is based on the spectral theory of the Laplace operator. We hope that such a method may be useful also in some other cases of the equation and space.

The paper consists, besides this introductory section, of three sections. In Section 2, we describe the structure of arbitrary rapidly decreasing function of the Laplace operator, showing that it is an integral operator and giving an explicit formula for its kernel. Next we use these results in Section 3 to derive the explicit representation formulae for the classical solution to the initial value problem for the wave equation in arbitrary dimensions. The final Section is an appendix and contains some explanation of several points in the paper.

2. Structure of Arbitrary Function of the Laplace Operator

Let A be the self-adjoint positive operator obtained as the closure of the symmetric operator A' determined in the Hilbert space $L^2(\mathbb{R}^n)$ by the differential expression

$$-\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{2.1}$$

on the domain of definition $D(A') = C_0^\infty(\mathbb{R}^n)$ that is the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Let E_μ denote the resolution of the identity (the spectral projection) for A:

$$Af = \int_0^\infty \mu dE_\mu f, \quad f \in D(A). \tag{2.2}$$

Next, let $g(t)$ be any infinitely differentiable even function on the axis $-\infty < t < \infty$ with compact support and

$$\widetilde{g}(\lambda) = \int_{-\infty}^\infty g(t) e^{i\lambda t} dt \tag{2.3}$$

its Fourier transform. Note that the function $\widetilde{g}(\lambda)$ tends to zero as $|\lambda| \to \infty$ ($\lambda \in \mathbb{R}$) faster than any negative power of $|\lambda|$. Consider the operator $\widetilde{g}(A^{1/2})$ defined according to the general theory of self-adjoint operators (see [3]):

$$\widetilde{g}\left(A^{1/2}\right) f = \int_0^\infty \widetilde{g}(\sqrt{\mu}) dE_\mu f, \quad f \in L^2(\mathbb{R}^n). \tag{2.4}$$

The following theorem describes the structure of the operator $\widetilde{g}(A^{1/2})$ showing that it is an integral operator and giving an explicit formula for its kernel in terms of the function $g(t)$.

Theorem 2.1. *The operator* $\tilde{g}(A^{1/2})$ *is an integral operator*

$$\tilde{g}\left(A^{1/2}\right)f(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^n). \tag{2.5}$$

Further, there is a smooth function $k(t)$ *defined on the interval* $0 \le t < \infty$ *such that*

$$\mathcal{K}(x,y) = k\left(|x-y|^2\right). \tag{2.6}$$

The function $k(t)$ *depends on the function* $g(t)$ *as follows. If one sets*

$$Q(t) = g\left(\sqrt{t}\right), \qquad \text{that is, } Q\left(t^2\right) = g(t), \quad 0 \le t < \infty, \tag{2.7}$$

then

$$k(t) = \begin{cases} \dfrac{(-1)^m}{\pi^m} Q^{(m)}(t) & \text{if } n = 2m+1, \\ \dfrac{(-1)^m}{\pi^m} \displaystyle\int_t^{\infty} \dfrac{Q^{(m)}(w)}{\sqrt{w-t}} dw & \text{if } n = 2m, \end{cases} \tag{2.8}$$

where $Q^{(m)}(t)$ *denotes the* m*th order derivative of* $Q(t)$*. Further, if* $\operatorname{supp} g(t) \subset (-a,a)$*, then* $\operatorname{supp} k(t) \subset [0,a^2)$*. For any solution* $\psi(x,\lambda)$ *of the equation*

$$-\Delta\psi(x,\lambda) = \lambda^2\psi(x,\lambda), \tag{2.9}$$

the equality

$$\int_{\mathbb{R}^n} k\left(|x-y|^2\right)\psi(y,\lambda)dy = \tilde{g}(\lambda)\psi(x,\lambda) \tag{2.10}$$

holds for $\lambda \in \mathbb{R}$*.*

Proof. First we consider the case $n = 1$. In this case, the statements of the theorem take the following form: $k(t) = Q(t) = g(\sqrt{t})$ for $0 \le t < \infty$; the operator $\tilde{g}(A^{1/2})$ is an integral operator of the form

$$\tilde{g}\left(A^{1/2}\right)f(x) = \int_{-\infty}^{\infty} g(x-y)f(y)dy, \tag{2.11}$$

and for any solution $\psi(x,\lambda)$ of the equation

$$-\psi''(x,\lambda) = \lambda^2\psi(x,\lambda), \tag{2.12}$$

the equality

$$\int_{-\infty}^{\infty} g(x-y)\psi(y,\lambda)dy = \tilde{g}(\lambda)\psi(x,\lambda) \tag{2.13}$$

holds.

To prove the last statements note that, in the case $n = 1$, the operator A is generated in the Hilbert space $L^2(-\infty,\infty)$ by the operation $-d^2/dx^2$ and the operator $A^{1/2}$ by the operation id/dx. The resolvent $R_\mu = (A-\mu I)^{-1}$ of the operator A has the form

$$R_\mu f(x) = \frac{i}{2\sqrt{\mu}} \int_{-\infty}^{\infty} e^{i|x-y|\sqrt{\mu}} f(y)dy, \tag{2.14}$$

while the spectral projection E_μ of the operator A has the form (see [3, page 201])

$$\begin{gathered} E_\mu f(x) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\mu}(x-y)}{\pi(x-y)} f(y)dy, \quad 0 \le \mu < \infty, \\ E_\mu = 0 \quad \text{for } \mu < 0. \end{gathered} \tag{2.15}$$

Therefore,

$$\begin{aligned} \tilde{g}\left(A^{1/2}\right)f(x) &= \int_0^{\infty} \tilde{g}(\sqrt{\mu})dE_\mu f(x) \\ &= \int_0^{\infty} \tilde{g}(\sqrt{\mu}) \left\{ \int_{-\infty}^{\infty} \frac{\cos\sqrt{\mu}(x-y)}{2\pi\sqrt{\mu}} f(y)dy \right\} d\mu \\ &= \int_{-\infty}^{\infty} \left\{ \frac{1}{\pi} \int_0^{\infty} \tilde{g}(\lambda)\cos\lambda(x-y)d\lambda \right\} f(y)dy = \int_{-\infty}^{\infty} g(x-y)f(y)dy, \end{aligned} \tag{2.16}$$

where we have used the inversion formula for the Fourier cosine transform. Therefore, (2.11) is proved. To prove (2.13) note that the general solution of (2.12) is

$$\psi(x,\lambda) = \begin{cases} c_1\cos\lambda x + c_2\sin\lambda x & \text{if } \lambda \ne 0, \\ c_1 + c_2 x & \text{if } \lambda = 0, \end{cases} \tag{2.17}$$

where c_1 and c_2 are arbitrary constants. Then, we have, for $\lambda \neq 0$,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x-y)\psi(y,\lambda)dy &= c_1 \int_{-\infty}^{\infty} g(x-y)\cos\lambda y\,dy + c_2 \int_{-\infty}^{\infty} g(x-y)\sin\lambda y\,dy \\
&= c_1 \int_{-\infty}^{\infty} g(t)\cos\lambda(x-t)dt + c_2 \int_{-\infty}^{\infty} g(t)\sin\lambda(x-t)dt \\
&= c_1 \int_{-\infty}^{\infty} g(t)(\cos\lambda x\cos\lambda t + \sin\lambda x\sin\lambda t)dt \\
&\quad + c_2 \int_{-\infty}^{\infty} g(t)(\sin\lambda x\cos\lambda t - \sin\lambda t\cos\lambda x)dt \\
&= c_1\cos\lambda x \int_{-\infty}^{\infty} g(t)\cos\lambda t\,dt + c_2\sin\lambda x \int_{-\infty}^{\infty} g(t)\cos\lambda t\,dt \\
&= (c_1\cos\lambda x + c_2\sin\lambda x)\int_{-\infty}^{\infty} g(t)\cos\lambda t\,dt = \psi(x,\lambda)\tilde{g}(\lambda),
\end{aligned} \tag{2.18}$$

where we have used the fact that the function $g(t)$ is even and therefore

$$\int_{-\infty}^{\infty} g(t)\sin\lambda t\,dt = 0. \tag{2.19}$$

The same result can be obtained similarly for $\lambda = 0$. Thus, (2.13) is also proved.

Now we consider the case $n \geq 2$. We shall use the integral representation

$$R_\mu f(x) = \int_{\mathbb{R}^n} r(x,y;\mu) f(y)dy \tag{2.20}$$

of the resolvent $R_\mu = (A - \mu I)^{-1}$ of the operator A. As is known [4, Section 13.7, Formula (13.7.2)],

$$r(x,y;\mu) = \frac{i\mu^{(n-2)/4}}{2^{(n+2)/2}\pi^{(n-2)/2}|x-y|^{(n-2)/2}} H^{(1)}_{(n-2)/2}\left(|x-y|\sqrt{\mu}\right), \tag{2.21}$$

where $H^{(1)}_\nu(z)$ is the Hankel function of the first kind of order ν. Next, according to the general spectral theory of self-adjoint operators [3, page 150, Formula (11)], we have

$$dE_\mu f(x) = \frac{1}{2\pi i}\left(R_{\mu+i0} - R_{\mu-i0}\right) f(x)d\mu. \tag{2.22}$$

Therefore, from (2.4) it follows that the representation (2.5) holds with

$$\mathcal{K}(x,y) = \frac{1}{2\pi i}\int_0^{\infty} \tilde{g}(\sqrt{\mu})\left[r(x,y;\mu+i0) - r(x,y;\mu-i0)\right]d\mu. \tag{2.23}$$

Now the representation (2.6), which expresses that $\mathcal{K}(x,y)$ is a function of $|x-y|^2$, follows from (2.23) by (2.21).

To prove (2.10) we use (2.23). By virtue of (2.23),

$$\begin{aligned}\int_{\mathbb{R}^n} k\left(|x-y|^2\right)\psi(y,\lambda)dy &= \int_{\mathbb{R}^n} \mathcal{K}(x,y)\psi(y,\lambda)dy \\ &= \lim_{\varepsilon\to+0}\int_{\mathbb{R}^n}\left\{\frac{1}{2\pi i}\int_0^\infty \widetilde{g}(\sqrt{\mu})\left[r(x,y;\mu+i\varepsilon)\right.\right. \\ &\qquad\left.\left. -r(x,y;\mu-i\varepsilon)\right]d\mu\right\}\psi(y,\lambda)dy \\ &= \psi(x,\lambda)\lim_{\varepsilon\to+0}\frac{\varepsilon}{\pi}\int_0^\infty \frac{\widetilde{g}(\sqrt{\mu})}{\left(\mu-\lambda^2\right)^2+\varepsilon^2}d\mu = \psi(x,\lambda)\widetilde{g}(\lambda),\end{aligned} \tag{2.24}$$

see Appendix. Here we have used the fact that from (2.9) it follows that

$$(-\Delta-z)\psi(x,\lambda) = \left(\lambda^2-z\right)\psi(x,\lambda), \tag{2.25}$$

that is,

$$\psi(x,\lambda) = \left(\lambda^2-z\right)(-\Delta-z)^{-1}\psi(x,\lambda), \tag{2.26}$$

and therefore

$$\int_{\mathbb{R}^n} r(x,y;z)\psi(y,\lambda)dy = \frac{1}{\lambda^2-z}\psi(x,\lambda). \tag{2.27}$$

Finally, to deduce the explicit formulae (2.7), (2.8), we take $\psi(x,\lambda) = e^{i\lambda x_1}$ in (2.10). Then, putting $\widetilde{x} = (x_2,\ldots,x_n)$, we can write

$$\int_{\mathbb{R}^n} k\left(|x_1-y_1|^2+|\widetilde{x}-\widetilde{y}|^2\right)e^{i\lambda y_1}dy_1d\widetilde{y} = \widetilde{g}(\lambda)e^{i\lambda x_1}. \tag{2.28}$$

If we set

$$(x_1-y_1)^2 = w, \tag{2.29}$$

then the left-hand side of (2.28) equals

$$\int_{-\infty}^{\infty}\left\{\int_{\mathbb{R}^{n-1}} k\left(w+|\widetilde{x}-\widetilde{y}|^2\right)d\widetilde{y}\right\}e^{i\lambda y_1}dy_1. \tag{2.30}$$

On the other hand,

$$\int_{\mathbb{R}^{n-1}} k\left(w+|\tilde{x}-\tilde{y}|^2\right)d\tilde{y} = \int_0^\infty \left\{\int_{|\tilde{x}-\tilde{y}|=r} k\left(w+|\tilde{x}-\tilde{y}|^2\right)dS\right\}dr$$
$$= \int_0^\infty k\left(w+r^2\right)\left\{\int_{|\tilde{x}-\tilde{y}|=r} dS\right\}dr = \sigma_{n-1}\int_0^\infty r^{n-2}k\left(w+r^2\right)dr$$
$$= \frac{1}{2}\sigma_{n-1}\int_w^\infty (t-w)^{(n-3)/2}k(t)dt, \tag{2.31}$$

where

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{2.32}$$

is the surface area of the $(n-1)$-dimensional unit sphere (Γ is the gamma function) and dS denotes the surface element of the sphere $\{\tilde{y} \in \mathbb{R}^{n-1} : |\tilde{x}-\tilde{y}| = r\}$. Therefore, setting

$$Q(w) = \frac{1}{2}\sigma_{n-1}\int_w^\infty (t-w)^{(n-3)/2}k(t)dt, \tag{2.33}$$

we get that (2.28) takes the form

$$\int_{-\infty}^\infty Q(w)e^{i\lambda y_1}dy_1 = \tilde{g}(\lambda)e^{i\lambda x_1}. \tag{2.34}$$

Substituting here the expression of w given in (2.29) and making then the change of variables $x_1 - y_1 = t$, we obtain

$$\int_{-\infty}^\infty Q\left(t^2\right)e^{i\lambda t}dt = \tilde{g}(\lambda) = \int_{-\infty}^\infty g(t)e^{i\lambda t}dt. \tag{2.35}$$

Hence (2.7) follows. Further, it is not difficult to check that the formula (2.33) for $n \geq 2$ is equivalent to (2.8), see Appendix.

Since $g(t)$ is smooth and has a compact support, it follows from (2.7), (2.8) that the function $k(t)$ also is smooth and has a compact support; more precisely, if $\operatorname{supp} g(t) \subset (-a,a)$, then $\operatorname{supp} k(t) \subset [0,a^2)$. This implies, in particular, convergence of the integral in (2.10) for each fixed x. The theorem is proved. □

3. Derivation of Formulae (1.17), (1.18)

Consider the Cauchy problem (1.7), (1.8):

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \quad x \in \mathbb{R}^n, \ t > 0, \tag{3.1}$$

$$u(x,0) = \varphi(x), \qquad \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in \mathbb{R}^n, \tag{3.2}$$

where $u = u(x,t)$, $t \geq 0$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$.

For $\nu = (\nu_1, \ldots, \nu_n)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let us set

$$|\nu|^2 = \nu_1^2 + \cdots + \nu_n^2, \qquad (\nu, x) = \nu_1 x_1 + \cdots + \nu_n x_n. \tag{3.3}$$

Since

$$-\Delta e^{i(\nu,x)} = |\nu|^2 e^{i(\nu,x)}, \tag{3.4}$$

applying (2.9), (2.10), we get

$$\int_{\mathbb{R}^n} k\left(|x-y|^2\right) e^{i(\nu,y)} dy = \widetilde{g}(|\nu|) e^{i(\nu,x)} \quad (\nu \in \mathbb{R}^n). \tag{3.5}$$

Hence, by the inverse Fourier transform formula,

$$k\left(|x-y|^2\right) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widetilde{g}(|\nu|) e^{i(\nu,x)} e^{-i(\nu,y)} d\nu. \tag{3.6}$$

Multiplying both sides of the last equality by $\varphi(y)$ and then integrating on $y \in \mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n} k\left(|x-y|^2\right) \varphi(y) dy = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widetilde{g}(|\nu|) e^{i(\nu,x)} \left[\int_{\mathbb{R}^n} \varphi(y) e^{-i(\nu,y)} dy\right] d\nu. \tag{3.7}$$

Substituting here for $\widetilde{g}(|\nu|)$ its expression

$$\widetilde{g}(|\nu|) = 2\int_0^\infty g(t) \cos(|\nu|t) dt \tag{3.8}$$

and setting

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\cos|\nu|t) e^{i(\nu,x)} \left[\int_{\mathbb{R}^n} \varphi(y) e^{-i(\nu,y)} dy\right] d\nu, \tag{3.9}$$

we obtain

$$\int_{\mathbb{R}^n} k\left(|x-y|^2\right)\varphi(y)\,dy = 2\int_0^\infty g(t)u(x,t)\,dt. \tag{3.10}$$

Obviously, the function $u(x,t)$ defined by (3.9) is the solution of problem (3.1), (3.2). Next we will transform the left-hand side of (3.10) using Theorem 2.1.

First we consider the case $n = 1$. In this case, (3.10) takes the form

$$\int_{-\infty}^{\infty} k\left(|x-y|^2\right)\varphi(y)\,dy = 2\int_0^\infty g(t)u(x,t)\,dt \tag{3.11}$$

and from (2.7), (2.8) we have

$$k\left(t^2\right) = Q\left(t^2\right) = g(t). \tag{3.12}$$

Therefore, making the change of variables $y - x = t$ and taking into account the evenness of the function $g(t)$, we can write

$$\begin{aligned}\int_{-\infty}^{\infty} k\left(|x-y|^2\right)\varphi(y)\,dy &= \int_{-\infty}^{\infty} k\left(t^2\right)\varphi(x+t)\,dt \\ &= \int_{-\infty}^{\infty} g(t)\varphi(x+t)\,dt = \int_0^\infty g(t)\left[\varphi(x+t)+\varphi(x-t)\right]dt.\end{aligned} \tag{3.13}$$

Substituting this in the left-hand side of (3.11), we obtain

$$\int_0^\infty g(t)\left[\varphi(x+t)+\varphi(x-t)\right]dt = 2\int_0^\infty g(t)u(x,t)\,dt. \tag{3.14}$$

Hence, by the arbitrariness of the smooth even function $g(t)$ with compact support, we get

$$u(x,t) = \frac{\varphi(x+t)+\varphi(x-t)}{2}. \tag{3.15}$$

Further assume that $n \geq 2$. Making the change of variables

$$y - x = t\omega, \quad 0 \leq t < \infty, \qquad |\omega| = 1, \quad \omega = (\omega_1,\ldots,\omega_n), \qquad dy = t^{n-1}dt\,dS_\omega, \tag{3.16}$$

where dS_ω is the surface element of the unit sphere $\{\omega \in \mathbb{R}^n : |\omega| = 1\}$, we get

$$\int_{\mathbb{R}^n} k\left(|x-y|^2\right)\varphi(y)\,dy = \int_0^\infty t^{n-1}k\left(t^2\right)\left\{\int_{|\omega|=1}\varphi(x+t\omega)\,dS_\omega\right\}dt. \tag{3.17}$$

Further, making in the right-hand side of (3.17) the change of variables

$$x + t\omega = y, \qquad dS_y = t^{n-1} dS_\omega, \tag{3.18}$$

where dS_y is the surface element of the sphere $\{y \in \mathbb{R}^n : |y - x| = t\}$, we have

$$t^{n-1} \int_{|\omega|=1} \varphi(x + t\omega) dS_\omega = \int_{|y-x|=t} \varphi(y) dS_y =: P_\varphi(x,t). \tag{3.19}$$

Therefore,

$$\int_{\mathbb{R}^n} k\left(|x-y|^2\right)\varphi(y)dy = \int_0^\infty k\left(t^2\right)P_\varphi(x,t)dt, \tag{3.20}$$

and (3.10) becomes

$$\int_0^\infty k\left(t^2\right)P_\varphi(x,t)dt = 2\int_0^\infty g(t)u(x,t)dt. \tag{3.21}$$

Consider the cases of odd and even n separately.

Let $n = 2m + 1 (m \in \mathbb{N})$. Then, by (2.8) we have

$$k\left(t^2\right) = \frac{(-1)^m}{\pi^m} Q^{(m)}\left(t^2\right) \tag{3.22}$$

and it follows from (2.7) (by successive differentiation) that

$$Q^{(m)}\left(t^2\right) = \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^m g(t). \tag{3.23}$$

Therefore,

$$k\left(t^2\right) = \frac{(-1)^m}{2^m \pi^m}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m g(t), \tag{3.24}$$

and (3.21) takes the form

$$\frac{(-1)^m}{2^m \pi^m}\int_0^\infty \left\{\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m g(t)\right\}P_\varphi(x,t)dt = 2\int_0^\infty g(t)u(x,t)dt. \tag{3.25}$$

Further, integrating m times by parts, we get

$$\int_0^\infty \left\{\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m g(t)\right\}P_\varphi(x,t)dt = R(x,t)\Big|_{t=0}^{t=\infty} + (-1)^m \int_0^\infty g(t)\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^m P_\varphi(x,t)dt, \tag{3.26}$$

where

$$R(x,t) = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{t} \left\{ \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{m-k} g(t) \right\} \left(\frac{\partial}{\partial t} \frac{1}{t} \right)^{k-1} P_{\varphi}(x,t)$$
$$= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{t} \left\{ \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{m-k} g(t) \right\} \left(\frac{\partial}{\partial t} \frac{1}{t} \right)^{k-1} t^{2m} \int_{|\omega|=1} \varphi(x+t\omega) dS_{\omega}. \tag{3.27}$$

Since $g(t)$ is identically zero for large values of t, we have from (3.27) that $R(x,\infty) = 0$. Also, it follows directly from (3.27) that $R(x,0) = 0$. Therefore, (3.25) becomes

$$\frac{1}{2^m \pi^m} \int_0^{\infty} g(t) \left(\frac{\partial}{\partial t} \frac{1}{t} \right)^m P_{\varphi}(x,t) dt = 2 \int_0^{\infty} g(t) u(x,t) dt. \tag{3.28}$$

Since in (3.28) $g(t)$ is arbitrary smooth even function with compact support, we obtain that

$$u(x,t) = \frac{1}{2^{m+1} \pi^m} \left(\frac{\partial}{\partial t} \frac{1}{t} \right)^m P_{\varphi}(x,t). \tag{3.29}$$

This coincides with (1.17) by (3.19).

Now let us consider the case $n = 2m$ ($m \in \mathbb{N}$). In this case, by (2.8) we have

$$k\left(r^2\right) = \frac{(-1)^m}{\pi^m} \int_{r^2}^{\infty} \frac{Q^{(m)}(w)}{\sqrt{w - r^2}} dw = \frac{(-1)^m}{\pi^m} \int_r^{\infty} \frac{Q^{(m)}(t^2) 2t}{\sqrt{t^2 - r^2}} dt, \tag{3.30}$$

and therefore

$$\int_0^{\infty} k\left(r^2\right) P_{\varphi}(x,r) dr = \frac{(-1)^m}{\pi^m} \int_0^{\infty} \left\{ \int_r^{\infty} \frac{Q^{(m)}(t^2) 2t}{\sqrt{t^2 - r^2}} dt \right\} P_{\varphi}(x,r) dr$$
$$= \frac{(-1)^m}{\pi^m} \int_0^{\infty} \left\{ \int_r^{\infty} \frac{Q^{(m)}(t^2) 2t}{\sqrt{t^2 - r^2}} dt \right\} \left\{ \int_{|y-x|=r} \varphi(y) dS_y \right\} dr$$
$$= \frac{(-1)^m}{\pi^m} \int_0^{\infty} \left\{ \int_r^{\infty} Q^{(m)}\left(t^2\right) 2t \left[\int_{|y-x|=r} \frac{\varphi(y) dS_y}{\sqrt{t^2 - |y-x|^2}} \right] dt \right\} dr$$
$$= \frac{(-1)^m}{\pi^m} \int_0^{\infty} Q^{(m)}\left(t^2\right) 2t \left\{ \int_0^t \left[\int_{|y-x|=r} \frac{\varphi(y) dS_y}{\sqrt{t^2 - |y-x|^2}} \right] dr \right\} dt. \tag{3.31}$$

Hence, setting

$$H_{\varphi}(x,t) := \int_0^t \left[\int_{|y-x|=r} \frac{\varphi(y) dS_y}{\sqrt{t^2 - |y-x|^2}} \right] dr = \int_{|y-x|<t} \frac{\varphi(y) dy}{\sqrt{t^2 - |y-x|^2}}, \tag{3.32}$$

we get

$$\int_0^\infty k\left(r^2\right)P_\varphi(x,r)dr = \frac{(-1)^m}{\pi^m}\int_0^\infty Q^{(m)}\left(t^2\right)2tH_\varphi(x,t)dt. \tag{3.33}$$

Substituting this in the left-hand side of (3.21) (beforehand replacing t by r in the left side of (3.21)), we obtain

$$\frac{(-1)^m}{\pi^m}\int_0^\infty Q^{(m)}\left(t^2\right)2tH_\varphi(x,t)dt = 2\int_0^\infty g(t)u(x,t)dt \tag{3.34}$$

or, using (3.23),

$$\frac{(-1)^m}{2^{m-1}\pi^m}\int_0^\infty \left\{\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m g(t)\right\}tH_\varphi(x,t)dt = 2\int_0^\infty g(t)u(x,t)dt. \tag{3.35}$$

Further, integrating m times by parts, we get

$$\int_0^\infty \left\{\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m g(t)\right\}tH_\varphi(x,t)dt = L(x,t)|_{t=0}^{t=\infty} + (-1)^m\int_0^\infty g(t)\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^m tH_\varphi(x,t)dt, \tag{3.36}$$

where

$$L(x,t) = \sum_{k=1}^{m}(-1)^{k-1}\left\{\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-k} g(t)\right\}\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^{k-1} tH_\varphi(x,t). \tag{3.37}$$

Since $g(t)$ is identically zero for large values of t, we have from (3.37) that $L(x,\infty) = 0$. Also, using the expression of $H_\varphi(x,t)$,

$$\begin{aligned}
H_\varphi(x,t) &= \int_0^t \left[\int_{|y-x|=r} \frac{\varphi(y)dS_y}{\sqrt{t^2-|y-x|^2}}\right]dr \\
&= \int_0^t r^{2m-1}\left[\int_{|\omega|=1} \frac{\varphi(x+r\omega)}{\sqrt{t^2-r^2}}dS_\omega\right]dr \\
&= \int_0^t \frac{r^{2m-1}}{\sqrt{t^2-r^2}}\left[\int_{|\omega|=1} \varphi(x+r\omega)dS_\omega\right]dr \\
&= \int_0^t \left(t^2-\xi^2\right)^{2m-2}\left[\int_{|\omega|=1} \varphi\left(x+\sqrt{t^2-\xi^2}\omega\right)dS_\omega\right]d\xi,
\end{aligned} \tag{3.38}$$

we can check directly from (3.37) that $L(x,0) = 0$. Therefore, (3.35) becomes

$$\frac{1}{2^{m-1}\pi^m}\int_0^\infty g(t)\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^m tH_\varphi(x,t)dt = 2\int_0^\infty g(t)u(x,t)dt. \tag{3.39}$$

Since in (3.39) $g(t)$ is arbitrary smooth even function with compact support, we obtain that

$$u(x,t) = \frac{1}{2^m\pi^m}\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^m tH_\varphi(x,t) = \frac{1}{2^m\pi^m}\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^{m-1}\frac{\partial}{\partial t}H_\varphi(x,t). \tag{3.40}$$

This coincides with (1.18) by (3.32).

Appendix

For reader's convenience, in this section we give some explanation of several points in the paper.

(1) Let us show how (2.33) for $n \geq 2$ implies (2.8).

Let $n = 2m + 1$, where $m \geq 1$. Then, since

$$\frac{1}{2}\sigma_{2m} = \frac{1}{2}\frac{2\pi^m}{\Gamma(m)} = \frac{\pi^m}{(m-1)!}, \tag{A.1}$$

Equation (2.33) takes the form

$$Q(w) = \pi^m\int_w^\infty \frac{(t-w)^{m-1}}{(m-1)!}k(t)dt. \tag{A.2}$$

Hence applying the differentiation formula

$$\frac{d}{dw}\int_w^\infty G(t,w)dt = -G(w,w) + \int_w^\infty \frac{\partial G(t,w)}{\partial w}dt \tag{A.3}$$

repeatedly, we find

$$Q^{(m)}(w) = \pi^m(-1)^m k(w) \tag{A.4}$$

which gives (2.8) for $n = 2m + 1$.

In the case $n = 2m$ with $m \geq 1$, (2.33) takes the form

$$Q(w) = \frac{1}{2}\sigma_{2m-1}\int_w^\infty (t-w)^{(2m-3)/2}k(t)dt. \tag{A.5}$$

Hence,

$$Q^{(m-1)}(w) = \frac{1}{2}\sigma_{2m-1}\int_w^\infty (-1)^{m-1}\frac{2m-3}{2}\frac{2m-5}{2}\cdots\frac{1}{2}(t-w)^{-1/2}k(t)dt. \tag{A.6}$$

Therefore, taking into account that by virtue of

$$\Gamma(x) = (x-1)\Gamma(x-1), \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{A.7}$$

we have

$$\begin{aligned}\frac{1}{2}\sigma_{2m-1} &= \frac{\pi^{(2m-1)/2}}{\Gamma((2m-1)/2)} = \frac{\pi^{(2m-1)/2}}{((2m-3)/2)((2m-5)/2)\cdots(1/2)\Gamma(1/2)} \\ &= \frac{\pi^{m-1}}{((2m-3)/2)((2m-5)/2)\cdots(1/2)},\end{aligned} \tag{A.8}$$

we get

$$Q^{(m-1)}(w) = (-1)^{m-1}\pi^{m-1}\int_w^\infty \frac{k(t)}{\sqrt{t-w}}dt. \tag{A.9}$$

In the right-hand side we replace t by u, then divide both sides by $\sqrt{w-t}$ and integrate on $w \in (t,\infty)$ to get

$$\begin{aligned}\int_t^\infty \frac{Q^{(m-1)}(w)}{\sqrt{w-t}}dw &= (-1)^{m-1}\pi^{m-1}\int_t^\infty \frac{1}{\sqrt{w-t}}\left\{\int_w^\infty \frac{k(u)}{\sqrt{u-w}}du\right\}dw \\ &= (-1)^{m-1}\pi^{m-1}\int_t^\infty k(u)\left\{\int_t^u \frac{dw}{\sqrt{(w-t)(u-w)}}\right\}du \\ &= (-1)^{m-1}\pi^{m}\int_t^\infty k(u)du,\end{aligned} \tag{A.10}$$

because for any $t < u$, using the change of variables $\sqrt{w-t} = \xi$, we have

$$\begin{aligned}\int_t^u \frac{dw}{\sqrt{(w-t)(u-w)}} &= 2\int_0^{\sqrt{u-t}} \frac{d\xi}{\sqrt{u-t-\xi^2}} \\ &= 2\arcsin \frac{\xi}{\sqrt{u-t}}\bigg|_{\xi=0}^{\xi=\sqrt{u-t}} = 2\arcsin 1 = \pi.\end{aligned} \tag{A.11}$$

Therefore, differentiating (A.10) with respect to t, we get

$$\begin{aligned}k(t) &= \frac{(-1)^m}{\pi^m}\frac{d}{dt}\int_t^\infty \frac{Q^{(m-1)}(w)}{\sqrt{w-t}}dw = \frac{(-1)^m}{\pi^m}\frac{d}{dt}\int_0^\infty \frac{Q^{(m-1)}(u+t)}{\sqrt{u}}du \\ &= \frac{(-1)^m}{\pi^m}\int_0^\infty \frac{Q^{(m)}(u+t)}{\sqrt{u}}du = \frac{(-1)^m}{\pi^m}\int_t^\infty \frac{Q^{(m)}(w)}{\sqrt{w-t}}dw.\end{aligned} \tag{A.12}$$

Thus, (2.8) is obtained also for $n = 2m$ with $m \geq 1$.

(2) Here we explain (2.24). Note that since the spectrum of the operator A is $[0,\infty)$ (zero is included into the spectrum), the spectral representation formula (2.4) should be understood in the sense of the formula

$$\tilde{g}\left(A^{(1/2)}\right)f = \int_{-\delta}^{\infty} \tilde{g}(\sqrt{\mu})\, dE_\mu f, \tag{A.13}$$

where δ is an arbitrary positive real number and the integral does not depend on $\delta > 0$ (E_μ is zero on $(-\infty, 0)$ because A is a positive operator). Therefore, for (2.24) we have to show that

$$\lim_{\varepsilon\to+0} \frac{\varepsilon}{\pi} \int_{-\delta}^{\infty} \frac{\tilde{g}(\sqrt{\mu})}{\left(\mu-\lambda^2\right)^2+\varepsilon^2} d\mu = \tilde{g}(\lambda), \quad \lambda \in \mathbb{R}, \tag{A.14}$$

for any $\delta > 0$.

Since for any $\varepsilon > 0$

$$\frac{\varepsilon}{\pi} \int_{-\delta}^{\infty} \frac{1}{\left(\mu-\lambda^2\right)^2+\varepsilon^2} d\mu = \frac{\varepsilon}{\pi} \int_{-\delta-\lambda^2}^{\infty} \frac{du}{u^2+\varepsilon^2} = \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan\frac{\delta+\lambda^2}{\varepsilon}\right), \tag{A.15}$$

we have

$$\lim_{\varepsilon\to+0} \frac{\varepsilon}{\pi} \int_{-\delta}^{\infty} \frac{1}{\left(\mu-\lambda^2\right)^2+\varepsilon^2} d\mu = 1, \quad \lambda \in \mathbb{R},$$
$$\frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{du}{u^2+\varepsilon^2} = 1. \tag{A.16}$$

Given $\alpha > 0$, we can choose a $\beta > 0$ such that

$$\left|\tilde{g}\left(\sqrt{u+\lambda^2}\right) - \tilde{g}(\lambda)\right| < \alpha \quad \text{for } u \in \Omega = \left\{u : -\delta-\lambda^2 < u < \infty,\ |u| < \beta\right\} \tag{A.17}$$

since the function $\tilde{g}(z)$ is continuous for $z = \lambda$ (we choose the continuous branch of the square root for which $\sqrt{1} = 1$). Further, we choose a number M such that

$$\left|\tilde{g}(z)\right| \le M \quad \text{for } |\mathrm{Im}\, z| \le C < \infty, \tag{A.18}$$

for sufficiently large positive number C. This is possible by (2.3) and the fact that $g(t)$ has a compact support. Let us set $\Omega' = (-\delta - \lambda^2, \infty) \setminus \Omega$. Then,

$$\begin{aligned}
&\left| \frac{\varepsilon}{\pi} \int_{-\delta}^{\infty} \frac{\tilde{g}(\sqrt{\mu})}{(\mu-\lambda^2)^2+\varepsilon^2} d\mu - \tilde{g}(\lambda) \frac{\varepsilon}{\pi} \int_{-\delta}^{\infty} \frac{1}{(\mu-\lambda^2)^2+\varepsilon^2} d\mu \right| \\
&\quad \le \frac{\varepsilon}{\pi} \int_{-\delta-\lambda^2}^{\infty} \frac{\left|\tilde{g}\left(\sqrt{u+\lambda^2}\right) - \tilde{g}(\lambda)\right|}{u^2+\varepsilon^2} du \\
&\quad = \frac{\varepsilon}{\pi} \int_{\Omega} \frac{\left|\tilde{g}\left(\sqrt{u+\lambda^2}\right) - \tilde{g}(\lambda)\right|}{u^2+\varepsilon^2} du + \frac{\varepsilon}{\pi} \int_{\Omega'} \frac{\left|\tilde{g}\left(\sqrt{u+\lambda^2}\right) - \tilde{g}(\lambda)\right|}{u^2+\varepsilon^2} du.
\end{aligned} \tag{A.19}$$

Further,

$$\begin{aligned}
\frac{\varepsilon}{\pi} \int_{\Omega} \frac{\left|\tilde{g}\left(\sqrt{u+\lambda^2}\right) - \tilde{g}(\lambda)\right|}{u^2+\varepsilon^2} du &< \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon}{u^2+\varepsilon^2} du = \alpha, \\
\frac{\varepsilon}{\pi} \int_{\Omega'} \frac{\left|\tilde{g}\left(\sqrt{u+\lambda^2}\right) - \tilde{g}(\lambda)\right|}{u^2+\varepsilon^2} du &\le \frac{2M}{\pi} \int_{|u|\ge\beta} \frac{\varepsilon}{u^2+\varepsilon^2} du \\
&= \frac{4M}{\pi} \int_{\beta}^{\infty} \frac{\varepsilon}{u^2+\varepsilon^2} du = \frac{4M}{\pi} \left(\frac{\pi}{2} - \arctan \frac{\beta}{\varepsilon} \right).
\end{aligned} \tag{A.20}$$

For fixed β, the last expression tends to zero as $\varepsilon \to +0$; hence, and by (A.16), (A.19), and (A.20) we get (A.14).

(3) The formula (2.14) follows from (2.21) for $n = 1$ noting that

$$H^{(1)}_{-(1/2)}(z) = \left(\frac{2}{\pi z} \right)^{1/2} e^{iz}. \tag{A.21}$$

(4) The difference between operators $(\partial/\partial t \;\; 1/t)^m$ (formulae (1.17), (1.18)) and $(1/t \;\; \partial/\partial t)^m$ (formula (3.25)) is given by

$$\left(\frac{\partial}{\partial t} \frac{1}{t} \right)^m = \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{m-1} \frac{1}{t}. \tag{A.22}$$

(5) The explicit formula for the solution of the wave equation in the case n even can be derived from the case n odd by a known computation called the "method of descent" (see [1]).

(6) Since for $\operatorname{supp} g(t) \subset (-a, a)$, $a > 0$, we have $\operatorname{supp} k(t) \subset [0, a^2)$, and on the left-hand side of (2.10) the integral is taken in fact over the ball $\{y \in \mathbb{R}^n : |y - x| < a\}$, for fixed x. Therefore, this integral is finite for each $x \in \mathbb{R}^n$ and any solution $\psi(x, \lambda)$ of (2.9). We proved (2.10) for $\lambda \in \mathbb{R}$. If the solution $\psi(x, \lambda)$ is an analytic function of $\lambda \in \mathbb{C}$, then (2.10) will be held also for complex values of λ by the uniqueness of analytic continuation.

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