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Research Article

Implicit and Explicit Iterations with Meir-Keeler-Type Contraction for a Finite Family of Nonexpansive Semigroups in Banach Spaces

Jiancai Huang,[1] Shenghua Wang,[2] and Yeol Je Cho[3]

[1] *School of Control and Computer Engineering, North China Electric Power University, Baoding 071003, China*
[2] *School of Mathematics and Physics, North China Electric Power University, Baoding 071003, China*
[3] *Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea*

Correspondence should be addressed to Yeol Je Cho, yjcho@gnu.ac.kr

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We introduce an implicit and explicit iterative schemes for a finite family of nonexpansive semigroups with the Meir-Keeler-type contraction in a Banach space. Then we prove the strong convergence for the implicit and explicit iterative schemes. Our results extend and improve some recent ones in literatures.

1. Introduction

Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a mapping. We call T nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. The set of all fixed points of T is denoted by $\text{Fix}(T)$, that is, $\text{Fix}(T) = \{x \in C : x = Tx\}$.

One parameter family $\mathcal{T} = \{T(t) : t \geq 0\}$ is said to a semigroup of nonexpansive mappings or nonexpansive semigroup on C if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in C$;

(2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(3) for each $t \geq 0$, $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$;

(4) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of all nonnegative reals, into C is continuous.

We denote by $\text{Fix}(\mathfrak{T})$ the set of all common fixed points of semigroup $\mathfrak{T}$, that is, $\text{Fix}(\mathfrak{T}) = \{x \in C : T(t)x = x,\ 0 \le t < \infty\}$ and $\mathbb{N}$ by the set of natural numbers.

Now, we recall some recent work on nonexpansive semigroup in literatures. In [1], Shioji and Takahashi introduced the following implicit iteration for a nonexpansive semigroup in a Hilbert space:

$$x_n = \alpha_n x + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0,\infty)$. Under the certain conditions on $\{\alpha_n\}$ and $\{t_n\}$, they proved that the sequence $\{x_n\}$ defined by (1.1) converges strongly to an element in $\text{Fix}(\mathfrak{T})$.

In [2], Suzuki introduced the following implicit iteration for a nonexpansive semigroup in a Hilbert space:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0,\infty)$. Under the conditions that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$, he proved that $\{x_n\}$ defined by (1.2) converges strongly to an element of $\text{Fix}(\mathfrak{T})$. Later on, Xu [3] extended the iteration (1.2) to a uniformly convex Banach space that admits a weakly sequentially continuous duality mapping. Song and Xu [4] also extended the iteration (1.2) to a reflexive and strictly convex Banach space.

In 2007, Chen and He [5] studied the following implicit and explicit viscosity approximation processes for a nonexpansive semigroup in a reflexive Banach space admitting a weakly sequentially continuous duality mapping:

$$\begin{aligned} x_n &= \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \\ y_{n+1} &= \beta_n f(y_n) + (1 - \beta_n)T(t_n)y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where f is a contraction, $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0,\infty)$. They proved the strong convergence for the above iterations under some certain conditions on the control sequences.

Recently, Chen et al. [6] introduced the following implicit and explicit iterations for nonexpansive semigroups in a reflexive Banach space admitting a weakly sequentially continuous duality mapping:

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)T(t_n)x_n, \\ x_n &= \beta_n f(x_n) + (1 - \beta_n)y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)T(t_n)x_n, \\ x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n)y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where f is a contraction, $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0,\infty)$. They proved that $\{x_n\}$ defined by (1.4) and (1.5) converges strongly to an element q of $\text{Fix}(\mathfrak{T})$, which is the unique solution of the following variation inequality problem:

$$\langle (f - I), j(x - q) \rangle \le 0, \quad \forall x \in \text{Fix}(\mathfrak{T}). \tag{1.6}$$

For more convergence theorems on implicit and explicit iterations for nonexpansive semigroups, refer to [7–13].

In this paper, we introduce an implicit and explicit iterative process by a generalized contraction for a finite family of nonexpansive semigroups in a Banach space. Then we prove the strong convergence for the iterations and our results extend the corresponding ones of Suzuki [2], Xu [3], Chen and He [5], and Chen et al. [6].

2. Preliminaries

Let E be a Banach space and E^* the duality space of E. We denote the normalized mapping from E to 2^{E^*} by J defined by

$$J(x) = \left\{ j \in E^* : \langle x, jx \rangle = \|x\|^2 = \left\| j \right\| \right\}, \quad \forall x \in E, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. For any $x, y \in E$ with $j(x) \in J(x)$ and $j(x+y) \in J(x+y)$, it is well known that the following inequality holds:

$$\|x\|^2 + 2\langle y, j(x) \rangle \leq \left\| x + y \right\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle. \tag{2.2}$$

The dual mapping J is called weakly sequentially continuous if J is single valued, and $\{x_n\} \rightharpoonup x \in E$, where $\rightharpoonup$ denotes the weak convergence, then $J(x_n)$ weakly star converges to $J(x)$ [14–16]. A Banach space E is called to satisfy Opial's condition [17] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$,

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \left\| x_n - y \right\|, \quad \forall y \in E \text{ with } x \neq y. \tag{2.3}$$

It is known that if E admits a weakly sequentially continuous duality mapping J, then E is smooth and satisfies Opial's condition [14].

A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an L-function if $\psi(0) = 0$, $\psi(t) > 0$ for any $t > 0$, and for every $t > 0$ and $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$, for all $t \in [s, u]$. This implies that $\psi(t) < t$ for all $t > 0$.

Let $f : C \to C$ be a mapping. f is said to be a (ψ, L)-contraction if there exists a L-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|f(x) - f(y)\| < \psi(\|x - y\|)$ for all $x, y \in C$ with $x \neq y$. Obviously, if $\psi(t) = kt$ for all $t > 0$, where $k \in (0, 1)$, then f is a contraction. f is called a Meir-Keeler-type mapping if for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x, y \in C$, if $\epsilon < \|x - y\| < \epsilon + \delta$, then $\|f(x) - f(y)\| < \epsilon$.

In this paper, we always assume that $\psi(t)$ is continuous, strictly increasing and $\lim_{t \to \infty} \eta(t) = \infty$, where $\eta(t) = t - \psi(t)$, is strictly increasing and onto.

The following lemmas will be used in next section.

Lemma 2.1 (see [18]). *Let (X, d) be a metric space and $f : X \to X$ be a mapping. The following assertions are equivalent:*

(i) *f is a Meir-Keeler-type mapping,*

(ii) *there exists an L-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is a (ψ, L)-contraction.*

Lemma 2.2 (see [19]). *Let E be a Banach space and C be a convex subset of E. Let $T : C \to C$ be a nonexpansive mapping and f be a (ψ, L)-contraction. Then the following assertions hold:*

(i) *$T \circ f$ is a (ψ, L)-contraction on C and has a unique fixed point in C;*

(ii) *for each $\alpha \in (0,1)$, the mapping $x \mapsto \alpha f(x) + (1-\alpha)Tx$ is of Meir-Keeler-type and it has a unique fixed point in C.*

Lemma 2.3 (see [20]). *Let E be a Banach space and C be a convex subset of E. Let $f : C \to C$ be a Meir-Keeler-type contraction. Then for each $\epsilon > 0$ there exists $r \in (0,1)$ such that, for each $x, y \in C$ with $\|x-y\| \geq \epsilon$, $\|f(x) - f(y)\| \leq r\|x-y\|$.*

Lemma 2.4 (see [21]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $T_m : C \to C$ be a nonexpansive mapping for each $1 \leq m \leq r$, where r is some integer. Suppose that $\cap_{m=1}^{r} \mathrm{Fix}(T_m)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{r} \lambda_n = 1$. Then the mapping $S : C \to C$ defined by*

$$Sx = \sum_{m=1}^{r} \lambda_m T_m x, \quad \forall x \in C, \tag{2.4}$$

is well defined, nonexpansive and $\mathrm{Fix}(S) = \cap_{m=1}^{r} \mathrm{Fix}(T_m)$ holds.

Lemma 2.5 (see [22]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad n \in \mathbb{N}, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\lim_{n\to\infty} \gamma_n = 0$;

(ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(iii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main Results

In this section, by a generalized contraction mapping we mean a Meir-Keeler-type mapping or (ψ, L)- contraction. In the rest of the paper we suppose that ψ from the definition of the (ψ, L)-contraction is continuous, strictly increasing and $\eta(t)$ is strictly increasing and onto, where $\eta(t) = t - \psi(t)$, for all $t \in \mathbb{R}^+$. As a consequence, we have the $\eta(t)$ is a bijection on $\mathbb{R}^+$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a reflexive Banach space E which admits a weakly sequentially continuous duality mapping J from E into E^*. For every $i = 1, \ldots, N (N \geq 1)$, let $\mathcal{T}_i = \{T_i(t) : t \geq 0\}$ be a semigroup of nonexpansive mappings on C such that $\mathcal{F} = \cap_{i=1}^{N} \mathrm{Fix}(\mathcal{T}_i) \neq \emptyset$ and $f : C \to C$ be a generalized contraction on C. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$ and $\{t_n\} \subset (0,\infty)$ be* *the sequences satisfying* $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\alpha_n/t_n) = 0$ *and* $\limsup_{n\to\infty}\beta_n < 1$. *Let* $\{x_n\}$ *be a sequence generated by*

$$x_n = \alpha_n f(x_n) + \frac{1-\alpha_n}{N}\sum_{i=1}^{N} y_{in},$$
$$y_{in} = \beta_n x_n + (1-\beta_n)T_i(t_n)x_n, \quad i = 1,\dots,N. \tag{3.1}$$

Then $\{x_n\}$ *converges strongly to a point* $x^* \in \mathcal{F}$, *which is the unique solution to the following variational inequality:*

$$\langle (f-I)x^*,\ j(x-x^*)\rangle \le 0, \quad \forall x \in \mathcal{F}. \tag{3.2}$$

Proof. First, we show that the sequence $\{x_n\}$ generated by (3.1) is well defined. For every $n \in \mathbb{N}$ and $i = 1,\dots,N$, let $U_{in} = \beta_n I + (1-\beta_n)T_i(t_n)$ and define $W_n : C \to C$ by

$$W_n x = \alpha_n f(x) + (1-\alpha_n)G_n x, \quad \forall x \in C, \tag{3.3}$$

where $G_n x = (1/N)\sum_{i=1}^{N} U_{in}x$. Since U_{in} is nonexpansive, G_n is nonexpansive. By Lemma 2.2 we see that W_n is a Meir-Keeler-type contraction for each $n \in \mathbb{N}$. Hence, each W_n has a unique fixed point, denoted as x_n, which uniquely solves the fixed point equation (3.3). Hence $\{x_n\}$ generated by (3.1) is well defined.

Now we prove that $\{x_n\}$ generated by (3.1) is bounded. For any $p \in \mathcal{F}$, we have

$$\|y_{in} - p\| \le \beta_n\|x_n - p\| + (1-\beta_n)\|T_i(t_n)x_n - p\| \le \|x_n - p\|. \tag{3.4}$$

Using (3.4), we get

$$\begin{aligned}
\|x_n - p\|^2 &= \left\langle \alpha_n f(x_n) + \frac{1-\alpha_n}{N}\sum_{i=1}^{N} y_{in} - p, j(x_n - p)\right\rangle \\
&= \alpha_n\langle f(x_n) - f(p), j(x_n - p)\rangle + \alpha_n\langle f(p) - p, j(x_n - p)\rangle \\
&\quad + \frac{1-\alpha_n}{N}\sum_{i=1}^{N}\langle y_{in} - p, j(x_n - p)\rangle \\
&\le \alpha_n\psi(\|x_n - p\|)\|x_n - p\| + \alpha_n\|f(p) - p\|\|x_n - p\| \\
&\quad + \frac{1-\alpha_n}{N}\sum_{i=1}^{N}\|y_{in} - p\|\|x_n - p\| \\
&= \alpha_n\psi(\|x_n - p\|)\|x_n - p\| + \alpha_n\|f(p) - p\|\|x_n - p\| \\
&\quad + (1-\alpha_n)\|x_n - p\|^2
\end{aligned} \tag{3.5}$$

and hence

$$\|x_n - p\| \le \psi(\|x_n - p\|) + \|f(p) - p\|, \tag{3.6}$$

which implies that

$$\eta(\|x_n - p\|) = \|x_n - p\| - \psi(\|x_n - p\|) \le \|f(p) - p\|. \tag{3.7}$$

Hence

$$\|x_n - p\| \le \eta^{-1}(\|f(p) - p\|). \tag{3.8}$$

This shows that $\{x_n\}$ is bounded, and so are $\{T_i(t_n)x_n\}$, $\{f(x_n)\}$ and $\{y_{in}\}$.

Since E is reflexivity and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$ for some $x^* \in C$ as $j \to \infty$. Now we prove that $x^* \in \mathcal{F}$. For any fixed $t > 0$, we have

$$\begin{aligned}
\sum_{i=1}^{N}\left\|x_{n_j} - T_i(t)x^*\right\| &\le \sum_{i=1}^{N}\left[\sum_{k=0}^{[t/t_{n_i}]-1}\left\|T_i\left((k+1)t_{n_j}\right)x_{n_j} - T_i\left(kt_{n_j}\right)x_{n_j}\right\|\right.\\
&\quad\left.+\left\|T_i\left(\left[\frac{t}{t_{n_j}}\right]t_{n_j}\right)x_{n_j} - T_i\left(\left[\frac{t}{t_{n_j}}\right]t_{n_j}\right)x^*\right\| + \left\|T_i\left(\left[\frac{t}{t_{n_j}}\right]t_{n_j}\right)x_{n_j} - T_i(t)x^*\right\|\right]\\
&\le \sum_{i=1}^{N}\left[\left[\frac{t}{t_{n_j}}\right]\left\|T_i\left(t_{n_j}\right)x_{n_j} - x_{n_j}\right\| + \left\|x_{n_j} - x^*\right\| + \left\|T_i\left(t - \left[\frac{t}{t_{n_j}}\right]t_{n_j}\right)x_{n_j} - x^*\right\|\right]\\
&\le \sum_{i=1}^{N}\left[\left[\frac{t}{t_{n_j}}\right]\left\|T_i\left(t_{n_j}\right)x_{n_j} - x_{n_j}\right\| + \left\|x_{n_j} - x^*\right\| + \max\left\{\|T_i(s)x^* - x^*\| : 0 \le s \le t_{n_j}\right\}\right]\\
&\le \frac{N\alpha_{n_j}\left[t/t_{n_j}\right]}{\left(1-\alpha_{n_j}\right)\left(\left(1-\beta_{n_j}\right)\right)}\left\|x_{n_j} - f\left(x_{n_j}\right)\right\| + N\left\|x_{n_j} - x^*\right\|\\
&\quad + \sum_{i=1}^{N}\max\left\{\|T_i(s)x^* - x^*\| : 0 \le s \le t_{n_j}\right\}\\
&\le \frac{Nt}{\left(1-\alpha_{n_j}\right)\left(1-\beta_{n_j}\right)}\frac{\alpha_{n_j}}{t_{n_j}}\left\|x_{n_j} - f\left(x_{n_j}\right)\right\| + N\left\|x_{n_j} - x^*\right\|\\
&\quad + \sum_{i=1}^{N}\max\left\{\|T_i(s)x^* - x^*\| : 0 \le s \le t_{n_j}\right\}.
\end{aligned} \tag{3.9}$$

By hypothesis on $\{t_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, we have

$$\lim_{j\to\infty} \frac{Nt}{\left(1-\alpha_{n_j}\right)\left(1-\beta_{n_j}\right)} \frac{\alpha_{n_j}}{t_{n_j}} = 0. \tag{3.10}$$

Further, from (3.9) we get

$$\limsup_{j\to\infty} \sum_{i=1}^{N} \left\| x_{n_j} - T_i(t)x^* \right\| \leq \limsup_{j\to\infty} N \left\| x_{n_j} - x^* \right\|. \tag{3.11}$$

Since E admits a weakly sequentially duality mapping, we see that E satisfies Opial's condition. Thus if $x^* \notin \mathcal{F}$, we have

$$\limsup_{j\to\infty} N \left\| x_{n_j} - x^* \right\| < \limsup_{j\to\infty} \sum_{i=1}^{N} \left\| x_{n_j} - T_i x^* \right\|. \tag{3.12}$$

This contradicts (3.11). So $x^* \in \mathcal{F}$.

In (3.5), replacing p with x^* and n with n_j, we see that

$$\begin{aligned}
\left\| x_{n_j} - x^* \right\|^2 &= \alpha_{n_j} \left\langle f\left(x_{n_j}\right) - f(x^*), j\left(x_{n_j} - x^*\right) \right\rangle + \alpha_{n_j} \left\langle f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right\rangle \\
&\quad + \frac{1-\alpha_{n_j}}{N} \sum_{i=1}^{N} \left\langle y_{in_j} - x^*, j\left(x_{n_j} - x^*\right) \right\rangle \\
&\leq \alpha_{n_j} \psi\left(\left\| x_{n_j} - x^* \right\|\right) \left\| x_{n_j} - x^* \right\| + \alpha_{n_j} \left\langle f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right\rangle \\
&\quad + \frac{1-\alpha_{n_j}}{N} \sum_{i=1}^{N} \left\| y_{in_j} - x^* \right\| \left\| x_{n_j} - x^* \right\| \\
&\leq \alpha_{n_j} \psi\left(\left\| x_{n_j} - x^* \right\|\right) \left\| x_{n_j} - x^* \right\| + \alpha_{n_j} \left\langle f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right\rangle \\
&\quad + \left(1-\alpha_{n_j}\right) \left\| x_n - p \right\|^2,
\end{aligned} \tag{3.13}$$

which implies that

$$\left\| x_{n_j} - x^* \right\| \left(\psi\left(\left\| x_{n_j} - x^* \right\|\right) - \left\| x_{n_j} - x^* \right\| \right) \leq \left\langle f(x^*) - x^*, j\left(x_{n_j} - x^*\right) \right\rangle. \tag{3.14}$$

Now we prove that $\{x_n\}$ is relatively sequentially compact. Since j is weakly sequentially continuous, we have

$$\lim_{j\to\infty} \left\| x_{n_j} - x^* \right\| \left(\psi\left(\left\| x_{n_j} - x^* \right\|\right) - \left\| x_{n_j} - x^* \right\| \right) \leq 0, \tag{3.15}$$

which implies that

$$\lim_{j\to\infty}\left\|x_{n_j}-x^*\right\|=0,\qquad \text{or}\ \lim_{j\to\infty}\left(\psi\left(\left\|x_{n_j}-x^*\right\|\right)-\left\|x_{n_j}-x^*\right\|\right)=0. \tag{3.16}$$

If $\lim_{j\to\infty}\|x_{n_j}-x^*\|=0$, then $\{x_n\}$ is relatively sequentially compact. If $\lim_{j\to\infty}(\psi(\|x_{n_j}-x^*\|)-\|x_{n_j}-x^*\|)=0$, we have $\lim_{j\to\infty}\|x_{n_j}-x^*\|=\lim_{j\to\infty}\psi(\|x_{n_j}-x^*\|)$. Since ψ is continuous, $\lim_{j\to\infty}\|x_{n_j}-x^*\|=\psi(\lim_{j\to\infty}\|x_{n_j}-x^*\|)$. By the definition of ψ, we conclude that $\lim_{j\to\infty}\|x_{n_j}-x^*\|=0$, which implies that $\{x_n\}$ is relatively sequentially compact.

Next, we prove that x^* is the solution to (3.2). Indeed, for any $x\in\mathcal{F}$, we have

$$\begin{aligned}\|x_n-x\|^2&=\left\langle\alpha_n\left(f(x_n)-x_n+x_n-x\right),j(x_n-x)\right\rangle+\frac{1-\alpha_n}{N}\sum_{i=1}^{N}\left\langle y_{in}-x,j(x_n-x)\right\rangle\\&=\alpha_n\left\langle f(x_n)-x_n,j(x_n-x)\right\rangle+\alpha_n\left\langle x_n-x,j(x_n-x)\right\rangle\\&\quad+\frac{1-\alpha_n}{N}\sum_{i=1}^{N}\left[\beta_n\left\langle x_n-x,j(x_n-x)\right\rangle+(1-\beta_n)\left\langle T_i(t_n)x_n-x,j(x_n-x^*)\right\rangle\right]\\&\le\alpha_n\left\langle f(x_n)-x_n,j(x_n-x)\right\rangle+\alpha_n\|x_n-x\|^2\\&\quad+\frac{1-\alpha_n}{N}\sum_{i=1}^{N}\left[\beta_n\|x_n-x\|^2+(1-\beta_n)\|T_i(t_n)x_n-x\|\|x_n-x\|\right]\\&\le\alpha_n\left\langle f(x_n)-x_n,j(x_n-x)\right\rangle+\alpha_n\|x_n-x\|^2\\&\quad+\frac{1-\alpha_n}{N}\sum_{i=1}^{N}\left[\beta_n\|x_n-x\|^2+(1-\beta_n)\|x_n-x\|^2\right]\\&=\alpha_n\left\langle f(x_n)-x_n,j(x_n-x)\right\rangle+\|x_n-x\|^2.\end{aligned}\tag{3.17}$$

Therefore,

$$\left\langle f(x_n)-x_n,j(x-x_n)\right\rangle\le 0. \tag{3.18}$$

Since $x_{n_j}\rightharpoonup x^*$ and j is weakly sequentially continuous, we have

$$\left\langle f(x^*)-x^*,j(x-x^*)\right\rangle=\lim_{j\to\infty}\left\langle f\left(x_{n_j}\right)-x_{n_j},j\left(x-x_{n_j}\right)\right\rangle\le 0. \tag{3.19}$$

This shows that x^* is the solution of the variational inequality (3.2).

Finally, we prove that x^* is the unique solution of the variational inequality (3.2). Assume that $\hat{x}\in\mathcal{F}$ with $\hat{x}\neq x^*$ is another solution of (3.2). Then there exists $\epsilon>0$ such that $\|\hat{x}-x^*\|>\epsilon$. By Lemma 2.3 there exists $r\in(0,1)$ such that $\|f(\hat{x})-f(x^*)\|\le r\|\hat{x}-x^*\|$. Since both $\hat{x}$ and x^* are the solution of (3.2), we have

$$\left\langle f(x^*)-x^*,j(\hat{x}-x^*)\right\rangle\le 0,\qquad \left\langle f(\hat{x})-\hat{x},j(x^*-\hat{x})\right\rangle\le 0. \tag{3.20}$$

Adding the above inequalities, we get

$$0 < (1-r)\epsilon^2 < (1-r)\|\hat{x} - x^*\|^2 \le \big\langle \big(I-f\big)x^* - \big(I-f\big)\hat{x}\big\rangle, j(x^* - \hat{x}) \le 0, \tag{3.21}$$

which is a contradiction. Therefore, we must have $\hat{x} = x^*$, which implies that x^* is the unique solution of (3.2).

In a similar way it can be shown that each cluster point of sequence $\{x_n\}$ is equal to x^*. Therefore, the entire sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

If letting $\beta_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.1, then we get the following.

Corollary 3.2. *Let C be a nonempty closed convex subset of a reflexive Banach space E which admits a weakly sequentially continuous duality mapping J from E into E^*. For every $i = 1, \ldots, N$ ($N \ge 1$), let $\mathfrak{T}_i = \{T_i(t) : t \ge 0\}$ be a semigroup of nonexpansive mappings on C such that $\mathfrak{F} = \cap_{i=1}^N \operatorname{Fix}(\mathfrak{T}_i) \neq \emptyset$ and $f : C \to C$ be a generalized contraction on C. Let $\{\alpha_n\} \subset [0,1)$ and $\{t_n\} \subset (0,\infty)$ be sequences satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\alpha_n/t_n) = 0$. Let $\{x_n\}$ be a sequence generated by*

$$x_n = \alpha_n f(x_n) + \frac{1-\alpha_n}{N}\sum_{i=1}^{N} T_i(t_n)x_n. \tag{3.22}$$

Then $\{x_n\}$ converges strongly to a point $x^* \in \mathfrak{F}$, which is the unique solution to the following variational inequality:

$$\big\langle (f-I)x^*, j(x-x^*)\big\rangle \le 0, \quad \forall x \in \mathfrak{F}. \tag{3.23}$$

Theorem 3.3. *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E which admits a weakly sequentially continuous duality mapping J from E into E^*. For every $i = 1, \cdots, N (N \ge 1)$, let $\mathfrak{T}_i = \{T_i(t) : t \ge 0\}$ be a semigroup of nonexpansive mappings on C such that $\mathfrak{F} = \cap_{i=1}^N \operatorname{Fix}(\mathfrak{T}_i) \neq \emptyset$ and $f : C \to C$ be a generalized contraction on C. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$ and $\{t_n\} \subset (0,\infty)$ be the sequences satisfying $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty}(\beta_n/t_n) = 0$. Let $\{x_n\}$ be a sequence generated*

$$\begin{aligned} y_{in} &= \alpha_n x_n + (1-\alpha_n)T_i(t_n)x_n, \quad i = 1, \ldots, N, \\ x_{n+1} &= \beta_n f(x_n) + \frac{1-\beta_n}{N}\sum_{i=1}^{N} y_{in}, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.24}$$

Then $\{x_n\}$ converges strongly to a point $x^* \in \mathfrak{F}$, which is the unique solution of variational inequality (3.2).

Proof. Let $p \in \mathfrak{F}$ and $M = \max\{\|x_1 - p\|, \eta^{-1}(\|f(p) - p\|\}$. Now we show by induction that

$$\|x_n - p\| \le M, \quad \forall n \in \mathbb{N}. \tag{3.25}$$

It is obvious that (3.25) holds for $n = 1$. Suppose that (3.25) holds for some $n = k$, where $k > 1$. Observe that

$$\begin{aligned}\|y_{ik} - p\| &= \|\alpha_k(x_k - p) + (1-\alpha_k)(T_i(t_k)x_k - p)\| \\ &\le \alpha_k\|x_k - p\| + (1-\alpha_k)\|T_i(t_k)x_k - p\| \le \|x_k - p\|.\end{aligned} \tag{3.26}$$

Now, by using (3.24) and (3.26), we have

$$\begin{aligned}\|x_{k+1} - p\| &= \left\|\beta_k(f(x_k) - p) + \frac{1-\beta_k}{N}\sum_{i=1}^{N}(y_{ik} - p)\right\| \\ &\le \beta_k\|f(x_k) - f(p)\| + \beta_k\|f(p) - p\| + \frac{1-\beta_k}{N}\sum_{i=1}^{N}\|y_{ik} - p\| \\ &\le \beta_k\psi(\|x_k - p\|) + \beta_k\|f(p) - p\| + \frac{1-\beta_k}{N}\sum_{i=1}^{N}\|x_k - p\| \\ &= \beta_k\psi(\|x_k - p\|) + \beta_k\|f(p) - p\| + (1-\beta_k)\|x_k - p\| \\ &= \beta_k\psi(\|x_k - p\|) + \beta_k\eta\left(\eta^{-1}\|f(p) - p\|\right) + (1-\beta_k)\|x_k - p\| \\ &\le \beta_k\psi(M) + \beta_k\eta(M) + (1-\beta_k)M \\ &= \beta_k\psi(M) + \beta_k(M - \psi(M)) + (1-\beta_k)M = M.\end{aligned} \tag{3.27}$$

By induction we conclude that (3.25) holds for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{y_{in}\}$, $\{T_i(t_n)x_n\}$.

For each $i = 1, \ldots, N$ and $n \in \mathbb{N}$, define the mapping $U(t_n) = (1/N)\sum_{i=1}^{N} S_i(t_n)$, where $S_i(t_n) = \alpha_n I + (1-\alpha_n)T_i(t_n)$. Then we rewrite the sequence (3.24) to

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)U(t_n)x_n. \tag{3.28}$$

Obviously, each $U(t_n)$ is nonexpansive. Since $\{x_n\}$ is bounded and E is reflexive, we may assume that some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to p. Next we show that $p \in \mathcal{F}$. Put $x_j = x_{n_j}$, $\beta_j = \beta_{n_j}$, and $t_j = t_{n_j}$ for each $j \in \mathbb{N}$. Fix $t > 0$. By (3.28) we have

$$\begin{aligned}\|x_j - U(t)p\| = &\sum_{k=0}^{[t/t_j]-1}\|U((k+1)t_j)x_j - U(kt_j)x_j\| \\ &+ \left\|U\left(\left[\frac{t}{t_j}\right]t_j\right)x_j - U\left(\left[\frac{t}{t_j}\right]t_j\right)p\right\| + \left\|U\left(\left[\frac{t}{t_j}\right]t_j\right)p - U(t)p\right\|\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{t}{t_j}\right]\|U(t_j)x_j - x_{j+1}\| + \|x_{j+1} - p\| + \left\|U\left(t - \left[\frac{t}{t_j}\right]t_j\right)p - p\right\| \\
&= \left[\frac{t}{t_j}\right]\beta_j\|U(t_j)x_j - f(x_j)\| + \|x_{j+1} - p\| + \left\|U\left(t - \left[\frac{t}{t_j}\right]t_j\right)p - p\right\| \\
&\leq \frac{t\beta_j}{t_j}\|U(t_j)x_j - f(x_j)\| + \|x_{j+1} - p\| + \max\{\|U(s)p - p\| : 0 \leq s \leq t_j\}.
\end{aligned} \tag{3.29}$$

So, for all $j \in \mathbb{N}$, we have

$$\limsup_{j\to\infty}\|x_j - U(t)p\| \leq \limsup_{j\to\infty}\|x_{j+1} - p\| = \limsup_{j\to\infty}\|x_j - p\|. \tag{3.30}$$

Since E has a weakly sequentially continuous duality mapping satisfying Opials' condition, this implies $p = U(t)p$. By Lemma 2.4, we have $\mathrm{Fix}(U(t)) = \cap_{i=1}^{N}\mathrm{Fix}(T_i(t))$ for each $t > 0$. Therefore, $p \in \mathcal{F}$. In view of the variational inequality (3.2) and the assumption that duality mapping J is weakly sequentially continuous, we conclude that

$$\limsup_{n\to\infty}\langle (f - I)q, j(x_{n+1} - q)\rangle = \lim_{j\to\infty}\left\langle (f - I)q, j\left(x_{n_j+1} - q\right)\right\rangle = \langle (I - f)q, j(p - q)\rangle \leq 0. \tag{3.31}$$

Finally, we prove that $x_n \to q$ as $n \to \infty$. Suppose that $\|x_n - q\| \nrightarrow 0$. Then there exists $\epsilon > 0$ and subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - q\| \geq \epsilon$ for all $j \in \mathbb{N}$. Put $x_j = x_{n_j}$, $\beta_j = \beta_{n_j}$ and $t_j = t_{n_j}$. By Lemma 2.3 one has $\|f(x_j) - f(q)\| \leq r\|x_j - q\|$ for all $j \in \mathbb{N}$. Now, from (2.2) and (3.28) we have

$$\begin{aligned}
\|x_{j+1} - q\|^2 &= \|(1 - \beta_n)(U(t_j)x_j - q) + \beta_n(f(x_j) - q)\|^2 \\
&\leq (1 - \beta_j)^2\|U(t_j)x_j - q\|^2 + 2\beta_j\langle f(x_j) - q, j(x_{j+1} - q)\rangle \\
&\leq (1 - \beta_j)^2\|x_j - q\|^2 + 2\beta_n\langle f(x_j) - f(q), j(x_{j+1} - q)\rangle + 2\beta_j\langle f(q) - q, j(x_{j+1} - q)\rangle \\
&\leq (1 - \beta_j)^2\|x_j - q\|^2 + 2\beta_j r\|x_j - q\|\|x_{j+1} - q\| + 2\beta_n\langle f(q) - q, j(x_{j+1} - q)\rangle \\
&\leq (1 - \beta_j)^2\|x_j - q\|^2 + \beta_j r\left(\|x_j - q\|^2 + \|x_{j+1} - q\|^2\right) + 2\beta_j\langle f(q) - q, j(x_{j+1} - q)\rangle \\
&= \left((1 - \beta_j)^2 + \beta_j r\right)\|x_j - q\|^2 + \beta_j r\|x_{j+1} - q\|^2 + 2\beta_j\langle f(q) - q, j(x_{j+1} - q)\rangle.
\end{aligned} \tag{3.32}$$

It follows that

$$\begin{aligned}\|x_{j+1}\| &\le \frac{1-(2-r)\beta_j+\beta_j^2}{1-\beta_j r}\|x_j-q\|^2+\frac{2\beta_j}{1-\beta_j r}\langle f(q)-q,\ j(x_{j+1}-q)\rangle\\ &\le \frac{1-\beta_j r-2(1-r)\beta_j}{1-\beta_j r}\|x_j-q\|^2+\frac{2\beta_j}{1-\beta_j r}\langle f(q)-q,\ j(x_{j+1}-q)\rangle+\beta_j^2 M\\ &=\left(1-\frac{2(1-r)\beta_j}{1-\beta_j r}\right)\|x_j-q\|^2+\frac{2\beta_j}{1-\beta_j r}\langle f(q)-q,\ j(x_{j+1}-q)\rangle+\beta_j^2 M\\ &\le (1-2(1-r)\beta_j)\|x_j-q\|^2+\beta_j\left(\frac{2}{1-r}\langle f(q)-q, j(x_{j+1}-q)\rangle+\beta_j M\right),\end{aligned} \tag{3.33}$$

where M is a constant.

Let $\gamma_j = 2(1-r)\beta_j$ and $\delta_j = \beta_j((2/(1-r))\langle f(q)-q, j(x_{j+1}-q)\rangle + \beta_j M)$. It follows from (3.33) that

$$\|x_{j+1}-q\| \le (1-\gamma_j)\|x_j-q\| + \delta_j. \tag{3.34}$$

It is easy to see that $\gamma_j \to 0$, $\sum_{j=1}^{\infty}\gamma_j = \infty$ and (noting (3.28))

$$\begin{gathered}\limsup_{j\to\infty}\frac{\delta_j}{\gamma_j} = \limsup \frac{1}{(1-r)^2}\langle f(q)-q, j(x_{j+1}-q)\rangle + \frac{M}{2(1-r)}\beta_j,\\ \limsup_{n\to\infty}\frac{1}{(1-r)^2}\langle f(q)-q, j(x_{j+1}-q)\rangle \le 0.\end{gathered} \tag{3.35}$$

Using Lemma 2.5, we conclude that $\|x_j - q\| \to 0$ as $j \to \infty$. It is a contradiction. Therefore, $x_n \to q$ as $n \to \infty$. This completes the proof. □

If letting $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.3, then we get the following.

Corollary 3.4. *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E which admits a weakly sequentially continuous duality mapping J from E into E^*. For every $i = 1,\dots,N(N \ge 1)$, let $\mathfrak{T}_i = \{T_i(t) : t \ge 0\}$ be a semigroup of nonexpansive mappings on C such that $\mathcal{F} = \cap_{i=1}^{N}\operatorname{Fix}(\mathfrak{T}_i) \neq \emptyset$ and $f : C \to C$ be a generalized contraction on C. Let $\{\beta_n\} \subset [0,1)$ and $\{t_n\} \subset (0,\infty)$ be sequences satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\beta_n/t_n) = 0$. Let $\{x_n\}$ be a sequence generated*

$$x_{n+1} = \beta_n f(x_n) + \frac{1-\beta_n}{N}\sum_{i=1}^{N} T_i(t_n)x_n, \quad \forall n \in \mathbb{N}. \tag{3.36}$$

Then $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of variational inequality (3.2).

Remark 3.5. Theorem 3.1 and Corollary 3.2 extend the corresponding ones of Suzuki [2], Xu [3], and Chen and He [5] from one nonexpansive semigroup to a finite family of nonexpansive semigroups. But Theorem 3.3 and Corollary 3.4 are not the extension of Theorem 3.2 of Chen and He [5] since Banach space in Theorem 3.3 and Corollary 3.4 is required to be strictly convex. But if letting $N = 1$ in Theorem 3.3 and Corollary 3.4, we can remove the restriction on strict convexity and hence they extend Theorem 3.2 of Chen and He [5] from a contraction to a generalized contraction.

Remark 3.6. Our Theorem 3.1 extends and improves Theorems 3.2 and 4.2 of Song and Xu [4] from a nonexpansive semigroup to a finite family of nonexpansive semigroups and a contraction to a generalized contraction. Our conditions on the control sequences are different with ones of Song and Xu [4].

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