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Research Article

Jacobi Elliptic Solutions for Nonlinear Differential Difference Equations in Mathematical Physics

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We put a direct new method to construct the rational Jacobi elliptic solutions for nonlinear differential difference equations which may be called the rational Jacobi elliptic functions method. We use the rational Jacobi elliptic function method to construct many new exact solutions for some nonlinear differential difference equations in mathematical physics via the lattice equation and the discrete nonlinear Schrodinger equation with a saturable nonlinearity. The proposed method is more effective and powerful to obtain the exact solutions for nonlinear differential difference equations.

1. Introduction

It is well known that the investigation of differential difference equations (DDEs) which describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, and many others, has played an important role in the study of modern physics. Unlike difference equations which are fully discredited, DDEs are semidiscredited with some (or all) of their special variables discredited, while time is usually kept continuous. DDEs also play an important role in numerical simulations of nonlinear partial differential equations (NLPDEs), queuing problems, and discretization in solid state and quantum physics.

Since the work of Fermi et al. in the 1960s [1], DDEs have been the focus of many nonlinear studies. On the other hand, a considerable number of well-known analytic methods are successfully extended to nonlinear DDEs by researchers [2–17]. However, no method obeys the strength and the flexibility for finding all solutions to all types of nonlinear DDEs.

Zhang et al. [18] and Aslan [19] used the (G'/G)-expansion method in some physically important nonlinear DDEs. Xu and Li [12] constructed the Jacobi elliptic solutions for nonlinear DDEs. Recently, S. Zhang and H.-Q. Zhang [20] and Gepreel [21] have used the Jacobi elliptic function method for constructing new and more general Jacobi elliptic function solutions of the integral discrete nonlinear Schrödinger equation. The main objective of this paper is to put a direct new method to construct the rational Jacobi elliptic solutions for nonlinear DDEs. We use this method to calculate the exact wave solutions for some nonlinear DDEs in mathematical physics via the lattice equation and the discrete nonlinear Schrodinger equation with a saturable nonlinearity.

2. Description of the Rational Jacobi Elliptic Functions Method

In this section, we would like to outline an algorithm for using the rational Jacobi elliptic functions method to solve nonlinear DDEs. For a given nonlinear DDEs

$$\begin{aligned}\Delta\Big(&u_{n+p_1}(x),\dots,u_{n+p_k}(x),u'_{n+p_1}(x),\dots,u'_{n+p_k}(x),\dots,u^{(r)}_{n+p_1}(x),\dots,u^{(r)}_{n+p_k}(x),\\ &v_{n+p_1}(x),\dots,v_{n+p_k}(x),v'_{n+p_1}(x),\dots,v'_{n+p_k}(x),\dots,v^{(r)}_{n+p_1}(x),\dots,v^{(r)}_{n+p_k}(x),\dots\Big)=0,\end{aligned}\tag{2.1}$$

where $\Delta = (\Delta_1,\dots,\Delta_g)$, $x = (x_1,x_2,\dots,x_m)$, $n = (n_1,\dots,n_Q)$, and $g,m,Q,p_1,\dots,p_k$ are integers, $u_i^{(r)}$, $v_i^{(r)}$ denotes the set of all rth order derivatives of u_i, v_i with respect to x.

The main steps of the algorithm for the rational Jacobi elliptic functions method to solve nonlinear DDEs are outlined as follows.

Step 1. We seek the traveling wave solutions of the following form:

$$u_n(x) = U(\xi_n), \qquad v_n(x) = V(\xi_n),\dots,\tag{2.2}$$

where

$$\xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{m} c_j x_j + \xi_0,\tag{2.3}$$

d_i $(i = 1,\dots,Q)$, c_j, $(j = 1,\dots,m)$, and the phase ξ_0 are constants to be determined later. The transformations in (2.2) are reduced (2.1) to the following ordinary differential difference equations

$$\begin{aligned}\Omega\Big(&U(\xi_{n+p_1}),\dots,U(\xi_{n+p_k}),U'(\xi_{n+p_1}),\dots,U'(\xi_{n+p_k}),\dots,U^{(r)}(\xi_{n+p_1}),\dots,U^{(r)}(\xi_{n+p_k}),\\ &V(\xi_{n+p_1}),\dots,V(\xi_{n+p_k}),V'(\xi_{n+p_1}),\dots,V'(\xi_{n+p_k}),\dots,V^{(r)}_{n+p_1}(\xi_{n+p_1}),\dots,V^{(r)}_{n+p_k}(\xi_{n+p_k}),\dots\Big)=0,\end{aligned}\tag{2.4}$$

where $\Omega = (\Omega_1,\dots,\Omega_g)$. The transformations in (2.2) help in the calculation of the iteration relations between $u_n(x)$, $u_{n-1}(x)$, and $u_{n+1}(x)$. For example, Langmuir chains equation

$du_n(t)/dt = u_n(t)(u_{n+1}(t) - u_{n-1}(t))$ under the wave transformation $u_n(t) = U(\xi_n)$, $\xi_n = dn + ct + \xi_0$ takes the form $cU'(\xi_n) = U(\xi_n)(U(\xi_n + d) - U(\xi_n - d))$.

Step 2. We suppose the rational series expansion solutions of (2.4) in the following form:

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{F'(\xi_n)}{F(\xi_n)}\right)^i, \qquad V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(\frac{F'(\xi_n)}{F(\xi_n)}\right)^i, \ldots, \tag{2.5}$$

where α_i $(i = 0, 1, \ldots, K)$, and β_i $(i = 0, 1, \ldots, L)$ are constants to be determined later, and $F(\xi_n)$ satisfies a discrete Jacobi elliptic differential equation

$$F'^2(\xi_n) = e_0 + e_1 F^2(\xi_n) + e_2 F^4(\xi_n), \tag{2.6}$$

where e_0, e_1, and e_2 are arbitrary constants.

Step 3. Since the general solution of the proposed (2.6) is difficult to obtain and so the iteration relations corresponding to the general exact solutions. So that we discuss the solutions of the proposed discrete Jacobi elliptic differential equation (2.6) at some special cases to e_0, e_1 and e_2 to cover all the Jacobi elliptic functions as follows:

Type 1. if $e_0 = 1$, $e_1 = -(1 + m^2)$, $e_2 = m^2$. In this case (2.6) has the solution $F(\xi_n) = sn(\xi_n, m)$, where $sn(\xi_n, m)$ is the Jacobi elliptic sine function, and m is the modulus.

The Jacobi elliptic functions satisfy the following properties:

$$\begin{gathered}
[sn(\xi_n, m)]' = cn(\xi_n, m)dn(\xi_n, m), \qquad [cn(\xi_n, m)]' = -sn(\xi_n, m)dn(\xi_n, m), \\
[dn(\xi_n, m)]' = -m^2 sn(\xi_n, m)cn(\xi_n, m), \qquad [cs(\xi_n, m)]' = -ns(\xi_n, m)ds(\xi_n, m), \\
[sd(\xi_n, m)]' = -nd(\xi_n, m)ds(\xi_n, m), \qquad [dc(\xi_n, m)]' = \left(1 - m^2\right)nc(\xi_n, m)sc(\xi_n, m),
\end{gathered} \tag{2.7}$$

where $cn(\xi_n, m)$, and $dn(\xi_n, m)$ are the Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind. The other Jacobi elliptic functions can be generated by $sn(\xi_n, m)$, $cn(\xi_n, m)$, and $dn(\xi_n, m)$ as follows:

$$\begin{gathered}
cd(\xi_n, m) = \frac{cn(\xi_n, m)}{dn(\xi_n, m)}, \quad dc(\xi_n, m) = \frac{dn(\xi_n, m)}{cn(\xi_n, m)}, \quad nc(\xi_n, m) = \frac{1}{cn(\xi_n, m)}, \quad nd(\xi_n, m) = \frac{1}{dn(\xi_n, m)}, \\
cs(\xi_n, m) = \frac{cn(\xi_n, m)}{sn(\xi_n, m)}, \quad sc(\xi_n, m) = \frac{sn(\xi_n, m)}{cn(\xi_n, m)}, \quad sd(\xi) = \frac{sn(\xi_n, m)}{dn(\xi_n, m)}, \quad ds(\xi_n, m) = \frac{dn(\xi_n, m)}{sn(\xi_n, m)},
\end{gathered} \tag{2.8}$$

$$\begin{aligned}
sn(\xi_1 \pm \xi_2, m) &= \frac{sn(\xi_1, m)cn(\xi_2, m)dn(\xi_2, m) \pm sn(\xi_2, m)cn(\xi_1, m)dn(\xi_1, m)}{1 - m^2 sn^2(\xi_1, m)sn^2(\xi_2, m)}, \\
cn(\xi_1 \pm \xi_2, m) &= \frac{cn(\xi_1, m)cn(\xi_2, m) \mp sn(\xi_1, m)sn(\xi_2, m)dn(\xi_1, m)dn(\xi_2, m)}{1 - m^2 sn^2(\xi_1, m)sn^2(\xi_2, m)}, \\
dn(\xi_1 \pm \xi_2, m) &= \frac{dn(\xi_1, m)dn(\xi_2, m) \mp sn(\xi_1, m)sn(\xi_2, m)cn(\xi_1, m)cn(\xi_2, m)}{1 - m^2 sn^2(\xi_1, m)sn^2(\xi_2, m)}.
\end{aligned} \tag{2.9}$$

In this case from using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} \right)^i,$$
$$V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(\frac{cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} \right)^i, \ldots. \tag{2.10}$$

Further by using the properties of Jacobi elliptic functions, the iterative relations can be written in the following form:

$$U(\xi_{n\pm p}) = \sum_{i=0}^{K} \alpha_i \left(\frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})} \right)^i,$$
$$V(\xi_{n\pm p}) = \sum_{i=0}^{L} \beta_i \left(\frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})} \right)^i, \ldots, \tag{2.11}$$

where

$$\begin{aligned}\frac{F'(\xi_{n\pm d})}{F(\xi_{n\pm d})} = \frac{1}{M_1} \Big\{ &\pm cn(d, m) cn(\xi_n) dn(\xi_n, m) dn(d, m) \pm m^2 sn(d, m) sn(\xi_n, m) \\ &\mp 2m^2 sn(d, m) sn^3(\xi_n, m) \mp 2m^2 sn^3(d, m) sn(\xi_n, m) \pm m^2 sn^3(d, m) sn^3(\xi_n, m) \\ &+ sn(d, m) sn(\xi_n, m) \pm m^4 sn^3(d, m) sn^3(\xi_n, m) \\ &\mp m^2 sn^2(d, m) sn^2(\xi_n, m) dn(\xi_n, m) dn(d, m) cn(d, m) cn(\xi_n, m) \Big\},\end{aligned} \tag{2.12}$$

$$\begin{aligned}M_1 = &-cn(\phi, m) dn(\phi, m) sn(\xi_n, m) \mp sn(\phi, m) dn(\xi_n, m) cn(\xi_n, m) + m^2 sn^3(\xi_n, m) \\ &\times sn^2(\phi, m) cn(\phi, m) dn(\phi, m) \pm m^2 sn^2(\xi_n, m) sn^3(\phi, m) cn(\xi_n, m) dn(\xi_n, m),\end{aligned} \tag{2.13}$$

$d = p_{s1} d_1 + p_{s2} d_2 + \cdots + p_{sQ} d_Q$, p_{sj} is the jth component of shift vector p_s.

Type 2. if $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, $e_2 = -m^2$. In this case, (2.6) has the solution $F(\xi_n) = cn(\xi_n, m)$. From using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(-\frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} \right)^i,$$
$$V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(-\frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} \right)^i, \ldots. \tag{2.14}$$

Type 3. if $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, $e_2 = -1$. In this case, (2.6) has the solution $F(\xi_n) = dn(\xi_n, m)$. From using the properties of Jacobi elliptic functions the series expansion solutions (2.5) take the following form

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(-\frac{m^2 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} \right)^i,$$
$$V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(-\frac{m^2 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} \right)^i, \ldots. \tag{2.15}$$

Type 4. if $e_0 = 1 - m^2$, $e_1 = 2 - m^2$, $e_2 = 1$. In this case, (2.6) has the solution $F(\xi_n) = cs(\xi_n, m)$, then the series expansion solutions (2.5) take the following form

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(-\frac{ns(\xi_n, m) ds(\xi_n, m)}{cs(\xi_n, m)} \right)^i, \qquad V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(-\frac{ns(\xi_n, m) ds(\xi_n, m)}{cs(\xi_n, m)} \right)^i, \ldots. \tag{2.16}$$

Equation (2.16) can be written in the following form:

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(-\frac{dn(\xi_n, m)}{sn(\xi_n, m) cn(\xi_n, m)} \right)^i, \qquad V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(-\frac{dn(\xi_n, m)}{sn(\xi_n, m) cn(\xi_n, m)} \right)^i, \ldots. \tag{2.17}$$

Type 5. if $e_0 = 1$, $e_1 = 2m^2 - 1$, and $e_2 = m^2(m^2 - 1)$. In this case, (2.6) has the solution $F(\xi_n) = sd(\xi_n, m)$, then the series expansion solutions (2.5) take the following form

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{nd(\xi_n, m) cd(\xi_n, m)}{sd(\xi_n, m)} \right)^i, \qquad V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(\frac{nd(\xi_n, m) cd(\xi_n, m)}{sd(\xi_n, m)} \right)^i, \ldots. \tag{2.18}$$

Equation (2.18) can be written in the following form:

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)} \right)^i, \qquad V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(\frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)} \right)^i, \ldots. \tag{2.19}$$

Type 6. if $e_0 = m^2$, $e_1 = -(m^2 + 1)$, and $e_2 = 1$. In this case, (2.6) has the solution $F(\xi_n) = dc(\xi_n, m)$, then the series expansion solutions (2.5) take the following form

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{(1 - m^2) nc(\xi_n, m) sc(\xi_n, m)}{dc(\xi_n, m)} \right)^i,$$
$$V(\xi_n) = \sum_{i=0}^{L} \beta_i \left(\frac{(1 - m^2) nc(\xi_n, m) sc(\xi_n, m)}{dc(\xi_n, m)} \right)^i, \ldots. \tag{2.20}$$

Equation (2.20) can be written in the following form:

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{(1-m^2) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)} \right)^i, \qquad V_n(\xi_n) = \sum_{i=0}^{L} \beta_i \left(\frac{(1-m^2) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)} \right)^i, \ldots. \tag{2.21}$$

From the properties of the Jacobi elliptic functions, we can deduce the iterative relation to the above kind of solutions from Types 2–6 as we show in Type 1.

Equations (2.10)–(2.21) lead to getting all formulas of solutions from Types 1–6 as different. Consequently, we will discuss all solutions from Types 1–6.

Step 4. Determine the degree $K, L, \ldots$ of (2.5) by balancing the nonlinear term(s) and the highest-order derivatives of $U(\xi_n), V(\xi_n), \ldots$ in (2.4). It should be noted that the leading terms $U(\xi_{n\pm p})$, $V(\xi_{n\pm p})$, $\ldots$, $p \neq 0$ will not affect the balance because we are interested in balancing the terms of $F'(\xi_n)/F(\xi_n)$.

Step 5. Substituting $U(\xi_n)$, $V(\xi_n)$, and $\ldots$ in each type form 1–6 and the given values of K, L, and $\ldots$ into (2.4). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $dn(\xi_n, m)$, and $cn(\xi_n, m)$ together, the left hand side of (2.4) is converted into a polynomial in $sn(\xi_n, m)$, $dn(\xi_n, m)$, and $cn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for α_i, β_i, d_i, and c_i.

Step 6. Solving the over determined system of nonlinear algebraic equations by using Maple or Mathematica. We end up with explicit expressions for α_i, β_i, d_i, and c_j.

Step 7. Substituting α_i, β_i, d_i, and c_i into $U(\xi_n)$, $V(\xi_n)$, and $\ldots$ in the corresponding type from 1–6, we can finally obtain the exact solutions for (2.1).

3. Applications

In this section, we apply the proposed rational Jacobi elliptic functions method to construct the traveling wave solutions for some nonlinear DDEs via the lattice equation and the discrete nonlinear Schrodinger equation with a saturable nonlinearity which are very important in the mathematical physics and have been paid attention to by many researchers.

3.1. Example 1. The Lattice Equation

In this section, we study the lattice equation which takes the following form [22–25]

$$\frac{du_n(t)}{dt} = \left(\alpha + \beta u_n + \gamma u_n^2\right)(u_{n-1} - u_{n+1}), \tag{3.1}$$

where α, β, and γ are nonzero constants. The equation contains hybrid lattice equation, mKdV lattice equation, modified Volterra lattice equation, and Langmuir chain equation:

(i) (1+1) dimensional Hybrid lattice equation [25]:

$$\frac{du_n(t)}{dt} = \left(1 + \beta u_n + \gamma u_n^2\right)(u_{n-1} - u_{n+1}); \tag{3.2}$$

(ii) mKdV lattice equation [25]:

$$\frac{du_n(t)}{dt} = \left(\alpha - u_n^2\right)(u_{n-1} - u_{n+1}); \tag{3.3}$$

(iii) modified Volterra equation [24]:

$$\frac{du_n(t)}{dt} = u_n^2(u_{n-1} - u_{n+1}); \tag{3.4}$$

(iv) Langmuir chain equation [25]:

$$\frac{du_n(t)}{dt} = u_n(u_{n+1} - u_{n-1}). \tag{3.5}$$

According to the above steps, to seek traveling wave solutions of (3.1), we construct the transformation

$$u_n(t) = U(\xi_n), \qquad \xi_n = dn + c_1 t + \xi_0, \tag{3.6}$$

where d, c_1, and ξ_0 are constants. The transformation in (3.6) permits us to convert (3.1) into the following form:

$$c_1 U'(\xi_n) = \left(\alpha + \beta U(\xi_n) + \gamma U^2(\xi_n)\right)(U(\xi_n - d) - U(\xi_n + d)), \tag{3.7}$$

where $' = d/d\xi_n$. Considering the homogeneous balance between the highest-order derivative and the nonlinear term in (3.7), we get $K = 1$. Thus, the solution of (3.7) has the following form:

$$U(\xi_n) = \alpha_1\left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \alpha_0, \tag{3.8}$$

where α_0, and α_1 are constants to be determined later, and $F(\xi_n)$ satisfies a discrete Jacobi elliptic ordinary differential (2.6). When, we discuss the solutions of the Jacobi elliptic differential difference (2.6), we get the following types.

Type 1. If $e_0 = 1$, $e_1 = -(1+m^2)$, and $e_2 = m^2$. In this case, the series expansion solution of (3.7) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)}. \tag{3.9}$$

With help of Maple, we substitute (3.9) and (2.12) into (3.7), cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $dn(\xi_n, m)$, and $cn(\xi_n, m)$ together, the left hand side of (3.7) is converted into polynomial in $sn(\xi_n, m)$, $dn(\xi_n, m)$, and $cn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for α_0, α_1, d, and c_1. Solving the set of algebraic equations by using Maple or Mathematica, we have

$$\alpha_0 = -\frac{\beta}{2\gamma}, \qquad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}\, sn(d,m)}{2\gamma cn(d,m) dn(d,m)}, \qquad c_1 = -\frac{(4\alpha\gamma - \beta^2) sn(d,m)}{2\gamma cn(d,m) dn(d,m)}. \tag{3.10}$$

From (3.9) and (3.10), the solution of (3.7) takes the following form:

$$U(\xi_n) = \frac{\sqrt{\beta^2 - 4\alpha\gamma}\, sn(d,m) cn(\xi_n, m) dn(\xi_n, m)}{2\gamma cn(d,m) dn(d,m) sn(\xi_n, m)} - \frac{\beta}{2\gamma}, \tag{3.11}$$

where $\xi_n = dn - ((4\alpha\gamma - \beta^2) sn(d,m) / [2\gamma cn(d,m) dn(d,m)])t + \xi_0$.

Type 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, and $e_2 = -m^2$. In this case, the series expansion solution of (3.7) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n) dn(\xi_n)}{cn(\xi_n, m)}. \tag{3.12}$$

With the help of Maple, we substitute (3.12) into (3.7), cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $dn(\xi_n, m)$, and $cn(\xi_n, m)$ together, the left hand side of (3.7) is converted into polynomial in $sn(\xi_n, m)$, $dn(\xi_n, m)$, and $cn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for α_0, α_1, d, and c_1. Solving the set of algebraic equations by using Maple or Mathematica, we get

$$\alpha_0 = -\frac{\beta}{2\gamma}, \qquad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}\, sn(d,m) dn(d,m)}{2\gamma cn(d,m)}, \qquad c_1 = -\frac{(4\alpha\gamma - \beta^2) dn(d,m) sn(d,m)}{2\gamma cn(d,m)}. \tag{3.13}$$

In this case the solution of (3.7) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\beta^2 - 4\alpha\gamma}\, sn(d,m) dn(d,m) sn(\xi_n, m) dn(\xi_n, m)}{2\gamma cn(d,m) cn(\xi_n, m)}, \tag{3.14}$$

where $\xi_n = dn - ((4\alpha\gamma - \beta^2) dn(d,m) sn(d,m) / [2\gamma cn(d,m)])t + \xi_0$.

Type 3. if $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, and $e_2 = -1$. In this case, the series expansion solution of (3.7) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2\alpha_1 sn(\xi_n)cn(\xi_n)}{dn(\xi_n)}. \tag{3.15}$$

Consequently, by using Maple or Mathematica, we obtain the following results:

$$\alpha_0 = -\frac{\beta}{2\gamma}, \qquad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}sn(d,m)cn(d,m)}{2\gamma dn(d,m)}, \qquad c_1 = -\frac{(4\alpha\gamma - \beta^2)cn(d,m)sn(d,m)}{2\gamma dn(d,m)}. \tag{3.16}$$

In this case, the solution takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\beta^2 - 4\alpha\gamma}m^2 sn(d,m)cn(d,m)sn(\xi_n,m)cn(\xi_n,m)}{2\gamma dn(d,m)dn(\xi_n,m)}, \tag{3.17}$$

where $\xi_n = dn - ((4\alpha\gamma - \beta^2)cn(d,m)sn(d,m)/[2\gamma dn(d,m)])t + \xi_0$.

Type 4. if $e_0 = 1 - m^2$, $e_1 = 2 - m^2$, and $e_2 = 1$. In this case, the series expansion solution of (3.7) has the form:

$$U_n(\xi_n) = \alpha_0 - \frac{\alpha_1 ns(\xi_n)ds(\xi_n)}{cs(\xi_n)}. \tag{3.18}$$

Consequently, using the Maple or Mathematica we get the following results:

$$\alpha_0 = -\frac{\beta}{2\gamma}, \qquad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}sn(d,m)cn(d,m)}{2\gamma dn(d,m)}, \qquad c_1 = -\frac{(4\alpha\gamma - \beta^2)cn(d,m)sn(d,m)}{2\gamma dn(d,m)}. \tag{3.19}$$

In this case, the solution of (3.7) takes the following form:

$$U_n(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\beta^2 - 4\alpha\gamma}sn(d,m)cn(d,m)ns(\xi_n,m)ds(\xi_n,m)}{2\gamma dn(d,m)cs(\xi_n,m)}, \tag{3.20}$$

where $\xi_n = dn - ((4\alpha\gamma - \beta^2)cn(d,m)sn(d,m)/[2\gamma dn(d,m)])t + \xi_0$.

Type 5. if $e_0 = 1$, $e_1 = 2m^2 - 1$, and $e_2 = m^2(m^2 - 1)$. In this case, the series expansion solution of (3.7) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 nd(\xi_n)cd(\xi_n)}{sd(\xi_n)}. \tag{3.21}$$

Consequently, by using Maple or Mathematica, we get the following results:

$$\alpha_0 = -\frac{\beta}{2\gamma}, \qquad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}\, sn(d,m)dn(d,m)}{2\gamma cn(d,m)}, \qquad c_1 = \frac{(\beta^2 - 4\alpha\gamma)sn(d,m)dn(d,m)}{2\gamma cn(d,m)}. \tag{3.22}$$

In this case, the solution takes of (3.7) the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\beta^2 - 4\alpha\gamma}\, sn(d,m)dn(d,m)nd(\xi_n,m)cd(\xi_n,m)}{2\gamma cn(d,m)sd(\xi_n,m)}, \tag{3.23}$$

where $\xi_n = dn + ((\beta^2 - 4\alpha\gamma)sn(d,m)dn(d,m)/[2\gamma cn(d,m)])t + \xi_0$.

Type 6. if $e_0 = m^2$, $e_1 = -(m^2 + 1)$, and $e_2 = 1$. In this case, the series expansion solution of (3.7) has the form:

$$U(\xi_n) = \alpha_0 + \frac{(1 - m^2)\alpha_1 nc(\xi_n)sc(\xi_n)}{dc(\xi_n)}. \tag{3.24}$$

Consequently, by using Maple or Mathematica, we get the following results:

$$\alpha_0 = -\frac{\beta}{2\gamma}, \qquad \alpha_1 = \frac{\sqrt{\beta^2 - 4\alpha\gamma}\, sn(d,m)}{2\gamma cn(d,m)dn(d,m)}, \qquad c_1 = \frac{(\beta^2 - 4\alpha\gamma)sn(d,m)}{2\gamma cn(d,m)dn(d,m)}. \tag{3.25}$$

In this case, the solution of (3.7) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\beta^2 - 4\alpha\gamma}(1 - m^2)sn(d,m)nc(\xi_n,m)sc(\xi_n,m)}{2\gamma dn(d,m)cn(d,m)dc(\xi_n,m)}, \tag{3.26}$$

where $\xi_n = dn + ((\beta^2 - 4\alpha\gamma)sn(d,m)/[2\gamma cn(d,m)dn(d,m)])t + \xi_0$.

3.2. Example 2. The Discrete Nonlinear Schrodinger Equation

The discrete nonlinear Schrodinger equation (DNSE) is one of the most fundamental nonlinear lattice models [8]. It arises in nonlinear optics as a model of infinite wave guide arrays [26] and has been recently implemented to describe Bose-Einstein condensates in optical lattices. The class of DNSE model with saturable nonlinearity is also of particular interest in their own right, due to a feature first unveiled in [27]. In this section, we study the DNSE with a saturable nonlinearity [28, 29] having the form

$$i\frac{\partial \psi_n}{\partial t} + (\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \frac{\nu|\psi_n|^2}{1 + \mu|\psi_n|^2}\psi_n = 0, \tag{3.27}$$

which describes optical pulse propagations in various doped fibers, ψ_n is a complex valued wave function at sites n while ν and μ. We make the transformation

$$\psi_n = \phi(\xi_n)e^{-i(\sigma t+\rho)}, \qquad \xi_n = \alpha n + \beta, \tag{3.28}$$

where σ, ρ, α, and β are arbitrary real constants. The transformation (3.28) permits us converting (3.27) into the following nonlinear difference equation

$$(\sigma - 2)\phi(\xi_n) + \phi(\xi_{n+1}) + \phi(\xi_{n-1}) + \frac{\nu\phi^3(\xi_n)}{1+\mu\phi^2(\xi_n)} = 0. \tag{3.29}$$

We assume that (3.29) has a solution of the form:

$$\phi(\xi_n) = U(\xi_n) = \alpha_1\left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \alpha_0, \tag{3.30}$$

where α_1, and α_0 are constants to be determined later and $F(\xi_n)$ satisfying a discrete Jacobi elliptic differential equation (2.6). When, we discuss the solutions of (2.6), we have the following types.

Type 1. If $e_0 = 1$, $e_1 = -(1+m^2)$, and $e_2 = m^2$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = \alpha_1\frac{cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)} + \alpha_0. \tag{3.31}$$

With the help of Maple, we substitute (3.31) and (2.12) into (3.29), cleaning the denominator and collecting all terms with the same order of $cn(\xi_n, m)$, $dn(\xi_n, m)$, and $sn(\xi_n, m)$ together, the left hand side of (3.29) is converted into polynomial in $cn(\xi_n, m)$, $dn(\xi_n, m)$, and $sn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for α_0, α_1, σ, ρ, α, and β. Solving the set of algebraic equations by using Maple or Mathematica, we obtain

$$\begin{gathered}\alpha_0 = 0, \qquad \alpha_1 = \frac{sn(\alpha, m)}{\sqrt{-\mu}cn(\alpha, m)dn(\alpha, m)}, \qquad \nu = \frac{-2\mu(m^2sn^4(\alpha, m) - 1)}{cn^2(\alpha, m)dn^2(\alpha, m)}, \\ \sigma = \frac{-2sn^2(\alpha, m)\left[m^2cn^2(\alpha, m) + dn^2(\alpha, m)\right]}{cn^2(\alpha, m)dn^2(\alpha, m)}, \quad \mu < 0.\end{gathered} \tag{3.32}$$

In this case, the solution of (3.27) takes the following form:

$$\psi_n = \frac{sn(\alpha, m)cn(\xi_n, m)dn(\xi_n, m)}{\sqrt{-\mu}cn(\alpha, m)dn(\alpha, m)sn(\xi_n, m)}\operatorname{Exp}\left\{-i\left[\frac{-2tsn^2(\alpha, m)\left[m^2cn^2(\alpha, m) + dn^2(\alpha, m)\right]}{cn^2(\alpha, m)dn^2(\alpha, m)} + \rho\right]\right\}, \tag{3.33}$$

where $\xi_n = \alpha n + \beta$.

Type 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, and $e_2 = -m^2$. In this case the solution of (3.29) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)}. \tag{3.34}$$

Consequently, by using Maple or Mathematica, we get the following results:

$$\begin{gathered}
\alpha_0 = 0, \qquad \alpha_1 = \frac{sn(\alpha, m) dn(\alpha, m)}{\sqrt{-\mu} cn(\alpha, m)}, \qquad \nu = \frac{2\mu\left(m^2 sn^4(\alpha, m) - 2m^2 sn^2(\alpha, m) + 1\right)}{cn^2(\alpha, m)}, \\
\sigma = \frac{-2sn^2(\alpha, m)\left[m^2 sn^2(\alpha, m) + 1 - 2m^2\right]}{cn^2(\alpha, m)}, \quad \mu < 0.
\end{gathered} \tag{3.35}$$

In this case, the solution takes the following form:

$$\begin{aligned}
\psi_n = &-\frac{sn(\alpha, m) dn(\alpha, m) sn(\xi_n, m) dn(\xi_n, m)}{\sqrt{-\mu} cn(\alpha, m) cn(\xi_n, m)} \\
&\times \operatorname{Exp}\left\{-i\left[\frac{-2t sn^2(\alpha, m)\left[m^2 sn^2(\alpha, m) + 1 - 2m^2\right]}{cn^2(\alpha, m)} + \rho\right]\right\}.
\end{aligned} \tag{3.36}$$

Type 3. if $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, and $e_2 = -1$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2 \alpha_1 sn(\xi_n) cn(\xi_n)}{dn(\xi_n)}. \tag{3.37}$$

Consequently, by using Maple or Mathematica, we get the following results:

$$\begin{gathered}
\alpha_0 = 0, \qquad \alpha_1 = \frac{sn(\alpha, m) cn(\alpha, m)}{\sqrt{-\mu}\, dn(\alpha, m)}, \qquad \nu = \frac{2\mu\left(m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1\right)}{dn^2(\alpha, m)}, \\
\sigma = \frac{-2sn^2(\alpha, m)\left[m^2 sn^2(\alpha, m) - 2 + m^2\right]}{dn^2(\alpha, m)}, \quad \mu < 0.
\end{gathered} \tag{3.38}$$

In this case, the solution takes the following form:

$$\begin{aligned}
\psi_n = &-\frac{m^2 sn(\alpha, m) cn(\alpha, m) sn(\xi_n, m) cn(\xi_n, m)}{\sqrt{-\mu} dn(\alpha, m) dn(\xi_n, m)} \\
&\times \operatorname{Exp}\left\{-i\left[\frac{-2t sn^2(\alpha, m)\left[m^2 sn^2(\alpha, m) - 2 + m^2\right]}{dn^2(\alpha, m)} + \rho\right]\right\}.
\end{aligned} \tag{3.39}$$

Type 4. if $e_0 = 1 - m^2$, $e_1 = 2 - m^2$, and $e_2 = 1$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = \alpha_0 - \frac{\alpha_1 ns(\xi_n) ds(\xi_n)}{cs(\xi_n)}. \tag{3.40}$$

After some calculation, the solution of (3.27) takes the following form:

$$\begin{aligned}\varphi_n = &-\frac{sn(\alpha, m)cn(\alpha, m)ns(\xi_n, m)ds(\xi_n, m)}{\sqrt{-\mu}\ dn(\alpha, m)cn(\xi_n, m)}\\ &\times \operatorname{Exp}\left\{-i\left[\frac{-2t\ sn^2(\alpha, m)\left[m^2 sn^2(\alpha, m) - 2 + m^2\right]}{dn^2(\alpha, m)} + \rho\right]\right\},\end{aligned} \tag{3.41}$$

where $\nu = 2\mu(m^2\ sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1)/dn^2(\alpha, m)$.

Type 5. if $e_0 = 1$, $e_1 = 2m^2 - 1$, and $e_2 = m^2(m^2 - 1)$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 nd(\xi_n) cd(\xi_n)}{sd(\xi_n)}. \tag{3.42}$$

After some calculation, the solution of (3.27) takes the following form:

$$\begin{aligned}\varphi_n = &-\frac{sn(\alpha, m)dn(\alpha, m)cd(\xi_n, m)nd(\xi_n, m)}{\sqrt{-\mu} cn(\alpha, m)sd(\xi_n, m)}\\ &\times \operatorname{Exp}\left\{-i\left[\frac{-2tsn^2(\alpha, m)\left[m^2 sn^2(\alpha, m) + 1 - 2m^2\right]}{cn^2(\alpha, m)} + \rho\right]\right\},\end{aligned} \tag{3.43}$$

where $\nu = 2\mu(m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1)/cn^2(\alpha, m)$.

Type 6. if $e_0 = m^2$, $e_1 = -(m^2 + 1)$, and $e_2 = 1$. In this case, the series expansion solution of (3.29) has the form:

$$U(\xi_n) = \alpha_0 + \frac{(1 - m^2)\alpha_1\ nc(\xi_n) sc(\xi_n)}{dc(\xi_n)}. \tag{3.44}$$

After some calculation, the solution of (3.27) takes the following form:

$$\begin{aligned}\varphi_n = &\frac{(1 - m^2) sn(\alpha, m)nc(\xi_n, m)sc(\xi_n, m)}{\sqrt{-\mu}\ cn(\alpha, m)dn(\alpha, m)dc(\xi_n, m)}\\ &\times \operatorname{Exp}\left\{-i\left[\frac{-2tsn^2(\alpha, m)\left[m^2 cn^2(\alpha, m) + dn^2(\alpha, m)\right]}{cn^2(\alpha, m)dn^2(\alpha, m)} + \rho\right]\right\},\end{aligned} \tag{3.45}$$

where $\nu = -2\mu(m^2 sn^4(\alpha, m) - 1)/[cn^2(\alpha, m)dn^2(\alpha, m)]$.

Remark 3.1. (1) The formulas of the exact solutions from Types 1–6 are different, and consequently, we must discuss the exact solutions in all types from 1–6.

(2) The values of α_i, β_i, d_i, and c_i in Examples 1 and 2 have a unique determination in all types of this method.

4. Conclusion

In this paper, we put a direct method to calculate the rational Jacobi elliptic solutions for the nonlinear difference differential equations via the lattice equation and the discrete nonlinear Schrodinger equation with a saturable nonlinearity. As a result, many new and more rational Jacobi elliptic solutions are obtained, from which hyperbolic function solutions and trigonometric function solutions are derived when the modulus $m \to 1$ and $m \to 0$.

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