Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 727031, 10 pages
doi:10.1155/2012/727031

Research Article

Variational Homotopy Perturbation Method for Solving Fractional Initial Boundary Value Problems

Yanqin Liu[1, 2]

[1] *Department of Mathematics, Dezhou University, Dezhou 253023, China*
[2] *Nonlinear Dynamics and Chaos Group, School of Management, Tianjin University, Tianjin 30072, China*

Correspondence should be addressed to Yanqin Liu, yqlin8801@yahoo.cn

Received 12 March 2012; Accepted 5 April 2012

Academic Editor: Ibrahim Sadek

A variational homotopy perturbation method (VHPM) which is based on variational iteration method and homotopy perturbation method is applied to solve the approximate solution of the fractional initial boundary value problems. The nonlinear terms can be easily handled by the use of He's polynomials. It is observed that the variational iteration method is very efficient and easier to implements; illustrative examples are included to demonstrate the high accuracy and fast convergence of this new algorithm.

1. Introduction

Recently, it has turned out that many phenomena in engineering, physics, chemistry, and other sciences [1–3] can be described very successfully by models using mathematical tools for fractional calculus. The importance of obtaining the exact and approximate solutions of fractional nonlinear equations in physics and mathematics is still a significant problem that needs new methods to discover exact and approximate solutions. But these nonlinear fractional differential equations are difficult to get their exact solutions [4–7]. So, some semi-analytical techniques have also largely been used to solve these equations, such as, Adomian decomposition method [8, 9], variational iteration method [10–12], differential transform method [13, 14], Laplace decomposition method [15, 16], and homotopy perturbation method [17–23]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian polynomials, the Lagrange multiplier, divergent results, and huge computational work.

Variational homotopy perturbation method [24–26] has a very simple solution procedure and absorbs all of the positive features of variational iteration and homotopy perturbation methods and is highly compatible with the diversity of the physical problems. In this work, we will use variational homotopy perturbation method to solve fractional partial differential equations with initial and boundary conditions. The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. This paper considers the effectiveness of the variational homotopy perturbation method in solving fractional partial equations.

2. Description of the VHPM

To illustrate the basic idea of this method [24, 25], we consider a general fractional nonlinear nonhomogeneous partial differential equation with initial conditions of the following form:

$$\begin{gathered} D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \\ u(x,0) = h(x), \qquad u_t(x,0) = f(x), \end{gathered} \tag{2.1}$$

where $g(x,t)$ is the inhomogeneous term, N represents the general nonlinear differential operator, R is the linear differential operator, and $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of function $u(x,t)$ which is defined as

$${}_0D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(x,\tau)\,d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < \operatorname{Re}(\alpha) \le n, n \in N), \tag{2.2}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The properties of fractional derivative can be found in [1, 2]. According to variational iteration method [10, 11], we can construct a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left(\frac{\partial^m u_n}{\partial \xi^m}(x,\xi) + Ru_n(x,\xi) + N\tilde{u}_n(x,\xi) - g(x,\xi) \right) d\xi, \tag{2.3}$$

where the values of the natural number m can be 1 and 2 corresponding to $0 < \alpha \le 1$ and $1 < \alpha \le 2$, respectively, and λ is a Lagrange multiplier [11], which can be identified optimally via variational iteration method, for $m = 1$ and $\lambda = -1$ and for $m = 2$ and $\lambda = \xi - t$. The subscripts n denote the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation. That is, $\tilde{u}_n \delta = 0$, (2.3) is called a correction functional. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation $u_{n+1}, n \ge 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0; consequently, the solution is given by $u = \lim_{n\to\infty} u_n$. In the homotopy perturbation method, the basic assumption is that the solutions can be written as a power series in p:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{2.4}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u), \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter. $\mathcal{H}_n(u)$ is He's polynomials [26] that can be generated by

$$\mathcal{H}_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0,1,2\ldots. \tag{2.6}$$

The variational homotopy perturbation method is obtained by the elegant coupling of correction functional (2.3) of variational iteration method with He's polynomials and is given by

$$\begin{aligned}\sum_{n=0}^{\infty} p^n u_n(x,t) \\ = u_0(x,t) + p\int_0^t \lambda(\xi)\left(\sum_{n=0}^{\infty} p^n \frac{\partial^\alpha u_n}{\partial \xi^\alpha}(x,\xi) + \sum_{n=0}^{\infty} p^n Ru(x,\xi) + \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u) - g(x,\xi)\right)d\xi.\end{aligned} \tag{2.7}$$

Comparisons of like powers of p give solutions of various orders.

This method does not resort to linearization or assumptions of weak nonlinearity, the solutions generated in the form of general solution, and it is more realistic compared to the method of simplifying the physical problems.

3. Approximate Solutions of Fractional Equations

In order to assess the advantages and the accuracy of the variational homotopy perturbation method for fractional equations, we have applied it to the following several problems.

Case 1. Consider the one-dimensional fractional initial boundary value problem which describes the heat-like models [24]:

$$D_t^\alpha u(x,t) = \frac{1}{2}x^2 u_{xx}, \quad 0 < x < 1, \ t > 0, \tag{3.1}$$

with boundary conditions

$$\begin{aligned} u(0,t) &= 0, \qquad u(1,t) = e^t, \\ u(x,0) &= x^2, \end{aligned} \tag{3.2}$$

where $0 < \alpha \le 1$, and the correct functional is given as

$$u_{n+1}(x,t) = x^2 - \int_0^t \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha}(x,\xi) - \frac{1}{2}x^2 u_{nxx} \right) d\xi. \tag{3.3}$$

Applying the modified variational iteration method, one has

$$\begin{aligned}\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 - p\int_0^t \Bigg(\left(\frac{\partial^\alpha u_0}{\partial \xi^\alpha} + p\frac{\partial^\alpha u_1}{\partial \xi^\alpha} + p^2 \frac{\partial^\alpha u_2}{\partial \xi^\alpha} + \cdots \right) \\ -\frac{1}{2}\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \cdots \right)\Bigg) d\xi.\end{aligned} \tag{3.4}$$

Comparing the coefficient of like powers of p,

$$\begin{aligned} p^0 &: u_0(x,t) = x^2, \\ p^1 &: u_1(x,t) = x^2(1+t), \\ p^2 &: u_2(x,t) = x^2(1+t) + tx^2 + \frac{t^2x^2}{2} - \frac{t^{2-\alpha}x^2}{\Gamma(3-\alpha)}, \\ p^3 &: u_2(x,t) = x^2 + 3tx^2 + \frac{3t^2x^2}{2} + \frac{t^3x^2}{6} + \frac{t^{3-2\alpha}x^2}{\Gamma(4-2\alpha)} - \frac{3t^{2-\alpha}x^2}{\Gamma(3-\alpha)} - \frac{2t^{3-\beta}x^2}{\Gamma(4-\beta)}, \\ &\vdots \end{aligned} \tag{3.5}$$

and so on, in this manner the rest of component of the solution can be obtained. If we take $\alpha = 1$, the first few components of the solution of (3.1) are as follows:

$$\begin{aligned} u_0(x,t) &= x^2, \\ u_1(x,t) &= x^2(1+t), \\ p^2 : u_2(x,t) &= x^2\left(1 + t + \frac{t^2}{2!}\right), \\ p^3 : u_2(x,t) &= x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right), \\ &\vdots \end{aligned} \tag{3.6}$$

The solution of (3.1) in series form is given by

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right). \tag{3.7}$$

The series solution in a closed form is given by $u(x,t) = x^2e^t$, which was given in [24].

Case 2. Consider the following homogeneous fractional coupled Burger's equation [26]:

$$D_t^{\alpha} u - u_{xx} - 2uu_x + (uv)_x = 0, \tag{3.8}$$

$$D_t^{\beta} v - v_{xx} - 2vv_x + (uv)_x = 0, \tag{3.9}$$

with initial conditions

$$u(x,0) = \sin x, \tag{3.10}$$

$$v(x,0) = \sin x, \tag{3.11}$$

where $0 < \alpha,\ \beta \le 1$. The correction functional for the previous fractional coupled system is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial^{\alpha} u_n}{\partial \xi^{\alpha}} - \frac{\partial^2 u_n}{\partial x^2} - 2u_n (u_n)_x + (u_n v_n)_x \right) d\xi, \tag{3.12}$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial^{\beta} v_n}{\partial \xi^{\beta}} - \frac{\partial^2 v_n}{\partial x^2} - 2v_n (v_n)_x + (u_n v_n)_x \right) d\xi. \tag{3.13}$$

Applying the variational homotopy perturbation method using He's polynomials, we get

$$\begin{aligned}
u_0 + pu_1 + p^2 u_2 + \cdots &= u_0(x,t) - p \int_0^t \left(\left(\frac{\partial^{\alpha} u_0}{\partial \xi^{\alpha}} + p \frac{\partial^{\alpha} u_1}{\partial \xi^{\alpha}} + \cdots \right) - \left(\frac{\partial^2 u_0}{\partial x^2} \right) + p \frac{\partial^2 u_1}{\partial x^2} + \cdots \right) d\xi, \\
&\quad + p \int_0^t \Big(u_0 u_{0x} + p(u_0 u_{1x} + u_1 u_{0x}) + p^2 (u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} + \cdots) \Big) \\
&\qquad - \Big((u_0 v_0)_{xx} + p(u_1 v_0 + u_0 v_1)_{xx} + p^2 (u_0 v_2 + u_1 v_1 + u_0 v_2)_{xx} \Big) d\xi, \\
v_0 + pv_1 + p^2 v_2 + \cdots &= v_0(x,t) - p \int_0^t \left(\left(\frac{\partial^{\beta} v_0}{\partial \xi^{\beta}} + p \frac{\partial^{\beta} v_1}{\partial \xi^{\beta}} + \cdots \right) - \left(\frac{\partial^2 v_0}{\partial x^2} \right) + p \frac{\partial^2 v_1}{\partial x^2} + \cdots \right) d\xi \\
&\quad + p \int_0^t \Big(v_0 v_{0x} + p(v_0 v_{1x} + v_1 v_{0x}) + p^2 (v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x} + \cdots) \Big) \\
&\qquad - \Big((u_0 v_0)_{xx} + p(u_1 v_0 + u_0 v_1)_{xx} + p^2 (u_0 v_2 + u_1 v_1 + u_0 v_2)_{xx} \Big) d\xi.
\end{aligned} \tag{3.14}$$

Comparing the coefficient of like powders of p, one has

$$\begin{aligned}
&u_0(x,0) = \sin x, \\
&v_0(x,0) = \sin x, \\
&u_1(x,0) = \sin x - t \sin x,
\end{aligned}$$

$$
\begin{aligned}
v_1(x,0) &= \sin x - t\sin x,\\
u_2(x,0) &= \sin x - 2t\sin x + \frac{t^2\sin x}{2} + \frac{t^{2-\alpha}\sin x}{\Gamma(3-\alpha)},\\
v_2(x,0) &= \sin x - 2t\sin x + \frac{t^2\sin x}{2} + \frac{t^{2-\beta}\sin x}{\Gamma(3-\beta)},\\
u_3(x,0) &= \sin x - 3t\sin x + \frac{3t^2\sin x}{2} - \frac{t^3\sin x}{6} + \frac{t^{3-2\alpha}\sin x}{\Gamma(4-2\alpha)} + \frac{3t^{2-\alpha}\sin x}{\Gamma(3-\alpha)} - \frac{2t^{3-\alpha}\sin x}{\Gamma(4-\alpha)},\\
v_3(x,0) &= \sin x - 3t\sin x + \frac{3t^2\sin x}{2} - \frac{t^3\sin x}{6} + \frac{t^{3-2\beta}\sin x}{\Gamma(4-2\beta)} + \frac{3t^{2-\beta}\sin x}{\Gamma(3-\beta)} - \frac{2t^{3-\beta}\sin x}{\Gamma(4-\beta)}.
\end{aligned}
\tag{3.15}
$$

If we take $\alpha = \beta = 1$, the series solutions in closed-form are given by

$$
\begin{aligned}
u(x,t) &= \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) = \exp(-t)\sin x,\\
v(x,t) &= \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) = \exp(-t)\sin x,
\end{aligned}
\tag{3.16}
$$

which is in full agreement with the result given in [26].

Figure 1 depicts the third-order approximate solutions $u(x,t)$ or $v(x,t)$ for (3.8)–(3.11) by using the variational homotopy perturbation method when choosing $x = 0.8$. From the figure, it is clear to see the time evolution of fractional Burger equation and we know that the approximate solution of the model is continuous with the fractional parameter α. Figure 2 shows the third-order approximate solutions $u(x,t)$ or $v(x,t)$ for (3.8)–(3.11) when $t = 0.8$, and we also know that the solution of the fractional nonlinear equation changes with the parameters α.

Case 3. In this case, we consider the space fractional backward Kolmogorov equation as follows [27]:

$$
\frac{\partial u(x,t)}{\partial t} = \frac{(x+1)\partial^{\beta}u}{\partial x^{\beta}} + \frac{(x^2e^t)\partial^{2\beta}u}{\partial x^{2\beta}},
\tag{3.17}
$$

subject to the initial condition:

$$
u(x,0) = x+1,
\tag{3.18}
$$

where $0 < \beta \le 1$. Space fractional derivative is also the Caputo definition with respect to x and is defined as

$$
D_x^{\beta}u(x,t) = \frac{1}{\Gamma(m-\beta)}\int_0^x \frac{u^{(m)}(\xi,t)\,d\xi}{(x-\xi)^{\beta+1-m}} \quad (m-1 < \mathrm{Re}(\beta) \le m, m \in N).
\tag{3.19}
$$

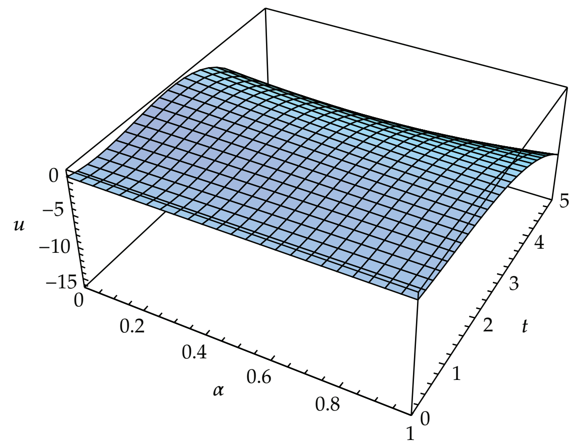

Figure 1: The surface of third-order approximate solution $u(x,t)$ of (3.8) and (3.9) when $x = 0.8$.

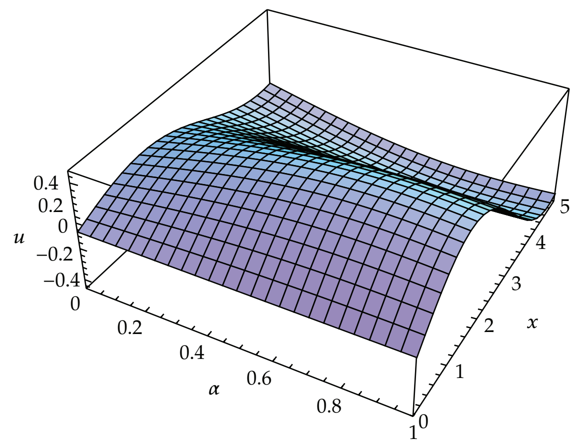

Figure 2: The surface of third-order approximate solution $u(x,t)$ of (3.8) and (3.9) when $t = 0.8$.

The correction functional for the previous space fractional system is given by

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{(x+1)\partial^{\beta} u_n}{\partial x^{\beta}} - \frac{(x^2 e^{\xi})\partial^{2\beta} u_n}{\partial x^{2\beta}} \right) d\xi. \tag{3.20}$$

Applying the modified variational iteration method, one has

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x,t) = x + 1 - \int_0^t \Bigg(&\frac{\partial \sum_{n=0}^{\infty} p^n u_n(x,t)}{\partial \xi} - \frac{(x+1)\partial^{\beta} \sum_{n=0}^{\infty} p^n u_n(x,t)}{\partial x^{\beta}} \\ &- \frac{(x^2 e^{\xi})\partial^{2\beta} \sum_{n=0}^{\infty} p^n u_n(x,t)}{\partial x^{2\beta}} \Bigg) d\xi. \end{aligned} \tag{3.21}$$

Comparing the coefficient of like powders of P, we can obtain the following approximations:

$$
\begin{aligned}
u_0 &= x+1,\\
u_1 &= 1+x-\frac{2x^{3-2\beta}}{\Gamma(2-2\beta)}+\frac{2e^t x^{3-2\beta}}{\Gamma(2-2\beta)}+\frac{tx^{1-\beta}}{\Gamma(2-\beta)}+\frac{tx^{2-\beta}}{\Gamma(2-\beta)},\\
u_2 &= 1+x+\frac{6x^{5-4\beta}}{\Gamma(4-4\beta)}-\frac{12e^t x^{5-4\beta}}{\Gamma(4-4\beta)}+\frac{6e^t x^{5-4\beta}}{\Gamma(4-4\beta)}-\frac{10x^{5-4\beta}\beta}{\Gamma(4-4\beta)}+\frac{20e^t x^{5-4\beta}\beta}{\Gamma(4-4\beta)}\\
&\quad-\frac{10e^{2t}x^{5-4\beta}\beta}{\Gamma(4-4\beta)}+\frac{4x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)}-\frac{8e^t x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)}+\frac{4e^{2t}x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)}+\frac{x^{3-3\beta}\beta}{\Gamma(2-3\beta)}-\frac{e^t x^{3-3\beta}}{\Gamma(2-3\beta)}\\
&\quad+\frac{e^t t x^{3-3\beta}}{\Gamma(2-3\beta)}+\frac{2x^{4-3\beta}\beta}{\Gamma(3-3\beta)}-\frac{2e^t x^{4-3\beta}}{\Gamma(3-3\beta)}+\frac{2e^t t x^{4-3\beta}}{\Gamma(3-3\beta)}-\frac{x^{4-3\beta}\beta}{\Gamma(3-3\beta)}+\frac{e^t x^{4-3\beta}\beta}{\Gamma(3-3\beta)}\\
&\quad+\frac{tx^{2-\beta}}{\Gamma(2-\beta)}-\frac{e^t t x^{4-3\beta}\beta}{\Gamma(3-3\beta)}+\frac{t^2x^{1-2\beta}}{2\Gamma(2-2\beta)}+\frac{t^2x^{2-2\beta}}{2\Gamma(2-2\beta)}-\frac{x^{3-2\beta}}{\Gamma(2-2\beta)}+\frac{e^t x^{3-2\beta}}{\Gamma(2-2\beta)}\\
&\quad+\frac{tx^{1-\beta}}{\Gamma(2-\beta)}+\frac{2e^t x^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)}-\frac{2x^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)}-\frac{2tx^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)}\\
&\quad-\frac{2x^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)}+\frac{2e^t x^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)}-\frac{2tx^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)}\\
&\quad+\frac{t^2x^{2-2\beta}\Gamma(3-\beta)}{2\Gamma(3-2\beta)}+\frac{t^2x^{3-2\beta}\Gamma(3-\beta)}{2\Gamma(3-2\beta)},
\end{aligned}
\tag{3.22}
$$

and so on; in the same manner the rest of components of the iteration formula (3.21) can be obtained using the Mathematica package. When fractional derivatives are $\beta = 1$, the exact solution of (3.17) was given in [27] using homotopy perturbation method and the approximate solution of (3.17) is

$$
\begin{aligned}
u_0 &= x+1,\\
u_1 &= (x+1)(1+t),\\
u_2 &= (x+1)\left(1+t+\frac{t^2}{2!}\right),\\
&\ \vdots
\end{aligned}
\tag{3.23}
$$

and so on. Hence, we have the closed form

$$
u(x,t)=(x+1)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)=\exp(t)(x+1),
\tag{3.24}
$$

which is the exact solution of the corresponding integer problem.

4. Conclusion

In this work, a variational homotopy perturbation method which is based on homotopy perturbation method and variational iteration method is used to solve fractional partial equations. The nonlinear terms can be easily handled by the use of He's polynomials. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the result, and the size reduction amounts to an improvement of the performance of the approach. The VHPM can be applied for some other engineering system with less computational work.

Acknowledgment

The authors would like to express their thanks to the referees for their fruitful advices and comments. This paper was supported by the National Science Foundation of Shandong Province (Grant nos. Y2007A06 and ZR2010Al019) and the China Postdoctoral Science Foundation (Grant no. 20100470783).

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.
[2] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, Singapore, 2000.
[3] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, p. 177, 2000.
[4] S. W. Wang and M. Y. Xu, "Axial Couette flow of two kinds of fractional viscoelastic fluids in an annulus," *Nonlinear Analysis. Real World Applications*, vol. 10, no. 2, pp. 1087–1096, 2009.
[5] X. Y. Jiang and M. Y. Xu, "Analysis of fractional anomalous diffusion caused by an instantaneous point source in disordered fractal media," *International Journal of Non-Linear Mechanics*, vol. 41, no. 1, pp. 156–165, 2006.
[6] J. H. Ma and Y. Q. Liu, "Exact solutions for a generalized nonlinear fractional Fokker-Planck equation," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 1, pp. 515–521, 2010.
[7] Y. Q. Liu and J. H. Ma, "Exact solutions of a generalized multi-fractional nonlinear diffusion equation in radical symmetry," *Communications in Theoretical Physics*, vol. 52, no. 5, pp. 857–861, 2009.
[8] G. Adomian, "A review of the decomposition method in applied mathematics," *Journal of Mathematical Analysis and Applications*, vol. 135, no. 2, pp. 501–544, 1988.
[9] A.-M. Wazwaz and S. M. El-Sayed, "A new modification of the Adomian decomposition method for linear and nonlinear operators," *Applied Mathematics and Computation*, vol. 122, no. 3, pp. 393–405, 2001.
[10] J. H. He, "Variational iteration method-a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.
[11] A.-M. Wazwaz, "The variational iteration method for analytic treatment of linear and nonlinear ODEs," *Applied Mathematics and Computation*, vol. 212, no. 1, pp. 120–134, 2009.
[12] A. Yildirim, "Variational iteration method for modified Camassa-Holm and Degasperis-Procesi equations," *International Journal for Numerical Methods in Biomedical Engineering*, vol. 26, no. 2, pp. 266–272, 2010.
[13] V. S. Ertürk, S. Momani, and Z. Odibat, "Application of generalized differential transform method to multi-order fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 8, pp. 1642–1654, 2008.
[14] A. Al-rabtah, V. S. Ertürk, and S. Momani, "Solutions of a fractional oscillator by using differential transform method," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1356–1362, 2010.
[15] E. Yusufoglu, "Numerical solution of Duffing equation by the Laplace decomposition algorithm," *Applied Mathematics and Computation*, vol. 177, no. 2, pp. 572–580, 2006.
[16] Y. Khan, "An effective modification of the laplace decomposition method for nonlinear equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 11-12, pp. 1373–1376, 2009.

[17] S. Momani and Z. Odibat, "Homotopy perturbation method for nonlinear partial differential equations of fractional order," *Physics Letters A*, vol. 365, no. 5-6, pp. 345–350, 2007.
[18] J.-H. He, "Recent development of the homotopy perturbation method," *Topological Methods in Nonlinear Analysis*, vol. 31, no. 2, pp. 205–209, 2008.
[19] M. A. Noor, "Iterative methods for nonlinear equations using homotopy perturbation technique," *Applied Mathematics & Information Sciences*, vol. 4, no. 2, pp. 227–235, 2010.
[20] M. A. Noor, "Some iterative methods for solving nonlinear equations using homotopy perturbation method," *International Journal of Computer Mathematics*, vol. 87, no. 1–3, pp. 141–149, 2010.
[21] Y. Khan and Q. Wu, "Homotopy perturbation transform method for nonlinear equations using He's polynomials," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 1963–1967, 2011.
[22] A. Yildirim, "Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method," *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3175–3180, 2008.
[23] A. Yildirim, "Application of He's homotopy perturbation method for solving the Cauchy reaction-diffusion problem," *Computers & Mathematics with Applications*, vol. 57, no. 4, pp. 612–618, 2009.
[24] M. A. Noor and S. T. Mohyud-Din, "Modified variational iteration method for heat and wave-like equations," *Acta Applicandae Mathematicae*, vol. 104, no. 3, pp. 257–269, 2008.
[25] M. A. Noor and S. T. Mohyud-Din, "Variational homotopy perturbation method for solving higher dimensional initial boundary value problems," *Mathematical Problems in Engineering*, Article ID 696734, 11 pages, 2008.
[26] S. T. Mohyud-Din, A. Yildirim, and M. M. Hosseini, "Variational iteration method for initial and boundary value problems using He's polynomials," *International Journal of Differential Equations*, Article ID 426213, 28 pages, 2010.
[27] A. Yildirim, "Application of the homotopy perturbation method for the Fokker-Planck equation," *International Journal for Numerical Methods in Biomedical Engineering*, vol. 26, no. 9, pp. 1144–1154, 2010.