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Research Article
A Sharp Double Inequality between Seiffert, Arithmetic, and Geometric Means

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For fixed $s \geq 1$ and any $t_1, t_2 \in (0, 1/2)$ we prove that the double inequality $G^*(t_1a + (1 - t_1)b, t_1b + (1 - t_1)a) < P(a, b) < G^*(t_2a + (1 - t_2)b, t_2b + (1 - t_2)a)A^{1-s}(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $t_1 \leq (1 - \sqrt{1 - (2/\pi)^{2/s}})/2$ and $t_2 \geq (1 - 1/\sqrt{3s})/2$. Here, $P(a, b)$, $A(a, b)$ and $G(a, b)$ denote the Seiffert, arithmetic, and geometric means of two positive numbers $a$ and $b$, respectively.

1. Introduction

The Seiffert mean $P(a, b)$ [1] of two distinct positive numbers $a$ and $b$ is defined by

$$P(a, b) = \frac{a - b}{4 \arctan \left( \frac{\sqrt{a/b}}{1 - \sqrt{a/b}} \right) - \pi}. \quad (1.1)$$

Recently, the Seiffert mean $P(a, b)$ has been the subject of intensive research. In particular, many remarkable inequalities for $P(a, b)$ can be found in the literature [2–17]. The Seiffert mean $P(a, b)$ can be rewritten as (see [6, (2.4)])

$$P(a, b) = \frac{a - b}{2 \arcsin((a - b)/(a + b))}. \quad (1.2)$$
Let \( A(a,b) = (a+b)/2 \), \( G(a,b) = \sqrt{ab} \) and \( H(a,b) = 2ab/(a+b) \) be the classical arithmetic, geometric, and harmonic means of two positive numbers \( a \) and \( b \), respectively. Then it is well known that inequalities \( H(a,b) < G(a,b) < P(a,b) < A(a,b) \) hold for all \( a, b > 0 \) with \( a \neq b \).

For \( a, \beta, \lambda, \mu \in (0, 1/2) \), Chu et al. [18, 19] proved that the double inequalities

\[
G(aa + (1-a)b, ab + (1-a)a) < P(a,b) < G(\beta a + (1-\beta)b, \beta b + (1-\beta)a),
\]

\[
H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < P(a,b) < H(\mu a + (1-\mu)b, \mu b + (1-\mu)a)
\]

(1.3)

hold for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha \leq (1 - \sqrt{1-4/\pi^2})/2 \), \( \beta \geq (3 - \sqrt{3})/6 \), \( \lambda \leq (1 - \sqrt{1-2/\pi^2})/2 \) and \( \mu \geq (6 - \sqrt{6})/12 \).

Let \( t \in (0, 1/2) \), \( s \geq 1 \) and

\[
Q_{t,s}(a,b) = G(ta + (1-t)b, tb + (1-t)a)A^{1-s}(a,b),
\]

(1.4)

then it is not difficult to verify that

\[
Q_{t,1}(a,b) = G(ta + (1-t)b, tb + (1-t)a),
\]

\[
Q_{t,2}(a,b) = H(ta + (1-t)b, tb + (1-t)a)
\]

(1.5)

and \( Q_{t,s}(a,b) \) is strictly increasing with respect to \( t \in (0, 1/2) \) for fixed \( a, b > 0 \) with \( a \neq b \).

It is natural to ask what are the largest value \( t_1 = t_1(s) \) and the least value \( t_2 = t_2(s) \) in \((0, 1/2)\) such that the double inequality \( Q_{t_1,s}(a,b) < P(a,b) < Q_{t_2,s}(a,b) \) holds for all \( a, b > 0 \) with \( a \neq b \) and \( s \geq 1 \). The main purpose of this paper is to answer this question.

### 2. Main Result

In order to establish our main result we need two lemmas, which we present in the following.

**Lemma 2.1.** If \( s \geq 1 \), then \( 1/(3s) + (2/\pi)^{2/s} < 1 \).

*Proof.* Consider the following:

\[
f(s) = \frac{1}{3s} + \left(\frac{2}{\pi}\right)^{2/s}.
\]

(2.1)
Then simple computations lead to

\[
\lim_{s \to +\infty} f(s) = 1, \tag{2.2}
\]

\[
f'(s) = \frac{2}{s^2} \log \frac{\pi}{2} \left[ \left( \frac{2}{\pi} \right)^{2/s} - \frac{1}{6 \log(\pi/2)} \right]
\geq \frac{2}{s^2} \log \frac{\pi}{2} \left[ \left( \frac{2}{\pi} \right)^{2} - \frac{1}{6 \log(\pi/2)} \right]
\]

\[
= \frac{24 \log(\pi/2) - \pi^2}{3 \pi^2 s^2}
\tag{2.3}
\]

for \( s \geq 1 \).

Computational and numerical experiments show that

\[
24 \log \left( \frac{\pi}{2} \right) - \pi^2 = 0.968 \cdots > 0. \tag{2.4}
\]

Inequalities (2.3) and (2.4) imply that \( f(s) \) is strictly increasing in \([1, +\infty)\). Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the monotonicity of \( f(s) \).

**Lemma 2.2.** Let \( 0 \leq u \leq 1, s \geq 1 \) and

\[
f_{u,s}(x) = \frac{s}{2} \log \left( 1 - ux^2 \right) - \log x + \log(\arcsin x). \tag{2.5}
\]

Then inequality \( f_{u,s}(x) > 0 \) holds for all \( x \in (0, 1) \) if and only if \( 3su \leq 1 \), and inequality \( f_{u,s}(x) < 0 \) holds for all \( x \in (0, 1) \) if and only if \( u + (2/\pi)^{2/s} \geq 1 \).

**Proof.** If \( u = 0 \), then we clearly see that \( 3su \leq 1 \), \( u + (2/\pi)^{2/s} \leq 1 \), and \( f_{0,s}(x) = \log[(\arcsin x)/x] > 0 \) for all \( s \geq 1 \) and \( x \in (0, 1) \). In the following discussion, we assume that \( 0 < u \leq 1 \).

From (2.5) and simple computations we have

\[
\lim_{x \to 0^+} f_{u,s}(x) = 0, \tag{2.6}
\]

\[
f'_{u,s}(x) = \frac{1}{\sqrt{1 - x^2} \arcsin x} - \frac{1 + u(s - 1)x^2}{x(1 - ux^2)} = \frac{1 + u(s - 1)x^2}{x(1 - ux^2) \arcsin x} g_{u,s}(x), \tag{2.7}
\]
where

\[
g_{u,s}(x) = \frac{x(1-ux^2)}{\sqrt{1-x^2[1+u(s-1)x^2]}} - \arcsin x,
\]
\[
g_{u,s}(0) = 0,
\]
\[
g'_{u,s}(x) = \frac{x^2}{(1-x^2)^{3/2}[1+u(s-1)x^2]^2} h_{u,s}(x),
\]

where

\[
h_{u,s}(x) = u^2(s-1)^2x^4 + u(-s^2u + us + 4s - 2)x^2 + 1 - 3su,
\]
\[
h_{u,s}(0) = 1 - 3su,
\]
\[
h_{u,s}(1) = us(1-u) + (1-u)^2.
\]

We divide the proof into four cases.

**Case 1 (3su \leq 1).** Then from (2.11) and (2.12) together with the fact that

\[
-us^2 + us + 4s - 2 = 2(s-1) + s(u + 2su + 1) + s(1-3su) > 0,
\]

we clearly see that

\[
h_{u,s}(0) \geq 0,
\]

and \(h_{u,s}(x)\) is strictly increasing in \([0,1]\).

Equation (2.12) and the monotonicity of \(h_{u,s}(x)\) imply that

\[
h_{u,s}(x) > 0
\]

for \(x \in (0,1)\).

Equation (2.10) and inequality (2.16) lead to the conclusion that \(g_{u,s}(x)\) is strictly increasing in \([0,1]\). Then from (2.9) we know that

\[
g_{u,s}(x) > 0
\]

for \(x \in (0,1)\).

It follows from (2.7) and inequality (2.17) that \(f_{u,s}(x)\) is strictly increasing in \((0,1]\).

Therefore, \(f_{u,s}(x) > 0\) for all \(x \in (0,1)\) follows from (2.6) and the monotonicity of \(f_{u,s}(x)\).

**Case 2 (3su > 1).** Then (2.12) and the continuity of \(h_{u,s}(x)\) imply that there exists \(0 < \lambda < 1\) such that

\[
h_{u,s}(x) < 0
\]

for \(x \in [0,\lambda)\).
Therefore, \( f_{u,s}(x) < 0 \) for \( x \in (0, \lambda) \) follows easily from (2.6), (2.7), (2.9) and (2.10) together with inequality (2.18).

**Case 3** \((u + (2/\pi)^{2/s}) \geq 1\). Then Lemma 2.1 and (2.12) lead to

\[
h_{u,s}(0) = 1 - 3su \leq 1 - 3s \left[ 1 - \left( \frac{2}{\pi} \right)^{2/s} \right] < 0. \tag{2.19}
\]

We divide the proof into two subcases.

**Subcase 3.1** \((u = 1)\). Then (2.13) becomes

\[
h_{u,s}(1) = 0. \tag{2.20}
\]

Let \( t = x^2 \), then from (2.11) we clearly see that the function \( h_{u,s} \) is a quadratic function of variable \( t \). It follows from inequality (2.19) and (2.20) that

\[
h_{u,s}(x) < 0
\]

for all \( x \in [0, 1) \).

Therefore, \( f_{u,s}(x) < 0 \) for \( x \in (0, 1) \) follows easily from (2.6), (2.7), (2.9) and (2.10) together with inequality (2.21).

**Subcase 3.2** \((0 < u < 1)\). Then from (2.5), (2.8), and (2.13) we have

\[
f_{u,s}(1) = \log \left[ \frac{\pi}{2} (1 - u)^{s/2} \right] \leq 0, \tag{2.22}
\]

\[
\lim_{x \to 1} g_{u,s}(x) = +\infty, \tag{2.23}
\]

\[
h_{u,s}(1) > 0. \tag{2.24}
\]

From (2.11), (2.19), and (2.24) we clearly see that there exists \( 0 < \lambda_1 < 1 \) such that \( h_{u,s}(x) < 0 \) for \( x \in [0, \lambda_1) \) and \( h_{u,s}(x) > 0 \) for \( x \in (\lambda_1, 1] \). Then (2.10) implies that \( g_{u,s}(x) \) is strictly decreasing in \([0, \lambda_1]\) and strictly increasing in \([\lambda_1, 1]\).

From (2.9) and (2.23) together with the piecewise monotonicity of \( g_{u,s}(x) \) we clearly see that there exists \( 0 < \lambda_2 < 1 \) such that \( g_{u,s}(x) < 0 \), for \( x \in (0, \lambda_2) \) and \( g_{u,s}(x) > 0 \) for \( x \in (\lambda_2, 1) \). Then (2.7) implies that \( f_{u,s}(x) \) is strictly decreasing in \((0, \lambda_2]\) and strictly increasing in \([\lambda_2, 1]\).

Therefore, \( f_{u,s}(x) < 0 \) for \( x \in (0, 1) \) follows from (2.6) and (2.22) together with the piecewise monotonicity of \( f_{u,s}(x) \).

**Case 4** \((u + (2/\pi)^{2/s}) < 1\). Then (2.5) leads to

\[
f_{u,s}(1) = \log \left[ \frac{\pi}{2} (1 - u)^{s/2} \right] > 0. \tag{2.25}
\]

From inequality (2.25) and the continuity of \( f_{u,s}(x) \) we know that there exists \( 0 < \mu < 1 \) such that \( f_{u,s}(x) > 0 \) for \( x \in (\mu, 1) \).
Theorem 2.3. If $t_1, t_2 \in (0, 1/2)$ and $s \geq 1$, then the double inequality
\[
Q_{t_1,s}(a, b) < P(a, b) < Q_{t_2,s}(a, b)
\]
holds for all $a, b > 0$ with $a \neq b$ if and only if $t_1 \leq (1 - \sqrt{1 - (2/\pi)^{2/s}})/2$ and $t_2 \geq (1 - 1/\sqrt{3s})/2$.

Proof. Since both $Q_{t,s}(a, b)$ and $P(a, b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $x = (a - b)/(a + b) \in (0, 1)$. Then from (1.2) and (1.4) we have
\[
\log \left( \frac{Q_{t,s}(a, b)}{P(a, b)} \right) = \log \left( \frac{Q_{t,s}(a, b)}{A(a, b)} \right) - \log \left( \frac{P(a, b)}{A(a, b)} \right) = \frac{s}{2} \log \left( 1 - (1 - 2t)^2 x^2 \right) - \log x + \log(\arcsin x).
\]
Therefore, Theorem 2.3 follows easily from Lemma 2.2 and (2.27). \qed

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References

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