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Research Article

A Note on Approximating Curve with 1-Norm Regularization Method for the Split Feasibility Problem

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Inspired by the very recent results of Wang and Xu (2010), we study properties of the approximating curve with 1-norm regularization method for the split feasibility problem (SFP). The concept of the minimum-norm solution set of SFP in the sense of 1-norm is proposed, and the relationship between the approximating curve and the minimum-norm solution set is obtained.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The problem under consideration in this paper is formulated as finding a point x satisfying the property:

$$x \in C, \quad Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Problem (1.1), referred to by Censor and Elfving [1] as the split feasibility problem (SFP), attracts many authors' attention due to its application in signal processing [1]. Various algorithms have been invented to solve it (see [2–13] and references therein).

Using the idea of Tikhonov's regularization, Wang and Xu [14] studied the properties of the approximating curve for the SFP. They gave the concept of the minimum-norm solution of the SFP (1.1) and proved that the approximating curve converges strongly to the minimum-norm solution of the SFP (1.1). Together with some properties of this approximating curve, they introduced a modification of Byrne's CQ algorithm [2] so that strong convergence is guaranteed and its limit is the minimum-norm solution of SFP (1.1).

In the practical application, H_1 and H_2 are often $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. Moreover, scientists and engineers are more willing to use 1-norm regularization method in the calculation process (see, e.g., [15–18]). Inspired by the above results of Wang and Xu [14], we study properties of the approximating curve with 1-norm regularization method. We also define the concept of the minimum-norm solution set of SFP (1.1) in the sense of 1-norm. The relationship between the approximating curve and the minimum-norm solution set is obtained.

2. Preliminaries

Let X be a normed linear space with norm $\|\cdot\|$, and let X^* be the dual space of X. We use the notation $\langle x, f\rangle$ to denote the value of $f \in X^*$ at $x \in X$. In particular, if X is a Hilbert space, we will denote it by H, and $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and its induced norm, respectively.

We recall some definitions and facts that are needed in our study.

Let P_C denote the *projection* from H onto a nonempty closed convex subset C of H; that is,

$$P_C x = \arg\min_{y\in C}\|x - y\|, \quad x \in H. \tag{2.1}$$

It is well known that $P_C x$ is characterized by the inequality

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C. \tag{2.2}$$

Definition 2.1. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex functional, $x_0 \in \operatorname{dom}(\varphi) = \{x \in X : \varphi(x) < +\infty\}$. Set

$$\partial\varphi(x_0) = \left\{\xi \in X^* : \varphi(x) \geq \varphi(x_0) + \langle x - x_0, \xi\rangle, \forall x \in X\right\}. \tag{2.3}$$

If $\partial\varphi(x_0) \neq \emptyset$, φ is said to be *subdifferentiable* at x_0 and $\partial\varphi(x_0)$ is called the *subdifferential* of φ at x_0. For any $\xi \in \partial\varphi(x_0)$, we say ξ is a *subgradient* of φ at x_0.

Lemma 2.2. *There holds the following property:*

$$\partial(\|x\|) = \begin{cases} \{x^* \in X^* : \|x^*\| = 1, \langle x, x^*\rangle = \|x\|\}, & x \neq 0, \\ \{x^* \in X^* : \|x^*\| \leq 1\}, & x = 0, \end{cases} \tag{2.4}$$

where $\partial(\|x\|)$ denotes the subdifferential of the functional $\|x\|$ at $x \in X$.

Proof. The process of the proof will be divided into two parts.

Case 1. In the case of $x = 0$, for any $x^* \in X^*$ such that $\|x^*\| \leq 1$ and any $y \in X$, there holds the inequality

$$\|y\| \geq \langle y, x^* \rangle = \|x\| + \langle y - x, x^* \rangle, \tag{2.5}$$

so we have $x^* \in \partial(\|x\|)$, and thus,

$$\{x^* \in X^* : \|x^*\| \leq 1\} \subset \partial(\|x\|). \tag{2.6}$$

Conversely, for any $x^* \in \partial(\|x\|)$, we have from the definition of subdifferential that

$$\begin{gathered} \|y\| \geq \|x\| + \langle y - x, x^* \rangle = \langle y, x^* \rangle, \quad \forall y \in X, \\ \|y\| = \| - y\| \geq \langle -y, x^* \rangle = -\langle y, x^* \rangle. \end{gathered} \tag{2.7}$$

Consequently,

$$\left|\langle y, x^* \rangle\right| \leq \|y\|, \quad \forall y \in X, \tag{2.8}$$

and this implies that $\|x^*\| \leq 1$. Thus, we have verified that

$$\partial(\|x\|) \subset \{x^* \in X^* : \|x^*\| \leq 1\}. \tag{2.9}$$

Combining (2.6) and (2.9), we immediately obtain

$$\partial(\|x\|) = \{x^* \in X^* : \|x^*\| \leq 1\}. \tag{2.10}$$

Case 2. If $x \neq 0$, for any $x^* \in \{x^* \in X^* : \|x^*\| = 1, \langle x, x^* \rangle = \|x\|\}$, we obviously have

$$\langle y - x, x^* \rangle = \langle y, x^* \rangle - \|x\| \leq \|y\| - \|x\|, \quad \forall y \in X, \tag{2.11}$$

which means that $x^* \in \partial(\|x\|)$, and thus,

$$\{x^* \in X^* : \|x^*\| = 1, \langle x, x^* \rangle = \|x\|\} \subset \partial(\|x\|). \tag{2.12}$$

Conversely, if $x^* \in \partial(\|x\|)$, we have

$$\langle -x, x^* \rangle \leq 0 - \|x\| = -\|x\|, \qquad \langle x, x^* \rangle \leq 2\|x\| - \|x\| = \|x\|; \tag{2.13}$$

hence,

$$\langle x, x^* \rangle = \|x\|. \tag{2.14}$$

On the other hand, using (2.14), we get

$$\|y\| \geq \|x\| + \langle y - x, x^* \rangle = \|x\| + \langle y, x^* \rangle - \langle x, x^* \rangle = \langle y, x^* \rangle, \quad \forall y \in X, \tag{2.15}$$

and consequently,

$$\begin{aligned} \|y\| = \|-y\| &\geq \|x\| + \langle -y - x, x^* \rangle \\ &= \|x\| - \langle y, x^* \rangle - \langle x, x^* \rangle \\ &= -\langle y, x^* \rangle; \end{aligned} \tag{2.16}$$

that is,

$$-\|y\| \leq \langle y, x^* \rangle. \tag{2.17}$$

Equation (2.17) together with (2.15) implies that

$$|\langle y, x^* \rangle| \leq \|y\|, \quad \forall y \in X; \tag{2.18}$$

hence, $\|x^*\| \leq 1$. Note that (2.14) implies that $\|x^*\| \geq \langle x, x^* \rangle / \|x\| = 1$; we assert that

$$\|x^*\| = 1. \tag{2.19}$$

Thus we have from (2.14) and (2.19) that

$$\{x^* \in X^* : \|x^*\| = 1, \langle x, x^* \rangle = \|x\|\} \supset \partial(\|x\|). \tag{2.20}$$

The proof is finished by combining (2.12) and (2.20). □

$\|\cdot\|_\infty$ and $\|\cdot\|_1$ will stand for ∞-norm and 1-norm of any Euclidean space; respectively, that is, for any $x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$, we have

$$\|x\|_\infty = \max_{1 \leq j \leq l} |x_j|, \qquad \|x\|_1 = \sum_{j=1}^{l} |x_j|. \tag{2.21}$$

Corollary 2.3. *In l-dimensional Euclidean space $\mathbb{R}^l$, there holds the following result:*

$$\begin{aligned} \partial(\|x\|_1) &= \begin{cases} \{\xi \in \mathbb{R}^l : \|\xi\|_\infty = 1, \langle x, \xi \rangle = \|x\|_1\}, & x \neq 0, \\ \{\xi \in \mathbb{R}^l : \|\xi\|_\infty \leq 1\}, & x = 0, \end{cases} \\ &= \begin{cases} \left\{\xi \in \mathbb{R}^l : \xi_i = \dfrac{x_i}{|x_i|}, \ \textit{if} \ x_i \neq 0; \ \xi_i \in [-1, 1], \ \textit{if} \ x_i = 0\right\}, & x \neq 0, \\ \{\xi \in \mathbb{R}^l : \|\xi\|_\infty \leq 1\}, & x = 0. \end{cases} \end{aligned} \tag{2.22}$$

Let H be a Hilbert space and $f : H \to \mathbb{R}$ a functional. Recall that

(i) *f is convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, for all $0 < \lambda < 1$, for all $x, y \in H$;*

(ii) *f is strictly convex if $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$, for all $0 < \lambda < 1$, for all $x, y \in H$ with $x \neq y$;*

(iii) *f is coercive if $f(x) \to \infty$ whenever $\|x\| \to \infty$. See [19] for more details about convex functions.*

The following lemma gives the optimality condition for the minimizer of a convex functional over a closed convex subset.

Lemma 2.4 (see [20]). *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let $f : H \to \mathbb{R}$ be a convex and subdifferentiable functional. Then $x \in C$ is a solution of the problem*

$$\min_{x \in C} f(x) \tag{2.23}$$

if and only if there exists some $\xi \in \partial f(x)$ satisfying the following optimality condition:

$$\langle \xi, v - x \rangle \geq 0, \quad \forall v \in C. \tag{2.24}$$

3. Main Results

It is well known that SFP (1.1) is equivalent to the minimization problem

$$\min_{x \in C} \left\| (I - P_Q) A x \right\|^2. \tag{3.1}$$

Using the idea of Tikhonov's regularization method, Wang and Xu [14] studied the minimization problem in Hilbert spaces:

$$\min_{x \in C} \left\| (I - P_Q) A x \right\|^2 + \alpha \|x\|^2, \tag{3.2}$$

where $\alpha > 0$ is the regularization parameter.

In what follows, H_1 and H_2 in SFP (1.1) are restricted to $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, and $\|\cdot\|$ will stand for the usual 2-norm of any Euclidean space $\mathbb{R}^l$; that is, for any $x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$,

$$\|x\| = \sqrt{x_1^2 + \cdots + x_l^2}. \tag{3.3}$$

Inspired by the above work of Wang and Xu, we study properties of the approximating curve with 1-norm regularization scheme for the SFP, that is, the following minimization problem:

$$\min_{x \in C} \frac{1}{2} \left\| (I - P_Q) A x \right\|^2 + \alpha \|x\|_1, \tag{3.4}$$

where $\alpha > 0$ is the regularization parameter. Let

$$f_\alpha(x) = \frac{1}{2}\left\|(I - P_Q)Ax\right\|^2 + \alpha\|x\|_1. \tag{3.5}$$

It is easy to see that f_α is convex and coercive, so problem (3.4) has at least one solution. However, the solution of problem (3.4) may not be unique since f_α is not necessarily strictly convex. Denote by S_α the solution set of problem (3.4); thus we can assert that S_α is a nonempty closed convex set but may contain more than one element. The following simple example illustrates this fact.

Example 3.1. Let $C = \{(x, y) : x + y = 1\}$, $Q = \{(x, y) : x + y = 1/2\}$ and

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{3.6}$$

Then $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a bounded linear operator. Obviously, $S_\alpha = \{(x, y) : x + y = 1, x \geq 0, y \geq 0\}$ and it contains more than one element.

Proposition 3.2. *For any $\alpha > 0$, $x_\alpha \in S_\alpha$ if and only if there exists some $\xi \in \partial(\|x\|_1)$ satisfying the following inequality:*

$$\left\langle A^*(I - P_Q)Ax_\alpha + \alpha\xi, v - x_\alpha \right\rangle \geq 0, \quad \forall v \in C. \tag{3.7}$$

Proof. Let

$$f(x) = \frac{1}{2}\left\|(I - P_Q)Ax\right\|^2, \tag{3.8}$$

then

$$f_\alpha(x) = f(x) + \alpha\|x\|_1. \tag{3.9}$$

Since f is convex and differentiable with gradient

$$\nabla f(x) = A^*(I - P_Q)Ax, \tag{3.10}$$

f_α is convex, coercive, and subdifferentiable with the subdifferential

$$\partial f_\alpha(x) = \partial f(x) + \alpha\partial(\|x\|_1); \tag{3.11}$$

that is,

$$\partial f_\alpha(x) = A^*(I - P_Q)Ax + \alpha\partial(\|x\|_1). \tag{3.12}$$

By Corollary 2.3 and Lemma 2.4, the proof is finished. □

Theorem 3.3. *Denote by x_α an arbitrary element of S_α, then the following assertions hold:*

(i) $\|x_\alpha\|_1$ *is decreasing for* $\alpha \in (0, \infty)$*;*

(ii) $\|(I - P_Q)Ax_\alpha\|$ *is increasing for* $\alpha \in (0, \infty)$*.*

Proof. Let $\alpha > \beta > 0$, for any $x_\alpha \in S_\alpha$, $x_\beta \in S_\beta$. We immediately obtain

$$\frac{1}{2}\left\|(I - P_Q)Ax_\alpha\right\|^2 + \alpha\|x_\alpha\|_1 \le \frac{1}{2}\left\|(I - P_Q)Ax_\beta\right\|^2 + \alpha\left\|x_\beta\right\|_1, \tag{3.13}$$

$$\frac{1}{2}\left\|(I - P_Q)Ax_\beta\right\|^2 + \beta\left\|x_\beta\right\|_1 \le \frac{1}{2}\left\|(I - P_Q)Ax_\alpha\right\|^2 + \beta\|x_\alpha\|_1. \tag{3.14}$$

Adding up (3.13) and (3.14) yields

$$\alpha\|x_\alpha\|_1 + \beta\left\|x_\beta\right\|_1 \le \alpha\left\|x_\beta\right\|_1 + \beta\|x_\alpha\|_1, \tag{3.15}$$

which implies $\|x_\alpha\|_1 \le \|x_\beta\|_1$. Hence (i) holds.

Using (3.14) again, we have

$$\frac{1}{2}\left\|(I - P_Q)Ax_\beta\right\|^2 \le \frac{1}{2}\left\|(I - P_Q)Ax_\alpha\right\|^2 + \beta\left(\|x_\alpha\|_1 - \left\|x_\beta\right\|_1\right), \tag{3.16}$$

which together with (i) implies

$$\left\|(I - P_Q)Ax_\beta\right\|^2 \le \left\|(I - P_Q)Ax_\alpha\right\|^2, \tag{3.17}$$

and hence (ii) holds. □

Let $\mathcal{F} = C \cap A^{-1}(Q)$, where $A^{-1}(Q) = \{x \in \mathbb{R}^N : Ax \in Q\}$. In what follows, we assume that $\mathcal{F} \ne \emptyset$; that is, the solution set of SFP (1.1) is nonempty. The fact that $\mathcal{F}$ is nonempty closed convex set thus allows us to introduce the concept of minimum-norm solution of SFP (1.1) in the sense of norm $\|\cdot\|$ (induced by the inner product).

Definition 3.4 (see [14]). An element $x^\dagger \in \mathcal{F}$ is said to be the *minimum-norm solution* of SFP (1.1) *in the sense of norm* $\|\cdot\|$ if $\|x^\dagger\| = \inf_{x\in\mathcal{F}}\|x\|$. In other words, $x^\dagger$ is the projection of the origin onto the solution set $\mathcal{F}$ of SFP (1.1). Thus there exists only one minimum-norm solution of SFP (1.1) in the sense of norm $\|\cdot\|$, which is always denoted by $x^\dagger$.

We can also give the concept of minimum-norm solution of SFP (1.1) in other senses.

Definition 3.5. An element $\tilde{x} \in \mathcal{F}$ is said to be a *minimum-norm solution* of SFP (1.1) *in the sense of 1-norm* if $\|\tilde{x}\|_1 = \inf_{x\in\mathcal{F}}\|x\|_1$. We use $\mathcal{F}_1$ to stand for all minimum-norm solutions of SFP (1.1) in the sense of 1-norm and $\mathcal{F}_1$ is called the minimum-norm solution set of SFP (1.1) in the sense of 1-norm.

Obviously, $\mathcal{F}_1$ is a closed convex subset of $\mathcal{F}$. Moreover, it is easy to see that $\mathcal{F}_1 \ne \emptyset$. Indeed, taking a sequence $\{x_n\} \subset \mathcal{F}$ such that $\|x_n\|_1 \to \inf_{x\in\mathcal{F}}\|x\|_1$ as $n \to \infty$, then $\{x_n\}$

is bounded. There exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Set $\overline{x} = \lim_{k\to\infty} x_{n_k}$, then $\overline{x} \in \mathcal{F}$ since $\mathcal{F}$ is closed. On the other hand, using lower semicontinuity of the norm, we have

$$\|\overline{x}\| \leq \lim_{k\to\infty} \|x_{n_k}\| = \inf_{x\in\mathcal{F}} \|x\|_1, \tag{3.18}$$

and this implies that $\overline{x} \in \mathcal{F}_1$.

However, $\mathcal{F}_1$ may contain more than one elements, in general (see Example 3.1, $\mathcal{F}_1 = \{(x, y) : x + y = 1, x, y \geq 0\}$).

Theorem 3.6. *Let $\alpha > 0$ and $x_\alpha \in S_\alpha$. Then $\omega(x_\alpha) \subset \mathcal{F}_1$, where $\omega(x_\alpha) = \{x : \exists\{x_{\alpha_k}\} \subset \{x_\alpha\}, x_{\alpha_k} \to x$ weakly$\}$.*

Proof. Taking $\tilde{x} \in \mathcal{F}_1$ arbitrarily, for any $\alpha \in (0, \infty)$, we always have

$$\frac{1}{2}\left\|(I - P_Q)Ax_\alpha\right\|^2 + \alpha\|x_\alpha\|_1 \leq \frac{1}{2}\left\|(I - P_Q)A\tilde{x}\right\|^2 + \alpha\|\tilde{x}\|_1. \tag{3.19}$$

Since $\tilde{x}$ is a solution of SFP (1.1), $\|(I - P_Q)A\tilde{x}\| = 0$. This implies that

$$\frac{1}{2}\left\|(I - P_Q)Ax_\alpha\right\|^2 + \alpha\|x_\alpha\|_1 \leq \alpha\|\tilde{x}\|_1, \tag{3.20}$$

then,

$$\|x_\alpha\|_1 \leq \|\tilde{x}\|_1; \tag{3.21}$$

thus $\{x_\alpha\}$ is bounded.

Take $\omega \in \omega(x_\alpha)$ arbitrarily, then there exists a sequence $\{\alpha_n\}$ such that $\alpha_n \to 0$ and $x_{\alpha_n} \to \omega$ as $n \to \infty$. Put $x_{\alpha_n} = x_n$. By Proposition 3.2, we deduce that there exists some $\xi_n \in \partial(\|x_n\|_1)$ such that

$$\left\langle A^*(I - P_Q)Ax_n + \alpha_n\xi_n, \tilde{x} - x_n\right\rangle \geq 0. \tag{3.22}$$

This implies that

$$\left\langle (I - P_Q)Ax_n, A(\tilde{x} - x_n)\right\rangle \geq \alpha_n\langle \xi_n, x_n - \tilde{x}\rangle. \tag{3.23}$$

Since $A\tilde{x} \in Q$, the characterizing inequality (2.2) gives

$$\left\langle (I - P_Q)Ax_n, A\tilde{x} - P_Q(Ax_n)\right\rangle \leq 0, \tag{3.24}$$

then,

$$\left\|(I - P_Q)Ax_n\right\|^2 \leq \left\langle (I - P_Q)Ax_n, A(x_n - \tilde{x})\right\rangle. \tag{3.25}$$

Combining (3.23) and (3.25), we have

$$\begin{aligned}\|(I-P_Q)Ax_n\|^2 &\le \alpha_n\langle \xi_n, \widetilde{x}-x_n\rangle \\ &\le \alpha_n\|\xi_n\|_\infty\|\widetilde{x}-x_n\|_1 \\ &\le 2\alpha_n\|\widetilde{x}\|_1.\end{aligned} \tag{3.26}$$

Consequently, we get

$$\lim_{n\to\infty}\|(I-P_Q)Ax_n\| = 0. \tag{3.27}$$

Furthermore, noting the fact that $x_n \to \omega$ and $I - P_Q$ and A are all continuous operators, we have $(I - P_Q)A\omega = 0$; that is, $A\omega \in Q$; thus, $\omega \in \mathcal{F}$. Since $\widetilde{x}$ is a minimum-norm solution of SFP (1.1) in the sense of 1-norm, using (3.21) again, we get

$$\|\omega\|_1 \le \liminf_{n\to\infty}\|x_n\|_1 \le \|\widetilde{x}\|_1 = \min\{\|x\|_1 : x \in \mathcal{F}\}. \tag{3.28}$$

Thus we can assert that $\omega \in \mathcal{F}_1$ and this completes the proof. □

Corollary 3.7. *If $\mathcal{F}_1$ contains only one element $\widetilde{x}$, then $x_\alpha \to \widetilde{x}$, ($\alpha \to 0$).*

Remark 3.8. It is worth noting that the minimum-norm solution of SFP (1.1) in the sense of norm $\|\cdot\|$ is very different from the minimum-norm solution of SFP (1.1) in the sense of 1-norm. In fact, $x^\dagger$ may not belong to $\mathcal{F}_1$! The following simple example shows this fact.

Example 3.9. Let $C = \{(x,y) : x+2y \ge 2, x \ge 0, y \ge 0\}$, $Q = \{(x,y) : x+y = 1, x \ge 0, y \ge 0\}$, and

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.29}$$

It is not hard to see that $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded linear operator and $A(x,y)^T = ((1/2)x, y)^T$, for all $(x,y) \in C$. Obviously, $\mathcal{F} = \{(x,y) : x+2y = 2, x \ge 0, y \ge 0\}$, $x^\dagger = (2/5, 4/5)$, but $\mathcal{F}_1 = \{(0,1)\}$. Hence, $x^\dagger \in \mathcal{F} \setminus \mathcal{F}_1$.

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