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Research Article

The Hypergroupoid Semigroups as Generalizations of the Groupoid Semigroups

Jeong Soon Han,[1] Hee Sik Kim,[2] and J. Neggers[3]

[1] *Department of Applied Mathematics, Hanyang University, Ansan 426-791, Republic of Korea*
[2] *Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea*
[3] *Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, USA*

Correspondence should be addressed to Hee Sik Kim, heekim@hanyang.ac.kr

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We introduce the notion of hypergroupoids $(H\mathrm{Bin}(X), \Box)$, and show that $(H\mathrm{Bin}(X), \Box)$ is a super-semigroup of the semigroup $(\mathrm{Bin}(X), \Box)$ via the identification $x \leftrightarrow \{x\}$. We prove that $(H\mathrm{Bin}^*(X), \ominus, [\emptyset])$ is a BCK-algebra, and obtain several properties of $(H\mathrm{Bin}^*(X), \Box)$.

1. Introduction

The notion of the semigroup $(\mathrm{Bin}(X), \Box)$ was introduced by Kim and Neggers [1]. Fayoumi [2] introduced the notion of the center $Z\mathrm{Bin}(X)$ in the semigroup $\mathrm{Bin}(X)$ of all binary systems on a set X, and showed that if $(X, \bullet) \in Z\mathrm{Bin}(X)$, then $x \neq y$ implies $\{x, y\} = \{x \bullet y, y \bullet x\}$. Moreover, she showed that a groupoid $(X, \bullet) \in Z\mathrm{Bin}(X)$ if and only if it is a locally zero groupoid. Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [3, 4]. Neggers and Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs [5]. The present authors [6] defined several special varieties of d-algebras, such as strong d-algebras, (weakly) selective d-algebras, and pre-d-algebras, discussed the associative groupoid product $(X; \Box) = (X; *)\Box(X; \circ)$, where $x \Box y = (x * y) \circ (y * x)$. They showed that the squared algebra $(X; \Box, 0)$ of a pre-d-algebra $(X; *, 0)$ is a strong d-algebra if and only if $(X; *, 0)$ is strong.

Zhan et al. [7] defined the T-fuzzy n-ary sub-hypergroups by using a norm T and obtained some related properties. Zhan, and Liu [8] introduced the notion of f-derivation of a BCI-algebras. They gave some characterizations of a p-semisimple BCI-algebras by using the idea of a regular f-derivation. Zhan et al. [9] defined the notion of hyperaction of a hypergroup as a generalization of the concept of action of a group. Recently, Davvaz and Leoreanu [10] published a beautiful book, *Hyperring Theory and Applications*, and provided useful information on the theory of the hypertheory.

In this paper we introduce the notion of hypergroupoids $(H\mathrm{Bin}(X), \square)$, and show that $(H\mathrm{Bin}(X), \square)$ is a super-semigroup of the semigroup $(\mathrm{Bin}(X), \square)$ via the identification $x \leftrightarrow \{x\}$. We prove that $(H\mathrm{Bin}^*(X), \ominus, [\emptyset])$ is a BCK-algebra, and obtain several properties of $(H\mathrm{Bin}^*(X), \square)$.

2. Preliminaries

Given a nonempty set X, we let $\mathrm{Bin}(X)$ the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and $(X, \bullet)$ of $\mathrm{Bin}(X)$, define a product "$\square$" on these groupoids as follows:

$$(X, *)\square(X, \bullet) = (X, \square), \tag{2.1}$$

where

$$x \square y = (x * y) \bullet (y * x), \tag{2.2}$$

for any $x, y \in X$. Using the notion, H. S. Kim and J. Neggers showed the following theorem.

Theorem 2.1 (see [1]). *$(Bin(X), \square)$ is a semigroup, that is, the operation "$\square$" as defined in general is associative. Furthermore, the left zero semigroup is an identity for this operation.*

3. Hypergroupoid Semigroups

Instead of a groupoid $(X, *)$ on X, we may also consider a *hypergroupoid* (X, φ) on X, where $\varphi : X \times X \to P^*(X)$ is a *hyperproduct* with $P^*(X)$, the set of all non-empty subsets of X. We denote the set of all hypergroupoids (X, φ) on X by $H\mathrm{Bin}(X)$, that is,

$$H\,\mathrm{Bin}(X) := \{(X, \varphi) \mid \varphi : \text{ a hypergroupoid on } X\}. \tag{3.1}$$

The product "$\square$" discussed in $\mathrm{Bin}(X)$ can be generalized in $H\mathrm{Bin}(X)$ as follows: given $(X, \varphi), (X, \psi) \in H\mathrm{Bin}(X)$, for any $x, y \in X$,

$$xy := (x\varphi y)\psi(y\varphi x). \tag{3.2}$$

If we identify $x \in X$ with $\{x\} \in P^*(X)$, then we have an inclusion: $X \subsetneq P^*(X)$ and thus for $\varphi(x, y) = x\varphi y \in P^*(X)$, we have $x\varphi y \subseteq X$ and hence also $x\varphi y \subseteq P^*(X)$ via this identification.

If $A, B \subseteq X$, then for the groupoid $(X, *) \in \mathrm{Bin}(X)$, we have

$$A * B := \{a * b \mid a \in A,\ b \in B\}, \tag{3.3}$$

hence $\{a\}*\{b\} = \{a*b\}$ in a natural way. Similarly, given a hypergroupoid $(X,\varphi) \in H\text{Bin}(X)$, $A\varphi B$ is defined by $A\varphi B = \cup\{x\varphi y \mid x \in A,\ y \in B\}$.

Given hypergroupoids $(X,\varphi), (X,\psi)$, we let $(X,\theta) := (X,\varphi)\Box(X,\psi)$. Then, for any x, $y \in X$, we have

$$\begin{aligned} x\theta y &= (x\varphi y)\psi(y\varphi x) \\ &= \cup\{a\psi b \mid a \in x\varphi y,\ b \in y\varphi x\}. \end{aligned} \tag{3.4}$$

Suppose that $(X,*)$ and $(X,\bullet)$ are groupoids and that we determine the following:

$$\begin{aligned} x\theta y &= (x*y)\bullet(y*x) \\ &= \cup\{a\bullet b \mid a \in \{x*y\},\ b \in \{y*x\}\} \\ &= \{(x*y)\bullet(y*x)\} \\ &= \{x\Box y\} = x\Box y, \end{aligned} \tag{3.5}$$

via the identification $x \leftrightarrow \{x\}$. Hence $(X,*)\Box(X,\bullet)$ is the same as a product of groupoids or as a product of hypergroupoids.

It can be shown that $(\text{Bin}(X),\Box) \to (H\text{Bin}(X),\Box)$ is an injection (an into homomorphism) via the identification $x \leftrightarrow \{x\}$ and the associated identification $x\theta y = \{x\Box y\} = x\Box y$.

Example 3.1. Let $X := \mathbf{R}^2$ and for any $x, y \in X$, let $x\varphi y$ denote the undirected line segment connecting x with y. Then $x\varphi x = \{x\}$ and $x\varphi y = y\varphi x$. Let $(X,\theta) := (X,\varphi)\Box(X,\varphi)$. Then $x\theta y = \cup\{a\varphi b \mid a \in x\varphi y,\ b \in y\varphi x\}$ for any $x, y \in X$. Since $x\varphi y = y\varphi x$, $a\varphi b \subseteq x\theta y$ for any $a, b \in x\varphi y$. Since $x, y \in x\varphi y$, $x\varphi y \subseteq x\theta y$. We claim that $x\theta y \subseteq x\varphi y$. If $\alpha \in x\theta y$, then $\alpha \in a\varphi b$ for some $a \in x\varphi y$ and $b \in y\varphi x$. Since $x\varphi y = y\varphi x$, $\alpha \in a\varphi b$ for some $a, b \in x\varphi y$, which shows that $\alpha \in x\varphi y$. This proves that $(X,\varphi) = (X,\theta) = (X,\varphi)\Box(X,\varphi)$, that is, (X,φ) is an idempotent hypergroupoid in $(H\text{Bin}(X),\Box)$.

Theorem 3.2. *$(H\text{Bin}(X),\Box)$ is a supersemigroup of the semigroup $(\text{Bin}(X),\Box)$ via the identification $x \leftrightarrow \{x\}$.*

Proof. Suppose that $(X,\varphi), (X,\psi)$ and (X,ω) are hypergroupoids and let $(X,\alpha) := (X,\psi)\Box(X,\omega)$ and $(X,\beta) := (X,\varphi)\Box(X,\psi)$. Then for any $x, y \in X$, we have $x\alpha y = (x\psi y)\omega(y\psi x)$ and $x\beta y = (x\varphi y)\psi(y\varphi x)$. Let $(X,\theta) := [(X,\varphi)\Box(X,\psi)]\Box(X,\omega)$. Then $(X,\theta) = (X,\beta)\Box(X,\omega)$ and hence we obtain the following

$$\begin{aligned} x\theta y &= (x\beta y)\omega(y\beta x) \\ &= [(x\varphi y)\psi(y\varphi x)]\omega[(y\varphi x)\psi(x\varphi y)]. \end{aligned} \tag{3.6}$$

If we let $(X,\mu) := (X,\varphi)\square[(X,\psi)W(X,\omega)]$, then $(X,\mu) = (X,\varphi)\square(X,\alpha)$ and hence $x\mu y = (x\varphi y)\alpha(y\varphi x)$ for any $x, y \in X$. Let $p := x\varphi y, q := y\varphi x$. Then

$$\begin{aligned} x\mu y &= p\alpha q \\ &= (p\psi q)\omega(q\psi p) \\ &= [(x\varphi y)\psi(y\varphi x)]\omega[(y\varphi x)\psi(x\varphi y)]. \end{aligned} \tag{3.7}$$

This proves that $(X,\theta) = (X,\mu)$, that is, $(H\mathrm{Bin}(X),\square)$ is a semigroup. □

Proposition 3.3. *The left-zero-semigroup* $(X,*)$, *that is,* $x * y = x$ *for any* $x, y \in X$, *is an identity of the semigroup* $(H\mathrm{Bin}(X),\square)$.

Proof. Let $(X,*)$ be a left-zero-semigroup. Then $(X,*) \in \mathrm{Bin}(X)$. By the identification $x \leftrightarrow \{x\}$, we have $(X,*) \in (H\mathrm{Bin}(X),\square)$. Given $(X,\nu) \in H\mathrm{Bin}(X)$, let $(X,\theta) := (X,*)\square(X,\nu)$. Then for any $x, y \in X$, we have

$$\begin{aligned} x\theta y &= (x * y)\nu(y * x) \\ &= \{x\}\nu\{y\} \\ &= \cup\{a\nu b \mid a \in \{x\},\ b \in \{y\}\} \\ &= x\nu y, \end{aligned} \tag{3.8}$$

that is, $(X,\theta) = (X,\nu)$. This proves that $(X,*)$ is a left identity on $H\mathrm{Bin}(X)$.

Similarly, if we let $(X,\theta) = (X,\nu)\square(X,*)$, then for any $x, y \in X$,

$$\begin{aligned} x\theta y &= (x\nu y) * (y\nu x) \\ &= \{a * b \mid a \in x\nu y,\ b \in y\nu x\} \\ &= \{a \mid a \in x\nu y\} \\ &= x\nu y, \end{aligned} \tag{3.9}$$

that is, $(X,\theta) = (X,\nu)$. This proves that $(X,*)$ is a right identity on $H\mathrm{Bin}(X)$. □

Given an element $(X,\varphi) \in H\mathrm{Bin}(X)$, $x\varphi y \in P^*(X)$, that is, $\emptyset \neq x\varphi y \subseteq X$. We extend (X,φ) to $(P^*(X),\widehat{\varphi})$ as

$$\widehat{\varphi} : P^*(X) \times P^*(X) \longrightarrow P^*(P^*(X)) \tag{3.10}$$

by $\widehat{\varphi}(A,B) := A\widehat{\varphi}B$, where $A\widehat{\varphi}B = \cup\{a\varphi b \mid a \in A,\ b \in B\}$. In particular,

$$\begin{aligned} \{x\}\widehat{\varphi}\{y\} &= \cup\{a\varphi b \mid a \in \{x\},\ b \in \{y\}\} \\ &= x\varphi y. \end{aligned} \tag{3.11}$$

This produces a mapping $\pi : H\text{Bin}(X) \to \text{Bin}P^*(X)$. Let $(X,\theta) := (X,\varphi)\square(X,\psi)$. Then $x\theta y = \cup\{a\psi b \mid a \in x\varphi y,\ b \in y\varphi x\}$ for any $x, y \in X$. Since $x\varphi y, y\varphi x \in P^*(X)$, we have

$$\begin{aligned}(x\varphi y)\widehat{\psi}(y\varphi x) &= \cup\{a\psi b \mid a \in x\varphi y,\ b \in y\varphi x\}\\ &= x\theta y.\end{aligned} \tag{3.12}$$

Since $x\varphi y = \{x\}\widehat{\varphi}\{y\}$ via the identification $x \leftrightarrow \{x\}$, we obtain

$$\begin{aligned}x\theta y &= (x\varphi y)\widehat{\psi}(y\varphi x)\\ &= (\{x\}\widehat{\varphi}\{y\})\widehat{\psi}(\{y\}\widehat{\varphi}\{x\})\\ &= x\widehat{\theta}y,\end{aligned} \tag{3.13}$$

where $(P^*(X),\widehat{\theta}) = (P^*(X),\widehat{\varphi})\square(P^*(X),\widehat{\psi})$ in $(\text{Bin}P^*(X),\square)$. We claim that π is a homomorphism. In fact, $\pi((X,\varphi)\square(X,\psi)) = \pi((X,\theta)) = (P^*(X),\widehat{\theta}) = (P^*(X),\widehat{\varphi})\square(P^*(X),\widehat{\psi}) = \pi((X,\varphi))\square\pi((X,\psi))$.

Given $H\text{Bin}(X)$, we may order it according to the rule

$$(X,\varphi) \le (X,\psi) \Longleftrightarrow x\varphi y \subseteq x\psi y, \quad \forall x, y \in X. \tag{3.14}$$

We define a mapping $[\emptyset] : X \times X \to P(X)$ by $[\emptyset](x,y) := \emptyset$ for all $x, y \in X$. If we let $H\text{Bin}^*(X) := H\text{Bin}(X) \cup \{(X,[\emptyset])\}$, then $(X,[\emptyset])$ is the minimal element of $(H\text{Bin}^*(X),\le)$.

Proposition 3.4. *Let* $(X,\varphi) \in H\text{Bin}(X)$ *and* $(X,*) \in \text{Bin}(X)$. *If* $(X,\varphi) \le (X,*)$, *then* $(X,\varphi) = (X,*)$.

Proof. If $(X,\varphi) \le (X,*)$, then $\emptyset \ne x\varphi y \subseteq \{x*y\}$ for any $x, y \in X$. It follows that $x\varphi y = \{x*y\} = x*y$, proving that $(X,\varphi) = (X,*)$. □

Proposition 3.5. *Let* $(X,*), (X,\bullet) \in Bin(X)$. *If* $(X,*) \le (X,\bullet)$, *then* $(X,*) = (X,\bullet)$, *that is,* $\text{Bin}(X)$ *is an antichain in* $(H\text{Bin}^*(X),\le)$.

Proof. If $(X,*) \le (X,\bullet)$, then $\{x*y\} \subseteq \{x\bullet y\}$ for any $x, y \in X$. It follows that $x*y = x\bullet y$ for any $x, y \in X$, proving that $(X,*) = (X,\bullet)$. □

4. BCK-Algebras on $H\text{Bin}^*(X)$

In this section we discuss BCK-algebras on $H\text{Bin}^*(X)$ by introducing a binary operation as follows: given hypergroupoids $(X,\varphi), (X,\psi) \in H\text{Bin}^*(X)$, we define a binary operation "$\ominus$" by

$$(X,\varphi) \ominus (X,\psi) := (X,\varphi \setminus \psi), \tag{4.1}$$

where $x(\varphi \setminus \psi)y := x\varphi y \setminus x\psi y$ for any $x, y \in X$.

Theorem 4.1. *$(HBin^*(X), \ominus, [\emptyset])$ is a BCK-algebra.*

Proof. For any $(X, \varphi) \in H\text{Bin}^*(X)$, since $x[\emptyset]y \setminus x\varphi y = \emptyset$ for any $x, y \in X$, we have $(X, [\emptyset]) \ominus (X, \varphi) = (X, [\emptyset])$.

Given $(X, \varphi) \in H\text{Bin}^*(X)$, since $x\varphi y \setminus x\varphi y = \emptyset$ for any $x, y \in X$, we have $(X, \varphi) \ominus (X, \varphi) = (X, [\emptyset])$.

Assume that $(X, \varphi) \ominus (X, \psi) = (X, [\emptyset]) = (X, \psi) \ominus (X, \varphi)$. Then $x\varphi y \setminus x\psi y = \emptyset$, $x\psi y \setminus x\varphi y = \emptyset$ for any $x, y \in X$, which shows that $x\varphi y = x\psi y$ for any $x, y \in X$, that is, $(X, \varphi) = (X, \psi)$.

Given $(X, \varphi), (X, \psi) \in H\text{Bin}^*(X)$, since $[x\varphi y \setminus [x\varphi y \setminus x\psi y]] \setminus x\psi y = \emptyset$ for any $x, y \in X$, we obtain $[(X, \varphi) \ominus [(X, \varphi) \ominus (X, \psi)]] \ominus (X, \psi) = (X, [\emptyset])$.

Given $(X, \varphi), (X, \psi), (X, \delta) \in H\text{Bin}^*(X)$, since $[(x\varphi y \setminus x\psi y) \setminus (x\varphi y \setminus x\delta y)] \setminus (x\delta y \setminus x\psi y) = \emptyset$ for any $x, y \in X$, we obtain $[((X, \varphi) \ominus (X, \psi)) \ominus ((X, \varphi) \ominus (X, \delta)] \ominus [(X, \delta) \ominus (X, \psi)] = (X, [\emptyset])$. This proves the theorem. □

5. Several Properties on $H\text{Bin}(X)$

In this section, we discuss some properties on $H\text{Bin}(X)$.

Proposition 5.1. *The product "□" is order-preserving, that is, if $(X, \varphi) \leq (X, \xi), (X, \psi) \leq (X, \omega)$, then $(X, \varphi)\square(X, \psi) \leq (X, \xi)\square(X, \omega)$.*

Proof. Let $(X, \varphi) \leq (X, \xi), (X, \psi) \leq (X, \omega)$ in $H\text{Bin}(X)$. If we let $(X, \theta) := (X, \varphi)\square(X, \psi)$ and $(X, \rho) := (X, \xi)\square(X, \omega)$, then for any $x, y \in X$,

$$\begin{aligned} x\theta y &= (x\varphi y)\psi(y\varphi x) \\ &\subseteq (x\xi y)\psi(y\xi x) \\ &\subseteq (x\xi y)\omega(y\xi x) \\ &= x\rho y, \end{aligned} \tag{5.1}$$

proving that $(X, \theta) \leq (X, \rho)$. □

We define a mapping $[X] : X \times X \to P(X)$ by $[X](x, y) := X$ for all $x, y \in X$. Then $(X, [X])$ is the maximal element of $(H\text{Bin}^*(X), \leq)$. Given $(X, \varphi) \in H\text{Bin}(X)$, if we let $(X, \theta) := (X, [X])\square(X, \varphi)$, then $x\theta y = (x[X]y)\varphi(y[X]x) = X\varphi X = \cup\{a\varphi b \mid a,\ b \in X\}$ for any $x, y \in X$.

Proposition 5.2. *If $(X, \varphi) \in HBin(X)$, then $(X, \varphi)\square(X, [X]) = (X, [X])$.*

Proof. Let $(X, \theta) := (X, \varphi)\square(X, [X])$. Then, for any $x, y \in X$, we have

$$\begin{aligned} x\theta y &= (x\varphi y)[X](y\varphi x) \\ &= \cup\{a[X]b \mid a \in x\varphi y,\ b \in y\varphi x\} \\ &= X, \end{aligned} \tag{5.2}$$

proving that $(X, \theta) = (X, [X])$. □

Given $(X, \varphi) \in H\text{Bin}^*(X)$, we define a hypergroupoid (X, φ^C) by $x\varphi^C y := X \setminus x\varphi y$, for any $x, y \in X$. We call it the *complementary hypergroupoid* of (X, φ^C).

For example, if $(X, \cdot, e)$ is a group, then $x \cdot^C y = X \setminus \{x \cdot y\}$, where $x, y \in X$. It follows that $x \cdot^C e = e \cdot^C x = X \setminus \{x\}$ and $x \cdot^C x^{-1} = x^{-1} \cdot^C x = X \setminus \{e\}$ for any $x \in X$.

A hypergroupoid (X, φ) is said to be a *complementary d-algebra* if there exists $0 \in X$ such that (i) $x\varphi x = X \setminus \{0\}$; (ii) $0\varphi x = X \setminus \{0\}$; (iii) $x\varphi y = y\varphi x = X \setminus \{x\}$ implies $x = y$, for any $x, y \in X$.

The following proposition can be easily seen.

Proposition 5.3. *Given $(X, \varphi) \in HBin^*(X)$, (X, φ) is a d-algebra if and only if (X, φ^C) is a complementary d-algebra.*

Example 5.4. Let $X := \mathbf{R}$ be the set of all real numbers and $f : X \to X$ be a mapping. Define a map $\varphi_f : X \times X \to P^*(X)$ by $\varphi_f(x, y) := [x - |f(y)|,\ x + |f(y)|]$. Then (X, φ_f) be a hypergroupoid for which $x\varphi_f y = [x - |f(y)|,\ x + |f(y)|]$ has a midpoint x where $x, y \in X$.

In particular, let $f(x) := x^2$ for any $x \in X$ and let $(X, \theta) := (X, \varphi_f)\Box(X, \varphi_f)$. Then $x\theta y = (x\varphi_f y)\varphi_f(y\varphi_f x) = \cup\{a\varphi_f b \mid a \in [x - |f(y)|,\ x + |f(y)|],\ b \in [y - |f(x)|,\ y + |f(x)|]\} = \cup\{[a - b^2,\ a + b^2] \mid a \in [x - y^2,\ x + y^2],\ b \in [y - x^2,\ y + x^2]\} = [x - 2y(y + x^2) - x^4,\ x + 2y(y + x^2) + x^4]$, an interval of length $y^2 + (y + x^2)^2 \geq 0$, where $x = y = 0$ implies $0\theta 0 = [0, 0] = \{0\}$, corresponding to 0 in the identification.

A hypergroupoid (X, φ) is said to be *left inclusive* if $x \in x\varphi y$ for any $x, y \in X$.

Note that the only left inclusive hypergroupoid which is a groupoid is the left-zero-semigroup. In fact, let $(X, *)$ be a left inclusive hypergroupoid which is a groupoid. Then $x \in \{x * y\}$ for any $x, y \in X$. It follows that $x = x * y$ for any $x, y \in X$, that is, $(X, *)$ is a left-zero-semigroup.

Proposition 5.5. *The left inclusive hypergroupoids on X relative to the product "$\Box$" on HBin(X) form a subsemigroup of $(H\text{Bin}(X), \Box)$.*

Proof. Let $(X, \varphi), (X, \psi)$ be left inclusive hypergroupoids and let $(X, \theta) := (X, \varphi)\Box(X, \psi)$. Then $x\theta y = (x\varphi y)\psi(y\varphi x) = \cup\{a\psi b \mid a \in x\varphi y,\ b \in y\varphi x\}$ for any $x, y \in X$. Since (X, φ) is left inclusive, $x \in x\varphi y$, $y \in y\varphi x$, and hence $x\psi y \subseteq x\theta y$ for any $x, y \in X$. Moreover, (X, ψ) is left inclusive implies that $x \in x\psi y$, which proves that $x \in x\theta y$. □

Proposition 5.6. *Let $(X, \varphi) \leq (X, \psi)$ in $HBin(X)$. If (X, φ) is left inclusive, then (X, ψ) is also left inclusive.*

Proof. Let $(X, \varphi) \leq (X, \psi)$. Then $x\varphi y \subseteq x\psi y$ for any $x, y \in X$. Since (X, φ) is left inclusive, we have $x \in x\varphi y \subseteq x\psi y$, proving the proposition. □

Proposition 5.6 means that the collection of all left inclusive hypergroupoids is a filter in the poset $(H\text{Bin}(X), \leq)$.

A hypergroupoid (X, φ) is said to be *left-self-avoiding* if $x \notin x\varphi y$ for any $x, y \in X$.

Proposition 5.7. *The complementary hypergroupoid (X, φ^C) of a left inclusive hypergroupoid (X, φ) is left-self-avoiding.*

Proof. Let (X, φ^C) be the complementary hypergroupoid of a left inclusive hypergroupoid (X, φ). Then $x\varphi^C y = X \setminus x\varphi y$ for any $x, y \in X$. Since (X, φ) is left inclusive, $x \in x\varphi y$ for any $x, y \in X$, and hence $x \notin x\varphi^C y$, proving the proposition. □

Proposition 5.8. *The complementary hypergroupoid (X, φ^C) of a left-self-avoiding hypergroupoid (X, φ) is left inclusive.*

Proof. Straightforward. □

Proposition 5.9. *Let $(X, \theta) = (X, \varphi)\Box(X, \psi)$ where (X, φ) is left inclusive and (X, θ) is left-self-avoiding. Then (X, ψ) is left-self-avoiding.*

Proof. Let (X, θ) be a left-self-avoiding hypergroupoid. Then (X, θ^C) is left inclusive by Proposition 5.8. It follows that $x \in x\theta^C y = X \setminus \cup\{a\psi b \mid a \in x\varphi y,\ b \in y\varphi x\}$. This means that $x \notin a\psi b$ for any $a \in x\varphi y$ and $b \in y\varphi x$ where $x, y \in X$. Since (X, φ) is left inclusive, $x \in x\varphi y$, $y \in y\varphi x$. Hence $x \notin x\psi y$, proving that (X, ψ) is left-self-avoiding. □

6. Conclusion

In this paper we have introduced the notion of hypergroupoids as a generalization of groupoids in a manner analogous to the introduction of the notion of hypergroups as a generalization of the notion of groups. Since the semigroup $(\mathrm{Bin}(X), \Box)$ can still benefit from more detailed investigation it follows that the same is even more true for $(H\mathrm{Bin}(X), \Box)$. In the latter case one must rely on proper adaptations obtained from $(\mathrm{Bin}(X), \Box)$ and certainly on results obtained from studies on hypergroupoids available in the literature [7–10] as a general plan for the organization of the subject, with parts to be completed as time and opportunity permits.

References

[1] H. S. Kim and J. Neggers, "The semigroups of binary systems and some perspectives," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 4, pp. 651–661, 2008.

[2] H. F. Fayoumi, "Locally-zero groupoids and the center of $Bin(X)$," *Korean Mathematical Society. Communications*, vol. 26, no. 2, pp. 163–168, 2011.

[3] K. Iséki, "On BCI-algebras," *Mathematics Seminar Notes*, vol. 8, no. 1, pp. 125–130, 1980.

[4] K. Iséki and S. Tanaka, "An introduction to the theory of BCK-algebras," *Mathematica Japonica*, vol. 23, no. 1, pp. 1–26, 1978/79.

[5] J. Neggers and H. S. Kim, "On d-algebras," *Mathematica Slovaca*, vol. 49, no. 1, pp. 19–26, 1999.

[6] J. S. Han, H. S. Kim, and J. Neggers, "Strong and ordinary d-algebras," *Journal of Multiple-Valued Logic and Soft Computing*, vol. 16, no. 3–5, pp. 331–339, 2010.

[7] J. Zhan, B. Davvaz, and K. P. Shum, "On probabilistic n-ary hypergroups," *Information Sciences*, vol. 180, no. 7, pp. 1159–1166, 2010.

[8] J. Zhan and Y. L. Liu, "On f-derivations of BCI-algebras," *Mathematica Slovaca*, vol. 49, pp. 19–26, 1999.

[9] J. Zhan, S. Sh. Mousavi, and M. Jafarpour, "On hyperactions of hypergroups," *University of Bucharest. Scientific Bulletin A*, vol. 73, no. 1, pp. 117–128, 2011.

[10] B. Davvaz and V. Leoreanu, *Hyperring Theory and Applications*, International Academic Press, Palm Harbor, Fla, USA, 2007.